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STRONGLY NONLINEAR VARIATIONAL PARABOLIC PROBLEMS IN WEIGHTED SOBOLEV SPACES

L. AHROUCH, E. AZROUL AND M. RHOUDAF

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DÉP. MATH. FACULTÉ DES SCIENCES DHAR-MAHRAZ, B.P 1796 ATLAS FÈS, MAROC.
rhoudaf_mohamed@yahoo.fr

ABSTRACT. In this paper , we study the existence of a weak solutions for the initial-boundary value problems of the strongly nonlinear degenerated parabolic equation,

$$\frac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f,$$

where A is a Leray-lions operator acted from $L^p(0,T,W_0^{1,p}(\Omega,w))$ into its dual. $g(x,t,u,\nabla u)$ is a nonlinear term with critical growth condition with respect to ∇u and no growth with respect to u. The source term f is assumed to belong to $L^{p'}(0,T,W^{-1,p'}(\Omega,w^*))$.

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1. INTRODUCTION

On a cylinder $Q = \Omega \times (0,T)$, over the bounded domain $\Omega \subset \mathbb{R}^N, T > 0$ we consider the parabolic initial-boundary value problem.

$$(P) = \begin{cases} \frac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f & \text{in } Q \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $Au = -\text{div}a(x,t,u,\nabla u)$ is a classical divergence operator of Leray-Lions type defined from $L^p(0,T,W_0^{1,p}(\Omega)), 1 < p < \infty$ into its dual space $L^{p'}(0,T,W^{-1,p'}(\Omega))$ and where the perturbation term g satisfies the "natural" growth condition (of order p)

$$|g(x,t,u,\nabla u)| \leq b(|u|)[c(x,t) + |\nabla u|^p],$$

and the sign condition

$$g(x,t,u,\nabla u)u \geq 0.$$

The right hand side f is assumed to belong to $L^{p'}(0,T,W^{-1,p'}(\Omega,w^*))$.
Landes-Mustonen have proved in [13] the existence of a solution for the parabolic initial-boundary value problems (P). Their result generalizes the analogous ones of Lions [16] and Landes [14] with $g \equiv 0$ and of Brezis [8] and Landes [15] with $g \equiv g(x,t,u)$ (see [6, 7, 9]).
In all the previous works, the principal part a of the operator A is supposed to satisfy the classical coercivity,

$$a(x,t,s,\xi)\xi \geq \alpha|\xi|^p, \tag{1.1}$$

where α is some strictly positive constant.
When the operator A becomes generated on the variable space x, that is (1.1) is replaced by the following (called degeneracy),

$$\sum_{i=1}^{N} a_i(x,t,s,\xi)\xi_i \geq \sum_{i=1}^{N} w_i(x)|\xi_i|^p, \tag{1.2}$$

when $w = \{w_i, 1 \leq i \leq N\}$ is a family of weight functions on Ω, (i.e., positive measurable functions defined in Ω), the operator like A is not coercive in the classical Sobolev space, thus we must change this setting by the more general one, called weighted Sobolev spaces $W_0^{1,p}(\Omega,w)$.
Note that little information is known about the degenerate parabolic problems. Similar problems for degenerate nonlinear elliptic equations have been studied in [2, 3, 4, 10].
It's our purpose in this paper to study the existence result for the strongly parabolic problem (P), in the setting of weighted Sobolev space where the principal part a and the nonlinearity g satisfy some general growth conditions (see (A_2) and (A_3) below). The simplest model of our problem is the following,

$$(P) \begin{cases} \frac{\partial u}{\partial t} - \text{div}(|x|^s|Du|^{p-2}Du) + c_0|u|^{p-2}u = f & \text{in } Q \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

For that, we must extend the classical Sobolev techniques of parabolic problems, on the general settings of weighted Sobolev space. Firstly, we have studied some functional properties of a time-regularization,

$$u_\mu = \mu\int_{-\infty}^{t} \tilde{u}(x,s)\exp(\mu(s-t))\,ds \ \text{ where } \ \tilde{u}(x,s) = u(x,s)\chi_{(0,T)}(s), \tag{1.3}$$

for a function u in $L^p(Q,w)$ (see section 3.1 below).
Secondly, in Section 3.2 we have established some embedding and some compactness results in weighted Sobolev space (which give in particularly some trace result and an extension of the Aubin's and Simon's results [18], and which, are sufficient to deal with our approximate problem (P_n).)
The last part of our paper is devoted to an example which illustrates our hypotheses.

2. BASIC ASSUMPTIONS

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $1 < p < \infty$ and $w = \{w_i(x),\ 1 \leq i \leq N\}$ be a vector of weight functions, i.e., every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω. Further, we suppose in all our considerations that, there exist

$$r_0 > \max(N,p) \text{ such that } w_i^{\frac{r_0}{r_0-p}} \in L^1_{loc}(\Omega), \tag{2.1}$$

and

$$w_i^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega), \tag{2.2}$$

for any $0 \leq i \leq N$.

We denote by $W^{1,p}(\Omega,w)$ the space of all real-valued functions $u \in L^p(\Omega,w_0)$ such that the derivatives in the sense of distributions fulfill

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega,w_i) \text{ for all } i=1,...,N,$$

which is a Banach space under the norm

$$\|u\|_{1,p,w} = \left[\int_\Omega |u(x)|^p w_0\ dx + \sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\ dx\right]^{\frac{1}{p}}. \tag{2.3}$$

The condition (2.1) implies that $C_0^\infty(\Omega)$ is a subset of $W^{1,p}(\Omega,w)$ and consequently, we can introduce the subspace $W_0^{1,p}(\Omega,w)$ of $W^{1,p}(\Omega,w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.3). Moreover, the condition (2.2) implies that $W^{1,p}(\Omega,w)$ as well as $W_0^{1,p}(\Omega,w)$ are reflexive Banach spaces.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega,w)$ is equivalent to $W^{-1,p'}(\Omega,w^*)$, where $w^* = \{w_i^* = w_i^{1-p'},\ \ i=0,...,N\}$ and where p' is the conjugate of p i.e. $p' = \frac{p}{p-1}$. For more details, we refer the reader to [11].
Now, since we have not some general compactness result in the compacted Sobolev spaces, we need to assume the following:

Assumption $(\mathbf{A_1})$ For $2 \leq p < \infty$, the expression

$$|||u||| = \left(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\ dx\right)^{\frac{1}{p}} \tag{2.4}$$

is a norm on $W_0^{1,p}(\Omega,w)$ and its equivalent to (2.2). There exists a weight function σ on Ω such that

$$\sigma \in L^1(\Omega) \text{ and } \sigma^{-1} \in L^1(\Omega). \tag{2.5}$$

The Hardy inequality,

$$\left(\int_\Omega |u(x)|^p\sigma\, dx\right)^{\frac{1}{p}} \le c\left(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\, dx\right)^{\frac{1}{p}}, \tag{2.6}$$

holds for every $u \in W_0^{1,p}(\Omega, w)$ with a constant $c > 0$ independent of u. Moreover, the embedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^p(\Omega, \sigma) \tag{2.7}$$

expressed by the inequality (2.6) is compact.

Note that $(W_0^{1,p}(\Omega, w), |||.|||)$ is a uniformly convex (and thus reflexive) Banach space.

Remark 2.1. Assume that $w_0(x) \equiv 1$ and there exists $\nu \in \left]\frac{N}{P}, +\infty\right[\cap \left[\frac{1}{P-1}, +\infty\right[$ such that

$$w_i^{\frac{N}{N-1}} \in L^1_{loc}(\Omega), w_i^{-\nu} \in L^1(\Omega) \text{ for all } i = 1, ..., N. \tag{2.8}$$

Note that the assumptions (2.1) and (2.8) imply that

$$|||u||| = \left(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\, dx\right)^{\frac{1}{p}} \tag{2.9}$$

is a norm defined on $W_0^{1,p}(\Omega, w)$ and it's equivalent to (2.3) and that, the embedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow\hookrightarrow L^p(\Omega) \tag{2.10}$$

is compact [see [11], p. 46].
Thus the hypotheses (A_1) is satisfied for $\sigma \equiv 1$.

Assumption$(\mathbf{A_2})$

$$|a_i(x, t, s, \xi)| \le \beta w_i^{\frac{1}{p}}(x)\ \ [c_1(x, t) + \sigma^{\frac{1}{p'}}|s|^{p-1} + \sum_{j=1}^N w_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}] \tag{2.11}$$

$$[a(x, t, s, \xi) - a(x, t, s, \eta)](\xi - \eta) > 0 \text{ for all } \xi \ne \eta \in \mathbb{R}^N \tag{2.12}$$

$$a(x, t, s, \xi).\xi \ge \alpha\sum_{i=1}^N w_i|\xi_i|^p \tag{2.13}$$

where $c_1(x, t)$ is a positive function in $L^{p'}(Q)$, and α, β are strictly positive constants.
Assumption$(\mathbf{A_3})$

$$|g(x, t, s, \xi)| \le b(|s|)(\sum_{i=1}^N w_i(x)|\xi_i|^p + c(x, t)) \tag{2.14}$$

$$g(x, t, s, \xi)s \ge 0. \tag{2.15}$$

where $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function and c is a positive function in $L^1(Q)$.

We recall that for $k > 1$ and s in $\mathbb{R}$ the truncation is defined as

$$T_k(s) = \begin{cases} s & \text{if } s \le k \\ k\frac{s}{|s|} & \text{if } |s| > k \end{cases}.$$

3. MAIN RESULTS

3.1. Characterization of the time mollification of a function u.

In order to deal with time derivative, we introduce a time mollification of a function u belonging in some weighted Lebesgue space. Thus we define for all $\mu \geq 0$ and all $(x,t) \in Q$,

$$u_\mu = \mu \int_{-\infty}^{t} \tilde{u}(x,s)\text{exp}(\mu(s-t))\ ds \ \text{ where } \ \tilde{u}(x,s) = u(x,s)\chi_{(0,T)}(s).$$

Proposition 3.1. *If $u \in L^p(Q, w_i)$ then u_μ is measurable in Q and $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ and*

$$\left(\int_Q |u_\mu|^p w_i(x)\ dx\ dt\right)^{\frac{1}{p}} \leq \left(\int_Q |u|^p w_i(x)\ dx\ dt\right)^{\frac{1}{p}},$$

i.e.,

$$\|u_\mu\|_{L^p(Q,w_i)} \leq \|u\|_{L^p(Q,w_i)}.$$

Proof. Since $(x,t,s) \to u(x,s)\text{exp}(\mu(s-t))$ is measurable in $\Omega \times [0,T] \times [0,T]$, u_μ then is measurable by Fubini's theorem.

By Jensen's integral inequality and since $\displaystyle\int_{-\infty}^{0} \mu \exp(\mu s)\ ds = 1$, we have

$$\begin{aligned}\left|\int_{-\infty}^{t} \mu\tilde{u}(x,s)\exp(\mu(s-t)\ ds\right|^p &= \left|\int_{-\infty}^{0} \mu\exp(\mu s)\tilde{u}(x,s+t)\ ds\right|^p \\ &\leq \int_{-\infty}^{0} \mu\exp(\mu s)|\tilde{u}(x,s+t)|^p\ ds\end{aligned}$$

which implies,

$$\begin{aligned}\int_Q |u_\mu|^p w_i(x)\ dx\ dt &\leq \int_{\Omega\times\mathbb{R}} \left(\int_{-\infty}^{0} \mu\exp(\mu s)|\tilde{u}(x,s+t)|^p\ ds\right) w_i(x)\ dx\ dt \\ &\leq \int_{-\infty}^{0} \mu\exp(\mu s)\left(\int_{\Omega\times\mathbb{R}} |\tilde{u}(x,s+t)|^p w_i(x)\ dx\ dt\right)\ ds \\ &\leq \int_{-\infty}^{0} \mu\exp(\mu s)\left(\int_Q |u(x,t)|^p w_i(x)\ dx\ dt\right)\ ds \\ &= \int_Q |u(x,t)|^p w_i(x)\ dx\ dt = \|u\|^p_{L^p(Q,w_i)}.\end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{\partial u_\mu}{\partial t} &= \lim_{\theta\to 0}\frac{1}{\theta}(e^{-\mu\theta}-1)u_\mu(x,t) \\ &+ \lim_{\theta\to 0}\frac{1}{\theta}\int_t^{t+\theta} u(x,s)e^{\mu(s-(t+\theta))}\ ds = -\mu u_\mu + \mu u.\end{aligned} \tag{3.1}$$

∎

Proposition 3.2. *If $u \in W_0^{1,p}(Q,w)$, then $u_\mu \to u$ in $W_0^{1,p}(Q,w)$ as $\mu \to +\infty$.*

Proof. First, by applying the statement of Proposition 3.1 for u_μ and for $\frac{\partial u_\mu}{\partial x_i}$ (remark that $\frac{\partial u_\mu}{\partial x_i} = (\frac{\partial u}{\partial x_i})_\mu$), we can easily prove that $u_\mu \in W_0^{1,p}(Q,w)$.
Now we can prove that $u_\mu \to u$ in $W_0^{1,p}(Q,w)$ as $\mu \to +\infty$.

Let $(\varphi_k) \subset D(Q)$ such that $\varphi_k \to u$ in $W_0^{1,p}(Q,w)$.
In virtue of Proposition 3.1 we have,

$$|(\varphi_k)_\mu(x,t) - \varphi_k(x,t)| = \frac{1}{\mu}\left|\frac{\partial \varphi_k}{\partial t}(x,t)\right| \leq \frac{1}{\mu}\left\|\frac{\partial \varphi_k}{\partial t}\right\|_\infty. \tag{3.2}$$

On the other hand,

$$\begin{aligned} \int_Q |u_\mu - u|^p w_i(x)\,dx\,dt &\leq \int_Q |u_\mu - (\varphi_k)_\mu|^p w_i(x)\,dx\,dt \\ &+ \int_Q |(\varphi_k)_\mu - \varphi_k|^p w_i(x)\,dx\,dt \\ &+ \int_Q |\varphi_k - u|^p w_i(x)\,dx\,dt. \end{aligned}$$

This implies that

$$\begin{aligned} \int_Q |u_\mu - u|^p w_i(x)\,dx\,dt &\leq \int_Q |(u - \varphi_k)_\mu|^p w_i(x)\,dx\,dt \\ &+ \int_Q |(\varphi_k)_\mu - \varphi_k|^p w_i(x)\,dx\,dt \\ &+ \int_Q |\varphi_k - u|^p w_i(x)\,dx\,dt. \end{aligned}$$

Thanks to Proposition 3.1 and the equality (3.2), we get

$$\begin{aligned} \int_Q |u_\mu - u|^p w_i(x)\,dx\,dt &\leq 2\int_Q |u - \varphi_k|^p w_i(x)\,dx\,dt \\ &+ \frac{T}{\mu^p}\int_{K_k} \left|\frac{\partial \varphi_k}{\partial t}\right|^p w_i(x)\,dx \\ &\leq 2\int_Q |u - \varphi_k|^p w_i(x)\,dx\,dt + \frac{T}{\mu^p}C_k, \end{aligned}$$

where $C_k = \|\frac{\partial \varphi_k}{\partial t}\|_\infty^p \int_{K_k} w_i(x)\,dx$ and where K_k is a compact set such that $\sup \varphi_k \subset K_k$.
Let $\epsilon > 0$, there exists k such that

$$\int_Q |u - \varphi_k|^p w_i(x)\,dx\,dt \leq \frac{\epsilon}{3}$$

and there exists μ_0 such that

$$\frac{T}{\mu^p}C_k \leq \frac{\epsilon}{3} \text{ for all } \mu \geq \mu_0.$$

Hence

$$\int_Q |u_\mu - u|^p w_i(x)\,dx\,dt \leq \epsilon,$$

which implies that

$$\|u_\mu - u\|_{L^p(Q,w_i)} \leq \epsilon. \tag{3.3}$$

Since $D_x^\alpha(u_\mu) = (D_x^\alpha u)_\mu$ (for all $|\alpha| \leq 1$), then by applying the same argument as above for each $D_x^\alpha u$, we conclude the desired result. ∎

Proposition 3.3. *If $u_n \to u$ in $W_0^{1,p}(Q,w)$, then $(u_n)_\mu \to u_\mu$ in $W_0^{1,p}(Q,w)$.*

Proof. Using Proposition 3.1 and

$$D_x^\alpha((u_n)_\mu) - D_x^\alpha(u_\mu) = (D_x^\alpha(u_n))_\mu - (D_x^\alpha(u))_\mu = (D_x^\alpha(u_n) - D_x^\alpha(u))_\mu,$$

we have,

$$\int_Q |D_x^\alpha((u_n)_\mu) - D_x^\alpha(u_\mu)|^p w_i(x)\,dx\,dt \le \int_Q |D_x^\alpha(u_n) - D_x^\alpha(u)|^p w_i(x)\,dx\,dt \to 0$$

as $n \to \infty$.
Then $(u_n)_\mu \to u_\mu$ in $W_0^{1,p}(Q,w)$ as $n \to \infty$. ∎

3.2. **Some weighted embedding and compactness results.**
In this section we establish some embedding and compactness results in weighted Sobolev Spaces which allow in particular to extend in the settings of weighted Sobolev spaces, some trace results and the Aubin's and Simon's results [18].
Let $V = W_0^{1,p}(\Omega,w)$, $H = L^2(\Omega,\sigma)$ and let $V^* = W^{-1,p'}(\Omega,w^*)$, with $(2 \le p < \infty)$.
Let $X = L^p(0,T,V)$. The dual space of X is $X^* = L^{p'}(0,T,V^*)$ where $\frac{1}{p'} + \frac{1}{p} = 1$ and denoting the space $W_p^1(0,T,V,H) = \{v \in X : v' \in X^*\}$ endowed with the norm

$$\|u\|_{w_p^1} = \|u\|_X + \|u'\|_{X^*}, \tag{3.4}$$

which is a Banach space. Here u' stands for the generalized derivative of u, i.e.,

$$\int_0^T u'(t)\varphi(t)\,dt = -\int_0^T u(t)\varphi'(t)\,dt \text{ for all } \varphi \in C_0^\infty(0,T).$$

Lemma 3.4. *The Banach space H is an Hilbert space and its dual H' can be identified with himself, $i.e., H' \simeq H$.*

Indeed, let

$$\begin{aligned} F\ &: H \times H \to \mathbb{R} \\ &(f,g) \mapsto \int_\Omega fg\sigma\,dx. \end{aligned}$$

Remark that F is a symmetric bilinear form which is also continuous and defined positively, since

$$\int_\Omega fg\sigma\,dx = \int_\Omega f\sigma^{\frac12} g\sigma^{\frac12}\,dx \le \left(\int_\Omega |f|^2\sigma\,dx\right)^{\frac12}\left(\int_\Omega |g|^2\sigma\,dx\right)^{\frac12}.$$

Then the Banach space H is a Hilbert space.
Finally by a standard argument, H is identified with its dual H' i.e., $H' \simeq H$.

Lemma 3.5. *The evolution triple $V \subseteq H \subseteq V^*$ is verified.*

Indeed, by the embedding assumption (2.7) and because $2 \le p < \infty$, and $\sigma \in L^1(\Omega)$, we can write

$$W_0^{1,p}(\Omega,w) \hookrightarrow\hookrightarrow L^p(\Omega,\sigma) \hookrightarrow H \simeq H' \hookrightarrow W^{-1,p'}(\Omega,w^*).$$

Lemma 3.6. *Assume that $V_1 \hookrightarrow\hookrightarrow B \hookrightarrow V_2$ (where V_1, B and V_2 are Banach spaces). Let F be bounded in $L^p(0,T,V_1)$ where $1 \le p < \infty$, and $\frac{\partial F}{\partial t} = \{\frac{\partial f}{\partial t} : \ f \in F\}$ be bounded in $L^1(0,T,V_2)$, then F is relatively compact in $L^p(0,T,B)$*

Proof. (see [18] Corollary 4 p.85) ∎

Lemma 3.7. *Assume that (2.1) and (2.5) hold true. Then, for every compact $K \subset \Omega$ we have,*

i) $W_0^{1,p}(K,w) \hookrightarrow\hookrightarrow L^p(K,\sigma) \hookrightarrow W^{-1,r'_0}(K)$ *with* $\frac{1}{r_0} + \frac{1}{r'_0} = 1.$

ii) $W^{-1,p'}(K,w^*) \hookrightarrow W^{-1,r_0'}(K)$.

Remark 3.1. Note that the statement of Lemma 3.7, holds true when the hypothesis (2.1) is replaced by $w_i^{\frac{N}{N-1}} \in L^1_{loc}(\Omega)$ (it suffices to take $r_0 = Np$).

Proof.
i) First, we claim that,

$$L^p(K,\sigma) \hookrightarrow L^1(K).$$

Indeed , we have

$$\int_K |f|\ dx = \int_K |f|\sigma^{\frac{1}{p}}\sigma^{\frac{-1}{p}}\ dx \leq \left(\int_K |f|^p\sigma\ dx\right)^{\frac{1}{p}} \left(\int_K \sigma^{\frac{-p'}{p}}\ dx\right)^{\frac{1}{p'}}.$$

On the other hand, since $1 \leq r_0' \leq \frac{N}{N-1}$ we can classically write,

$$L^1(K) \hookrightarrow W^{-1,r_0'}(K).$$

Then, we deduce from the above embedding that,

$$L^p(K,\sigma) \hookrightarrow W^{-1,r_0'}(K).$$

Hence, from (2.7), the assertion i) follows.
ii) In order to prove that,

$$W^{-1,p'}(K,w^*) \hookrightarrow W^{-1,r_0'}(K),$$

it suffices to show that

$$W_0^{1,r_0}(K) \hookrightarrow W_0^{1,p}(K,w).$$

Indeed, let f be an element of $W_0^{1,r_0}(K)$,
applying the Hölder's inequality for the exponents such that $\frac{p}{r_0} + \frac{r_0-p}{r_0} = 1$ we get,

$$\int_K |f|^p w\ dx \leq \left(\int_K |f|^{r_0}\ dx\right)^{\frac{p}{r_0}} \left(\int_K w_i^{\frac{r_0}{r_0-p}}\ dx\right)^{\frac{r_0-p}{r_0}}.$$

Taking also $\frac{\partial f}{\partial x_i}$ instead of f, we can deduce that $\|f\|_{W_0^{1,p}(K,w))} \leq C\|f\|_{W_0^{1,r_0}(K)}$. ∎

Remark 3.2. The statement of Lemma 3.6, can be applied for the triple $(V_1 = W_0^{1,p}(K,w), B = L^p(K,\sigma), V_2 = W^{-1,r_0'}(K))$.

Lemma 3.8. *Assume that*

$$\frac{\partial u_n}{\partial t} = h_n + k_n \ \textit{ in } \ D'(\Omega),$$

*where h_n and k_n are bounded respectively in $L^{p'}(0,T,W^{-1,p'}(\Omega,w^*)$ and in $L^1(Q)$.
If u_n is bounded in $L^p(0,T,W_0^{1,p}(\Omega,w))$, then $u_n \to u$ in $L^p_{loc}(Q,\sigma)$. Further $u_n \to u$ strongly in $L^1(Q)$.*

Proof. Consider $\phi(x,t) = \psi(x)\eta(t)$ with ψ in $D(\Omega)$ and η in $D(0,T)$ and set

$$v_n = \phi u_n, \ \ \alpha_n = \phi h_n + \frac{\partial \phi}{\partial t}u_n, \ \ \beta_n = \phi k_n.$$

Then for any bounded open subset K with supp$\psi \subset K \subset \Omega$ we have,

$$\begin{cases} \frac{\partial v_n}{\partial t} = \alpha_n + \beta_n \ \text{ in } \ D'(K\times(0,T)); \\ v_n \ \text{ bounded in } \ L^p(0,T,W_0^{1,p}(K,w)); \\ \alpha_n \ \text{ bounded in } \ L^{p'}(0,T,W^{-1,p'}(K,w^*)); \\ \beta_n \ \text{ bounded in } \ L^1(K\times(0,T)). \end{cases}$$

Let ρ_n be a mollification sequence such that

$$\|\bar{v}_n - v_n\|_{L^p(Q,\sigma)} \leq \frac{1}{n}, \tag{3.5}$$

where

$$\bar{v}_n = v_n * \rho_n, \ \ \bar{\alpha_n} = \alpha_n * \rho_n, \ \ \bar{\beta}_n = \beta_n * \rho_n.$$

The function $\bar{\beta}_n$ belongs to $L^1(0,T,L^1(K))$ and is bounded in this space while $\bar{\alpha}_n$ is bounded in $L^{p'}(0,T,W^{-1,p'}(K,w^*))$. Since $1 \leq r_0' \leq \frac{N}{N-1}$, we can write

$$L^1(K) \hookrightarrow W^{-1,r_0'}(K).$$

Then $\bar{\beta}_n$ is bounded in $L^1(0,T,W^{-1,r_0'}(K))$ and by ii) of Lemma 3.7, we have $\bar{\alpha}_n$ also bounded in $L^1(0,T,W^{-1,r_0'}(K))$. Thus we get,

$$\begin{cases} \frac{\partial \bar{v}_n}{\partial t} = \bar{\alpha}_n + \bar{\beta}_n \ \text{ in } \ D'(K \times (0,T)) \\ \bar{v}_n \ \text{ bounded in } \ L^p(0,T,W_0^{1,p}(K,w)) \\ \frac{\partial \bar{v}_n}{\partial t} \ \text{ bounded in } \ L^1(0,T,W^{-1,r_0'}(K)) \end{cases}$$

By $i)$ Lemma 3.7 and Lemma 3.6 we deduce that $\bar{v}_n$ is relatively compact in $L^p(0,T,L^p(K,\sigma))$. In view of (3.5) this implies that u_n is relatively compact in $L^p_{loc}(Q,\sigma))$.
Finally by using Hölders inequality and $\sigma^{-1} \in L^1(\Omega)$, it is easy to deduce that $u_n \to u$ strongly in $L^1(Q)$. ∎

Lemma 3.9. *Let $g \in L^r(Q,\gamma)$ and let $g_n \in L^r(Q,\gamma)$, with $\|g_n\|_{L^r(Q,\gamma)} \leq c, 1 < r < \infty$. If $g_n(x) \to g(x)$ a.e in Q, then $g_n \rightharpoonup g$ in $L^r(Q,\gamma)$, where $\rightharpoonup$ denotes weak convergence and γ is a weight function on Q.*

Proof. Since $g_n\gamma^{\frac{1}{r}}$ is bounded in $L^r(Q)$ and $g_n(x)\gamma^{\frac{1}{r}}(x) \to g\gamma^{\frac{1}{r}}$, a.e. in Q, then by Lemma 3.5 [17], we have

$$g_n\gamma^{\frac{1}{r}} \rightharpoonup g\gamma^{\frac{1}{r}} \ \text{ in } \ L^r(Q).$$

Moreover, for all $\varphi \in L^{r'}(Q,\gamma^{1-r'})$, we have $\varphi\gamma^{\frac{-1}{r}} \in L^{r'}(Q)$. Then

$$\int_Q g_n\varphi \, dx \to \int_Q g\varphi \, dx, \ \text{ i.e. } \ g_n \rightharpoonup g \ \text{ in } \ L^r(Q,\gamma).$$

∎

Lemma 3.10. *Assume that (A_1) and (A_2) are satisfied and let (u_n) be a sequence in $L^p(0,T,W_0^{1,p}(\Omega,w))$ such that $u_n \rightharpoonup u$ weakly in $L^p(0,T,W_0^{1,p}(\Omega,w))$ and*

$$\int_Q [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)][\nabla u_n - \nabla u] \, dxdt \to 0. \tag{3.6}$$

Then, $u_n \to u$ in $L^p(0,T,W_0^{1,p}(\Omega,w))$.

Proof. Let $D_n = [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)][\nabla u_n - \nabla u]$. Then by (2.12), D_n is a positive function and by (3.6), $D_n \to 0$ in $L^1(Q)$.
Extracting a subsequence, still denoted by u_n, and using (2.7) we can write

$$u_n \to u \ \text{ a.e. in } \ Q, \quad D_n \to 0 \ \text{ a.e. in } \ Q.$$

Then, there exists a subset B of Q, of zero measure such that,
for $(t,x) \in Q \setminus B$, $|u_n(x,t)| < \infty$, $|\nabla u(x,t)| < \infty$, $|c_1(x,t)| < \infty$, $w_i(x) > 0$ and $u_n(x,t) \to$

$u(x,t)$, $D_n(x,t) \to 0$.
We set $\epsilon_n = \nabla u_n(x,t)$ and $\epsilon = \nabla u(x,t)$. Then

$$\begin{aligned} D_n(x,t) &= [a(x,t,u_n,\epsilon_n) - a(x,t,u_n,\epsilon)](\epsilon_n - \epsilon) \\ &\geq \alpha \sum_{i=1}^{N} w_i |\epsilon_n^i|^p + \alpha \sum_{i=1}^{N} w_i |\epsilon^i|^p \\ &- \sum_{i=1}^{N} \beta w_i^{\frac{1}{p}} \left[c_1(x,t) + \sigma^{\frac{1}{p'}} |u_n|^{p-1} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}} |\epsilon_n^j|^{p-1} \right] |\epsilon^i| \\ &- \sum_{i=1}^{N} \beta w_i^{\frac{1}{p}} \left[c_1(x,t) + \sigma^{\frac{1}{p'}} |u_n|^{p-1} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}} |\epsilon^j|^{p-1} \right] |\epsilon_n^i| \end{aligned}$$

i.e,

$$D_n(x,t) \geq \alpha \sum_{i=1}^{N} w_i |\epsilon_n^i|^p - c_{x,t} \left[1 + \sum_{j=1}^{N} w_j^{\frac{1}{p'}} |\epsilon_n^j|^{p-1} + \sum_{i=1}^{N} w_i^{\frac{1}{p}} |\epsilon_n^i| \right], \tag{3.7}$$

where $c_{x,t}$ is a constant which depends on x and t, but does not depend on n.
Since $u_n(x,t) \to u(x,t)$, we have $|u_n(x,t)| \leq M_{x,t}$, where $M_{x,t}$ is some positive constant. Then by a standard argument $|\epsilon_n|$ is bounded uniformly with respect to n. Indeed, (3.7) becomes

$$D_n(x,t) \geq \sum_{i=1}^{N} |\epsilon_n^i|^p \left(\alpha w_i - \frac{c_{x,t}}{N|\epsilon_n^i|^p} - \frac{c_{x,t} w_i^{\frac{1}{p'}}}{|\epsilon_n^i|} - \frac{c_{x,t} w_i^{\frac{1}{p}}}{|\epsilon_n^i|^{p-1}} \right).$$

If $|\epsilon_n| \to \infty$ (for a subsequence) there exists at least one i_0 such that $|\epsilon_n^{i_0}| \to \infty$, which implies that $D_n(x,t) \to \infty$, which gives a contradiction.

Let now ϵ^* be a cluster point of ϵ_n. We have $|\epsilon^*| < \infty$ and by the continuity of a with respect to the two last variables we obtain

$$(a(x,t,u(x,t),\epsilon^*) - a(x,t,u(x,t),\epsilon))(\epsilon^* - \epsilon) = 0.$$

In view of (2.12) we have $\epsilon^* = \epsilon$. The uniqueness of the cluster point implies

$$\nabla u_n(x,t) \to \nabla u(x,t) \text{ a.e. in } Q.$$

Since the sequence $a(x,t,u_n,\nabla u_n)$ is bounded in $\prod_{i=1}^{N} L^{p'}(Q, w_i^*)$ and $a(x,t,u_n,\nabla u_n) \to a(x,t,u,\nabla u)$ a.e. in Q Lemma 3.9 implies

$$a(x,t,u_n,\nabla u_n) \rightharpoonup a(x,t,u,\nabla u) \text{ in } \prod_{i=1}^{N} L^{p'}(Q, w_i^*) \text{ and a.e. in } Q.$$

We set $\bar{y}_n = a(x,t,u_n,\nabla u_n)\nabla u_n$ and $\bar{y} = a(x,t,u,\nabla u)\nabla u$. As in the proof of Lemma 5 in [6] we can write $\bar{y}_n \to \bar{y}$ in $L^1(Q)$. By (2.13), we have

$$\alpha \sum_{i=1}^{N} w_i |\frac{\partial u_n}{\partial x_i}| \leq a(x,t,u_n,\nabla u_n)\nabla u_n.$$

Let $z_n = \sum_{i=1}^{N} w_i|\frac{\partial u_n}{\partial x_i}|^p,\ z = \sum_{i=1}^{N} w_i|\frac{\partial u}{\partial x_i}|^p,\ y_n = \frac{\bar{y}_n}{\alpha}$ and $y = \frac{\bar{y}}{\alpha}$. Then, by Fatou's Lemma we obtain

$$\int_Q 2y\ dxdt \le \lim_{n\to\infty}\inf \int_Q y + y_n - |z_n - z|\ dxdt,$$

i.e.

$$0 \le \lim_{n\to\infty}\sup \int_Q |z_n - z|\ dxdt,$$

hence,

$$0 \le \lim_{n\to\infty}\inf \int_Q |z_n - z|\ dxdt \le \lim_{n\to\infty}\sup \int_Q |z_n - z|\ dxdt \le 0.$$

This implies

$$\nabla u_n \to \nabla u \ \text{ in } \prod_{i=1}^{N} L^p(Q, w_i),$$

which with (2.4) completes the present proof. ∎

Now we recall the well-known general Sobolev embedding theorems for evolution equations (see[19]).

Lemma 3.11. [19]*Let $V \subseteq H \subseteq V^*$ be an evolution triple. Then the embedding*

$$W_p^1(0, T, V, H) \subseteq C([0, T]), H)$$

is continuous .

Lemma 3.12. [19] *Let Z_1, Y, Z_2 be real reflexive Banach space. Assume that the embedding $Z_1 \subseteq Y \subseteq Z_2$ are continuous, and the embedding $Z_1 \subseteq Y$ is compact,* $0 < T < \infty, 1 < p, q < \infty$. *Then $W = \{u \in L^p(0, T, Z_1) : u' \in L^q(0, T, Z_2)\}$ equipped with the norm*

$$\|u\|_w = \|u\|_{L^p(0,T,Z_1)} + \|u'\|_{L^q(0,T,Z_2)}$$

is a Banach space and the embedding $W \subseteq L^p(0, T, Y)$ is compact.

Definition 3.1. A monotone map $T : D(T) \to X^*$ is called maximal monotone if its graph

$$G(T) = \{(u, T(u)) \in X \times X^* \text{for all } u \in D(T)\}$$

is not a proper subset of any monotone set in $X \times X^*$.

Let us consider the operator $\frac{\partial}{\partial t}$ which induces a linear map L from the subset $D(L) = \{v \in X : v' \in X^*, v(0) = 0\}$ of X into X^* by

$$\langle Lu, v\rangle_X = \int_0^T \langle u'(t), v(t)\rangle_V\ dt \ \ u \in D(L),\ \ v \in X. \tag{3.8}$$

Lemma 3.13. [19] *L is a closed linear maximal monotone map.*

In our study we deal with mappings of the form $F = L + S$ where L is a given linear densely defined maximal monotone map from $D(L) \subset X$ to X^* and S is a bounded demicontinuous map of monotone type from X to X^*.

Definition 3.2. A mapping S is called pseudo-monotone with respect to $D(L)$, if for any sequence $\{u_n\}$ in $D(L)$ with $u_n \rightharpoonup u$ and $Lu_n \rightharpoonup Lu$ and $\lim_{n\to\infty}\sup\langle S(u_n), u_n - u\rangle \le 0$, we have $\lim_{n\to\infty}\langle S(u_n), u_n - u\rangle = 0$ and $S(u_n) \rightharpoonup S(u)$ as $n \to \infty$.

Consider the following non linear parabolic problem

$$(P):\begin{cases} \frac{\partial u}{\partial t}+A(u)+g(x,t,u,\nabla u)=f & \text{in } Q\\ u(x,t)=0 & \text{on } \partial\Omega\times(0,T)\\ u(x,0)=u_0(x) & \text{in } \Omega,\end{cases}$$

where u_0 is a given function in $L^2(\Omega,\sigma)$.

Definition 3.3. A function u is said to be a weak solution of the initial-boundary value problem (P) if $u\in C([0,T],H)\cap L^p(0,T,V)$, $\frac{\partial u}{\partial t}\in L^{p'}(0,T,V^*)$ and u satisfies the equation,

$$\frac{\partial u}{\partial t}+Au+G(u)=f \quad 0<t<T, u(0)=u_0,$$

where the operator $A+G:X\to X^*$ is defined by:

$$\langle (A+G)(u),v\rangle=\int_Q a(x,t,u,\nabla u)\nabla v\,dx\,dt+\int_Q g(x,t,u,\nabla u)v\,dx\,dt.$$

3.3. **The approximate problem .**
We consider the sequence of approximate equations,

$$(P_n):\begin{cases} \frac{\partial u_n}{\partial t}+A(u_n)+g_n(x,t,u_n,\nabla u_n)=f\\ u_n(0)=u_0^n\end{cases},$$

where

$$g_n(x,t,s,\xi)=\frac{g(x,t,s,\xi)}{1+\frac{1}{n}|g(x,t,s,\xi)|}\chi_{\Omega_n},\ \ u_0^n\in W_0^{1,p}(\Omega,w).$$

Note that Ω_n is a sequence of compacts covering the bounded open set Ω and χ_{Ω_n} denotes a characteristic function of $Omega_n$.

Recall that $g_n(x,t,s,\xi)$ satisfies the following conditions

$$g_n(x,t,s,\xi).s\geq 0,\ \ |g_n(x,t,s,\xi)|\leq g(x,t,s,\xi)\ \text{ and }\ |g_n(x,t,s,\xi)|\leq n.$$

We define the operator $G_n:X\to X^*$ by $\langle G_nu,v\rangle=\displaystyle\int_Q g_n(x,t,u,\nabla u)v\,dx\,dt.$

Remark 3.3. Note that in the remainder of this section we will consider the approximate problem (P_n) with $u_0^n=0$, without losing the generalities, since if $u_0^n\neq 0$, we will change $a(x,t,u,\nabla u)$ by $\bar a(x,t,u,\nabla u)=a(x,t,u+u_0,\nabla u+\nabla u_0)$ and $g(x,t,u,\nabla u)$ by $\bar g(x,t,u,\nabla u)=g(x,t,u+u_0,\nabla u+u_0)$.

Lemma 3.14. *The operator $A+G_n:X\to X^*$ is :*

a) *bounded and demicontinuous*
b) *pseudo-monotone with respect to $D(L)$*
c) *strongly coercive, i.e.,*

$$\frac{\langle (A+G_n)(u),u\rangle_X}{\|u\|_X}\to+\infty,\ \ \textit{as}\ \ \|u\|_X\to+\infty.$$

Proof.

a) We set $B_n=A+G_n$. Using (2.11) and Hölder's inequality we can show that A is bounded. Thanks of Hölder's inequality, for all $u\in X$ and all $v\in X$ we have

$$|\int_Q g_n(x,t,u,\nabla u)v\,dx\,dt|\leq C_n\|v\|^p_{L^p(Q,\sigma)}\leq C_n'\|v\|^p_X. \tag{3.9}$$

Then B_n is bounded.
In order to show that B_n is demicontinuous, let $v_\epsilon \longrightarrow v$ in X as $\epsilon \longrightarrow 0$, and prove that,

$$\langle B_n(v_\epsilon), \varphi \rangle \longrightarrow \langle B_n(v), \varphi \rangle \text{ for all } \varphi \in X.$$

Since, $a_i(x,t,v_\epsilon,\nabla v_\epsilon) \to a_i(x,t,v,\nabla v)$ as $\epsilon \longrightarrow 0$, for a.e. $x \in \Omega$, then by the growth conditions (2.11) and Lemma 3.9 we get

$$a_i(x,t,v_\epsilon,\nabla v_\epsilon) \rightharpoonup a_i(x,t,v,\nabla v) \text{ in } L^{p'}(Q, w_i^{1-p'}) \text{ as } \epsilon \longrightarrow 0.$$

Finally for all $\varphi \in X$,

$$\langle A(v_\epsilon), \varphi \rangle \to \langle A(v), \varphi \rangle \text{ as } \epsilon \longrightarrow 0.$$

On the other hand, $g_n(x,t,v_\epsilon,\nabla v_\epsilon) \to g_n(x,t,v,\nabla v)$ as $\epsilon \longrightarrow 0$ for a.e. (x,t) in Q. Also $g_n(x,t,v_\epsilon,\nabla v_\epsilon)_\epsilon$ is bounded in $L^{p'}(Q,\sigma^{1-p'})$ in fact,

$$\int_Q |g_n(x,t,v_\epsilon,\nabla v_\epsilon)|^{p'} \sigma^{1-p'}\, dx\, dt \le (n)^{p'} T \int_{\Omega_n} \sigma^{1-p'}\, dx \le c_n.$$

Then, Lemma 3.9 gives

$$g_n(x,t,v_\epsilon,\nabla v_\epsilon) \rightharpoonup g_n(x,t,v,\nabla v) \text{ in } L^{p'}(Q,\sigma^{1-p'}) \text{ as } \epsilon \longrightarrow 0.$$

Since $\varphi \in L^p(Q,\sigma)$, for all $\varphi \in X$ we have

$$\langle G_n(v_\epsilon), \varphi \rangle \to \langle G_n(v), \varphi \rangle \text{ as } \epsilon \longrightarrow 0.$$

b) Suppose that $\{u_j\}$ is any sequence in $D(L)$ with
   i) $u_j \rightharpoonup u$ weakly in X
   ii) $Lu_j \to Lu$ weakly in X^*,
   iii) $\lim\sup \langle A + G_n(u_j), u_j - u\rangle_X \le 0$.

Through the definition of the operator L, defined in (3.8), $\{u_j\}$ is a bounded sequence in $W_p^1(0,T,V,H)$.
By virtue of Lemma 3.12, we get,

$$u_j \to u \text{ strongly in } L^p(Q,\sigma).$$

On the other hand,

$$\langle G_n u_j, u_j - u\rangle = \int_Q g_n(x,t,u_j,\nabla u_j)(u_j - u)\, dx\, dt.$$

Thus Hölder's inequality and (i) imply,

$$\begin{aligned}\langle G_n u_j, u_j - u\rangle &\le \left(\int_Q |g_n|^{p'} \sigma^{1-p'}\, dx\, dt\right)^{\frac{1}{p'}} \|u_j - u\|_{L^p(Q,\sigma)} \\ &\le C_n \|u_j - u\|_{L^p(Q,\sigma)},\end{aligned}$$

i.e, $\langle G_n u_j, u_j - u\rangle \to 0$ as $j \to \infty$. Combining the last convergence with (iii), we obtain

$$\lim_{j\to\infty} \sup \langle A u_j, u_j - u\rangle \le 0.$$

Also, by the pseudo-monotonicity of A (see Proposition 1, [10]),we have

$$Au_j \rightharpoonup Au \text{ in } X^* \text{ and } \lim_{j\to\infty} \langle Au_j, u_j - u\rangle = 0.$$

Then,

$$\lim_{j\to\infty} \langle Au_j + G_n(u_j), u_j - u\rangle = 0.$$

On the other hand, $\lim_{j\to\infty}\langle Au_j, u_j - u\rangle = 0$, which implies that

$$\begin{aligned} 0 &= \lim_{j\to\infty}\int_Q a(x,t,u_j,\nabla u_j)\nabla(u_j-u)\,dx\,dt \\ &= \lim_{j\to\infty}\int_Q [a(x,t,u_j,\nabla u_j)-a(x,t,u_j,\nabla u)][\nabla u_j-\nabla u]\,dx\,dt \\ &+ \lim_{j\to\infty}\int_Q a(x,t,u_j,\nabla u)(\nabla u_j-\nabla u)\,dx\,dt. \end{aligned}$$

The last integral in the right hand tends to zero, since by the continuity of the Nemytskii operator, $a(x,t,u_j,\nabla u)\to a(x,t,u,\nabla u)$ in $\prod_{i=1}^{N} L^{p'}(Q, w_i^{1-p'})$ as $j\to+\infty$.

So,

$$\lim_{j\to\infty}\int_Q [a(x,t,u_j,\nabla u_j)-a(x,t,u_j,\nabla u)][\nabla u_j-\nabla u]\,dx\,dt = 0.$$

By Lemma 3.10 we have

$$\nabla u_j \to \nabla u \quad \text{a.e. in } Q.$$

Hence $g_n(x,t,u_j,\nabla u_j)\to g_n(x,t,u,\nabla u)$ a.e. in Q as $j\to\infty$ and since

$$|g_n(x,t,u_j,\nabla u_j)| \le n\chi_{\Omega_n} \in L^{p'}(Q,\sigma^{1-p'}),$$

by Lebesque's dominated convergence theorem, we obtain

$$g_n(x,t,u_j,\nabla u_j)\to g_n(x,t,u,\nabla u) \text{ in } L^{p'}(Q,\sigma^{1-p'}).$$

Finally,

$$(A+G_n)(u_j) \rightharpoonup (A+G_n)(u) \text{ in } X^*.$$

c) The strong coercivity follows from (2.13) and (2.15)

∎

Definition 3.4. A function u is said to be a weak solution for the problem (P_n) iff $u\in C([0,T],H)\cap D(L)$ and u satisfies the evolution equation

$$Lu + (A+G_n)(u) = f. \tag{3.10}$$

Theorem 3.15. *Assume that the conditions $(A_1)-(A_3)$ hold, then the problem (P_n) admits a weak solution for any $f\in X^*$.*

Proof. By virtue of Lemma 3.14, the operator $A+G_n : X\to X^*$ is pseudo-monotone with respect to $D(L)$, and the operator $A+G_n$ satisfies the strong coercivity condition, which implies that both of the conditions (i) and (ii) in Theorem 4 of [5] hold. So all the conditions of Theorem 4 in [5] are met. Therefore, there exists a solution $u_n\in D(L)$ of the evolution equation (3.10) for any $f\in X^*$. In order to prove that u_n is also a weak solution of the problem (P_n), we have to show that $u_n\in C([0,T],H)$. By the definition of $D(L)$ and Lemma 3.11, we obtain

$$D(L)\subseteq W_p^1(0,T,V,H)\subseteq C([0,T],H).$$

This implies that $u_n\in C([0,T],H)$. ∎

3.4. **Existence results of the general problem.**

Theorem 3.16. *Assume that the conditions* (A_1)-(A_3) *hold true. Then the problem* (P) *admits at least one weak solution* $u \in D(A) \cap L^p(0,T,W_0^{1,p}(\Omega,w)) \cap C([0,T],L^2(\Omega,\sigma))$ *such that* $g(x,t,u,\nabla u) \in L^1(Q), g(x,t,u,\nabla u)u \in L^1(Q)$.
Furthermore $u(x,0) = u_0(x)$ *for a.e.* $x \in \Omega$ *and we have*

$$\begin{aligned} - & \int_Q u\frac{\partial \varphi}{\partial t}\,dx\,dt + [\int_\Omega u(t)\varphi(t)\,dx]_0^T + \int_Q a(x,t,u,\nabla u)\nabla\varphi\,dx\,dt \\ + & \int_Q g(x,t,u,\nabla u)\varphi\,dx\,dt = \langle f,\varphi\rangle \end{aligned}$$

for all $\varphi \in L^p(0,T,W_0^{1,p}(\Omega,w)) \cap L^\infty(Q) \cap C^1([0,T],L^2(\Omega,\sigma))$, *for any* $f \in X^*$.

Proof. **Step 1: *A priori estimates.***
First, for all τ in $[0,T]$, we choose $u_n\chi_{[0,\tau]}$ as test function in (P_n), we have

$$\begin{aligned} & \frac{1}{2}\int_\Omega u_n^2(\tau)\,dx + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt + \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)u_n\,dx\,dt \\ = & \int_{Q_\tau} f u_n\,dx\,dt + c. \end{aligned}$$

Since g_n verifies sign condition, by using (2.13), for $\tau = T$ we obtain

$$\alpha\sum_{i=1}^N \int_Q w_i|\frac{\partial u_n}{\partial x_i}|^p \le \int_Q f u_n\,dx\,dt,$$

i.e.,

$$\alpha\|u_n\|^p_{L^p(0,T,V)} \le \|f\|_{L^{p'}(0,T,V^*)}\|u_n\|_{L^p(0,T,V)} \le c\|u_n\|_{L^p(0,T,V)}.$$

Then

$$\|u_n\|_{L^p(0,T,V)} \le \beta_1, \tag{3.11}$$

where β_1 is some positive constant. Then by (3.11) and (2.11) we conclude that $A(u_n)$ is bounded in $L^{p'}(0,T,V^*)$, and there exists $h \in \prod_{i=1}^N L^{p'}(Q,w_i)$, such that

$$a(x,t,u_n,\nabla u_n) \rightharpoonup h \quad \text{in} \quad \prod_{i=1}^N L^{p'}(Q,w_i), \tag{3.12}$$

and,

$$\int_Q g_n(x,t,u_n,\nabla u_n)u_n\,dx\,dt \le \beta_2, \tag{3.13}$$

where β_2 is some positive constant.
Moreover,

$$\|G_n(u_n)\|_{L^1(Q)} \le \beta_3. \tag{3.14}$$

Indeed, let $Q^{k,n} = \{(x,t) \in Q / \ \ |u_n(x,t)| \le k\}$. We get by (2.14) and (2.15),

$$\begin{aligned}
&\int_Q |g_n(x,t,u_n,\nabla u_n)| \, dx \, dt \\
= \;& \int_{\{|u_n|\le k\}} |g_n(x,t,u_n,\nabla u_n)| \, dx \, dt + \int_{\{|u_n|>k\}} |g_n(x,t,u_n,\nabla u_n)| \, dx \, dt \\
\le \;& b(k)(\int_Q \left[\sum_{i=1}^N w_i |\frac{\partial u_n}{\partial x_i}|^p + c(x,t) \right] \, dx \, dt + \frac{1}{k} \int_Q g_n(x,t,u_n,\nabla u_n) u_n \, dx \, dt \\
\le \;& \beta_3.
\end{aligned}$$

Finally, denoting $u_n' = (f + \text{div} a(x,t,u_n,\nabla u_n) + (-G_n(u_n))$ we observe that $h_n = f + \text{div} a(x,t,u_n,\nabla u_n)$ is bounded in $L^{p'}(0,T,V^*)$ and $k_n = -G_n(u_n)$ is bounded in $L^1(Q)$. Thus we can invoke a result of Lemma 3.8 to conclude that $\{u_n\}$ is relatively compact in $L^p_{loc}(Q,\sigma)$, and we can deduce $u_n \to u$ in $L^p_{loc}(Q,\sigma)$, and $u_n \to u$ strongly in $L^1(Q)$.

Step 2:*Basic convergence results*

Fix $k > 0$ and let $\varphi(s) = s e^{\delta s^2}, \delta > 0$. It is well known that when $\delta \ge (\frac{b(k)}{2\alpha})^2$ one has

$$\varphi'(s) - \frac{b(k)}{\alpha} |\varphi(s)| \ge \frac{1}{2} \text{ for all } s \in \mathbb{R}. \tag{3.15}$$

Let $\psi_i \in D(\Omega)$ be a sequence which converges strongly to u_0 in $L^2(\Omega,\sigma)$.
Set $w^i_\mu = (T_k(u))_\mu + e^{-\mu t} T_k(\psi_i)$ where $(T_k(u))_\mu$ is the mollification with respect to time of $T_k(u)$ see Section 3.1.
Note that w^i_μ is a smooth function having the following properties:

$$\begin{cases} \frac{\partial}{\partial t}(w^i_\mu) = \mu(T_k(u) - w^i_\mu), \ \ w^i_\mu(0) = T_k(\psi_i), \ \ |w^i_\mu| \le k \\ w^i_\mu \to T_k(u) \ \text{ in } \ L^p(0,T,W^{1,p}_0(\Omega,w)) \ \text{ as } \ \mu \to \infty \end{cases},$$

using in (P_n) the test function $z^{\mu,i}_n = \varphi(T_k(u_n) - w^i_\mu)$, we get for $\tau = T$.

$$\begin{aligned}
\langle u_n', z^{\mu,i}_n \rangle \;&+\; \int_Q a(x,t,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla w^i_\mu] \varphi'(T_k(u_n) - w^i_\mu) \, dxdt \\
&+\; \int_Q g_n(x,t,u_n,\nabla u_n) \varphi(T_k(u_n) - w^i_\mu) \, dxdt = \langle f, \varphi(T_k(u_n) - w^i_\mu) \rangle
\end{aligned}$$

which implies since $g_n(x,t,u_n,\nabla u_n)\varphi(T_k(u_n) - w^i_\mu) \ge 0$ on $\{|u_n| > k\}$:

$$\begin{aligned}
&\langle u_n', z^{\mu,i}_n \rangle + \int_Q a(x,t,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla w^i_\mu] \varphi'(T_k(u_n) - w^i_\mu) \, dxdt \\
&+ \int_{\{|u_n|\le k\}} g_n(x,t,u_n,\nabla u_n) \varphi(T_k(u_n) - w^i_\mu) \, dxdt \le \langle f, \varphi(T_k(u_n) - w^i_\mu) \rangle.
\end{aligned} \tag{3.16}$$

In the sequel and throughout the paper, we will omit for simplicity to denote $\epsilon(n,\mu,i)$ all quantities (possibly different) such that " $\lim\limits_{i\to\infty} \lim\limits_{\mu\to\infty} \lim\limits_{n\to\infty} \epsilon(n,\mu,i) = 0$" and this will be the order in which the parameters we use will tend to infinity, that is, first n, then μ and finally i. Similarly we will write only $\epsilon(n)$, or $\epsilon(n,\mu)$,... to mean that the limits are made only on the specified parameters.
We will deal with each term of (3.16). First of all, observe that

$$\langle f, \varphi(T_k(u_n) - w^i_\mu) \rangle = \epsilon(n,\mu), \tag{3.17}$$

since $T_k(u_n) - w_\mu^i \rightharpoonup T_k(u) - w_\mu^i$ weakly in $L^p(0,T,V)$ as $n \to \infty$, and $T_k(u) - w_\mu^i \to 0$ in $L^p(0,T,V)$ as $\mu \to +\infty$.
On the one hand, we have,

$$\begin{aligned}\langle \frac{\partial u_n}{\partial t}, z_n^{\mu,i}\rangle &= \int_Q \frac{\partial u_n}{\partial t}\varphi(T_k(u_n) - w_\mu^i)\, dx\, dt \\ &= \int_Q [(T_k(u_n))' + (G_k(u_n))']\varphi(T_k(u_n) - w_\mu^i)\, dx\, dt,\end{aligned}$$

where $G_k(s) = s - T_k(s)$. Hence

$$\begin{aligned}\langle \frac{\partial u_n}{\partial t}, z_n^{\mu,i}\rangle &= \int_Q (T_k(u_n) - w_\mu^i))'\varphi(T_k(u_n) - w_\mu^i)\, dx\, dt \\ &+ \int_Q (w_\mu^i)'\varphi(T_k(u_n) - w_\mu^i)\, dx\, dt \\ &+ \int_Q (G_k(u_n))'\varphi(T_k(u_n) - w_\mu^i)\, dx\, dt \\ &= I_1 + I_2 + I_3.\end{aligned}$$

Setting $\Phi(s) = \displaystyle\int_0^s \varphi(r)\, dr$, it is easy to see that, since $\Phi(s) \geq 0$

$$\begin{aligned}I_1 &= \int_\Omega \left[\int_0^T \Phi'(T_k(u_n) - w_\mu^i)(T_k(u_n) - w_\mu^i)'\, dt\right] dx \\ &= \left[\int_\Omega \Phi(T_k(u_n)(t) - w_\mu^i(t))\, dx\right]_0^T\end{aligned}$$

and

$$\begin{aligned}I_1 &\geq \int_\Omega \Phi(T_k(u_n)(0) - w_\mu^i(0))\, dx \\ &= -\int_\Omega \Phi(T_k(u_0) - T_k(\psi_i))\, ds\end{aligned}$$

$$\to 0 \text{ as } i \to \infty$$
$$\text{and } \quad I_1 \geq \epsilon(i).$$

About I_2, we have, since $(w_\mu^i)' = \mu(T_k(u_n) - w_\mu^i)$

$$\begin{aligned}I_2 &= \mu \int_Q (T_k(u) - w_\mu^i)\varphi(T_k(u_n) - w_\mu^i)\, dx\, dt \\ &= \mu \int_Q [(T_k(u) - T_k(u_n)) + (T_k(u_n) - w_\mu^i)\varphi(T_k(u_n) - w_\mu^i)]\, dx\, dt\end{aligned}$$

since $\varphi(s)s \geq 0$, then

$$I_2 \geq \mu \int_Q (T_k(u) - T_k(u_n))\varphi(T_k(u_n) - w_\mu^i)\, dx\, dt$$

$$\to 0 \text{ as } n \to +\infty$$

hence

$$I_2 \geq \epsilon(n).$$

Through integration, for I_3 we have,

$$\begin{aligned} I_3 &= -\int_Q G_k(u_n)\varphi'(T_k(u_n)-w_\mu^i)(T_k(u_n)-w_\mu^i)'\,dx\,dt \\ &\quad + \left[\int_Q G_k(u_n)\varphi(T_k(u_n)-w_\mu^i)\,dx\right]_0^T \end{aligned}$$

since $(T_k(u_n))' = 0$ on $\{|u_n| > k\}$ and $G_k(u_n) = 0$ on $\{|u_n| \le k\}$. Since

$$\left[\int_\Omega G_k(u_n)\varphi(T_k(u_n)-w_\mu^i)\,dx\right]_0^T \ge -\int_\Omega G_k(u_0)\varphi(T_k(u_0)-T_k(\psi_i))\,dx$$

we have

$$\begin{aligned} I_3 &\ge \int_Q G_k(u_n)\varphi'(T_k(u_n)-w_\mu^i)(w_\mu^i)'\,dx\,dt \\ &\quad -\int_\Omega G_k(u_0)\varphi(T_k(u_0)-T_k(\psi_i))\,dx \\ &= \mu\int_Q G_k(u_n)\varphi'(T_k(u_n)-w_\mu^i)(T_k(u)-w_\mu^i)\,dx\,dt \\ &\quad -\int_\Omega G_k(u_0)\varphi(T_k(u_0)-T_k(\psi_i))\,dx \\ &\to \mu\int_Q G_k(u)\varphi'(T_k(u)-w_\mu^i)(T_k(u)-w_\mu^i)\,dx\,dt \\ &\quad -\int_\Omega G_k(u_0)\varphi(T_k(u_0)-T_k(\psi_i))\,dx \;\text{ as }\; n\to\infty \\ &\ge -\int_\Omega G_k(u_0)\varphi(T_k(u_0)-T_k(\psi_i))\,dx \to 0 \;\text{ as }\; i\to\infty \end{aligned}$$

where we have used (recall $|w_\mu^i| \le k$)

$$\begin{aligned} &\int_Q G_k(u)\varphi'(T_k(u)-w_\mu^i)(T_k(u)-w_\mu^i)\,dx\,dt \\ &= \int_{\{u>k\}}(u-k)\varphi'(k-w_\mu^i)(k-w_\mu^i)\,dx\,dt \\ &\quad + \int_{\{u<-k\}}(u+k)\varphi'(-k-w_\mu^i)(-k-w_\mu^i)\,dx\,dt \ge 0 \end{aligned}$$

we deduce then that

$$I_3 \ge \epsilon(n,i).$$

Combining these estimates, we get

$$\langle\frac{\partial u_n}{\partial t}, \varphi(T_k(u_n)-w_\mu^i)\rangle \ge \epsilon(n,i). \tag{3.18}$$

On the other hand, splitting the second term of the left hand side of (3.16) where $|u_n| \leq k$ and $|u_n| > k$, we can write

$$\begin{aligned} (3.19) \qquad & \int_Q a(x,t,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla w_\mu^i]\varphi'(T_k(u_n) - w_\mu^i)\,dx\,dt \\ \geq & \int_{\{|u_n|\leq k\}} a(x,t,T_k(u_n),\nabla T_k(u_n)(\nabla T_k(u_n) - \nabla w_\mu^i)\varphi'(T_k(u_n) - w_\mu^i)\,dx\,dt \\ & -C_k \int_{\{|u_n|>k\}} |a(x,t,u_n,\nabla u_n)||\nabla w_\mu^i|\,dx\,dt \\ & = J_1 - C_k J_2 \end{aligned}$$

where $C_k = \varphi'(2k)$.
Now observe that

$$\begin{aligned} (3.20)\quad J_1 \;=\; & \int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))] \\ & \times[\nabla T_k(u_n) - \nabla T_k(u)]\varphi'(T_k(u_n) - w_\mu^i))\,dx\,dt \\ + & \int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u) - \nabla w_\mu^i]\varphi'(T_k(u_n) - w_\mu^i))\,dx\,dt \\ + & \int_Q a(x,t,T_k(u_n),\nabla T_k(u))[\nabla T_k(u_n) - \nabla T_k(u)]\varphi'(T_k(u_n) - w_\mu^i))\,dx\,dt \end{aligned}$$

by the continuity of the Nymetskii operator, we have for all $i = 1, ..., N$
$a_i(x,t,T_k(u_n),\nabla T_k(u))\varphi'(T_k(u_n) - w_\mu^i) \to a_i(x,t,T_k(u),\nabla T_k(u))\varphi'(T_k(u) - w_\mu^i)$ strongly in $L^{p'}(Q, w_i^{1-p'})$ and since $\frac{\partial T_k(u_n)}{\partial x_i} \rightharpoonup \frac{\partial T_k(u)}{\partial x_i}$ weakly in $L^p(Q, w_i)$, the third term of the right hand side of (3.20) tends to 0 as $n \to \infty$.
Thanks to (3.12) the second term of the right hand side of (3.20) tends to

$$\int_Q h_k[\nabla T_k(u) - \nabla w_\mu^i]\varphi'(T_k(u) - w_\mu^i)\,dx\,dt$$

so that, $\displaystyle\int_Q h_k[\nabla T_k(u) - \nabla w_\mu^i]\varphi'(T_k(u) - w_\mu^i)\,dx\,dt \to 0$ as $\mu \to \infty$
then we have,

$$\begin{aligned} J_1 \;=\; & \int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))] \\ & \times[\nabla T_k(u_n) - \nabla T_k(u)]\varphi'(T_k(u_n) - w_\mu^i)\,dx\,dt + \epsilon(n,\mu) \end{aligned}$$

and

$$\begin{aligned} J_2 \;=\; & \int_{\{|u_n|\geq k\}} |a(x,t,u_n,\nabla u_n)||\nabla w_\mu^i|(\varphi'(T_k(u_n) - w_\mu^i))\,dx\,dt \\ \to\; & \int_Q h|\nabla w_\mu^i|\varphi'(T_k(u) - w_\mu^i)\chi_{\{|u|>k\}}\,dx\,dt \;\text{ as }\; n \to \infty \\ \to\; & \int_Q h|\nabla T_k(u)|\varphi'(0)\chi_{\{|u|>k\}}\,dx\,dt = 0 \;\text{ as }\; \mu \to \infty. \end{aligned}$$

Therefore (3.19) yields

$$\begin{aligned}(3.21)\qquad &\int_Q a(x,t,u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla w_\mu^i]\varphi'(T_k(u_n)-w_\mu^i)\,dx\,dt\\ &\geq \int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))]\\ &\quad\times[\nabla T_k(u_n)-\nabla T_k(u)]\varphi'(T_k(u_n)-w_\mu^i)\,dx\,dt+\epsilon(n,\mu).\end{aligned}$$

For the third term of the left hand side of (3.16)

$$\begin{aligned}(3.22)\qquad &\left|\int_{\{|u_n|\leq k\}} g_n(x,t,u_n,\nabla u_n)\varphi(T_k(u_n)-w_\mu^i)\,dx\,dt\right|\\ &\leq b(k)\int_Q (c(x,t)+\sum_{i=1}^N w_i\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p)|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt\\ &\leq b(k)\int_Q c(x,t)|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt\\ &\quad+\tfrac{b(k)}{\alpha}\int_Q (a(x,t,T_k(u_n),\nabla T_k(u_n))|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt\end{aligned}$$

since $c(x,t)$ belongs to $L^1(Q)$. Furthermore,

$$(3.23)\qquad b(k)\int_Q c(x,t)|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt=\epsilon(n,\mu).$$

On the other hand, note that

$$\begin{aligned}(3.24)\qquad &\frac{b(k)}{\alpha}\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt\\ =\ &\frac{b(k)}{\alpha}\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))]\\ &\times[\nabla T_k(u_n)-\nabla T_k(u)]|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt\\ &+\frac{b(k)}{\alpha}\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u))|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt\\ &+\frac{b(k)}{\alpha}\int_Q a(x,t,T_k(u_n),\nabla T_k(u))[\nabla T_k(u_n)-\nabla T_k(u)]|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt\end{aligned}$$

By Lebesque's Theorem, we have

$$\nabla T_k(u)|\varphi(T_k(u_n)-w_\mu^i)|\to\nabla T_k(u)|\varphi(T_k(u)-w_\mu^i)|\ \text{ in }\ \prod_{i=1}^N L^p(Q,w_i).$$

Moreover, in view of (3.12) the second term of the right side of (3.24) tends to

$$\frac{b(k)}{\alpha}\int_Q h_k\nabla T_k(u)|\varphi(T_k(u)-w_\mu^i)|\,dx\,dt,$$

the third term of the right hand side of (3.24) tends to 0 since for all $i=1,....,N$

$$a_i(x,t,T_k(u_n),\nabla T_k(u))\varphi(T_k(u_n)-w_\mu^i)\to a_i(x,t,T_k(u),\nabla T_k(u))\varphi(T_k(u)-w_\mu^i)$$

strongly in $L^{p'}(Q, w_i^{1-p'})$, while

$$\frac{\partial(T_k(u_n))}{\partial x_i} \rightharpoonup \frac{\partial(T_k(u))}{\partial x_i} \text{ weakly in } L^p(Q, w_i).$$

From (3.22), (3.23) and (3.24), we obtain

$$\begin{aligned} &\int_{\{|u_n|<k\}} g_n(x,t,u_n,\nabla u_n)\varphi(T_k(u_n)-w_\mu^i)\,dx\,dt \\ \leq\ & \frac{b(k)}{\alpha}\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))] \\ & \times[\nabla T_k(u_n)-\nabla T_k(u)]|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt + \epsilon(n,\mu). \end{aligned} \tag{3.25}$$

By combining (3.16), (3.17), (3.18), (3.21) and (3.25) we get

$$\begin{aligned} &\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)] \\ &\qquad \times[\varphi'(T_k(u_n)-w_\mu^i) - \tfrac{b(k)}{\alpha}|\varphi(T_k(u_n)-w_\mu^i)|]\,dx\,dt \leq \epsilon(n,\mu,i) \end{aligned}$$

and so, because (3.15)

$$\begin{aligned} &\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]\,dx\,dt \\ &\qquad \leq 2\epsilon(n,\mu,i) \end{aligned}$$

and by passing to the limit sup over n, we get

$$\begin{aligned} & 0 \\ \leq\ & \lim_{n\to\infty}\sup \int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) \\ & -a(x,t,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]\,dx\,dt \\ \leq\ & \lim_{n\to\infty} 2\epsilon(n,\mu,i) \end{aligned}$$

in which we let successively $\mu \to \infty$ and $i \to \infty$ to obtain

$$\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))] \tag{3.26}$$

$$\times \ [\nabla T_k(u_n)-\nabla T_k(u)]\,dx\,dt \to 0 \quad \text{as } n\to\infty \tag{3.27}$$

which implies that by using Lemma 3.10

$$T_k(u_n) \to T_k(u) \text{ strongly in } L^p(0,T,W_0^{1,p}(\Omega,w)) \ \ \forall k \geq 0 \tag{3.28}$$

an thus, there exists a subsequence also denoted by u_n such that

$$\nabla u_n \to \nabla u \text{ a.e. in } Q. \tag{3.29}$$

We then deduce that, for all $k > 0$ $a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u))$ and $a(x,t,u_n,\nabla u_n) \rightharpoonup a(x,t,u,\nabla u)$ weakly in $\prod_{i=1}^N L^{p'}(Q,w_i)$.

Step 3: ***Equi-integrability of the nonlinearities***
Since g_n verifies the sign condition, then by (3.13) we deduce,

$$0 \leq \int_Q g_n(x,t,u_n,\nabla u_n)u_n \, dx \, dt \leq \gamma.$$

For any measurable subset E of Q and any $m > 0$, we have

$$\begin{aligned} & \int_E |g_n(x,t,u_n,\nabla u_n)| \, dx \, dt \\ = & \int_{E\cap X_m^n} |g_n(x,t,u_n,\nabla u_n)| \, dx \, dt \\ & + \int_{E\cap Y_m^n} |g_n(x,t,u_n,\nabla u_n)| \, dx \, dt \end{aligned}$$

where

$$X_m^n = \{(x,t) \in [0,T] \times \Omega, |u_n(x,t)| \leq m\},$$

and

$$Y_m^n = \{(x,t) \in Q, |u_n(x,t)| \geq m\}.$$

From these expressions,

$$\begin{aligned} \int_E |g_n(x,t,u_n,\nabla u_n)| \, dx \, dt \ & \leq \ \int_{E\cap X_m^n} |g_n(x,t,T_m(u_n),\nabla T_m(u_n))| \, dx \, dt \\ & + \ \frac{1}{m}\int_Q g_n(x,t,u_n,\nabla u_n)u_n \, dx \, dt \\ & \leq \ b(m)\int_E (\sum_{i=1}^N w_i |\frac{\partial T_m(u_n)}{\partial x_i}|^p + \int_E c(x,t)) \, dx \, dt + \gamma\frac{1}{m}. \end{aligned}$$

Since the sequence $(T_m(u_n))$ converge strongly and the fact that $c(x,t) \in L^1(Q)$,
there exists $\theta > 0$ such that

$$|E| < \theta \Rightarrow \int_E |g_n(x,t,u_n,\nabla u_n)| \, dx \, dt \leq \epsilon \quad \forall \ n.$$

This shows that $g_n(x,t,u_n,\nabla u_n)$ is uniformly equi-integrable in Q as required.
Step 4: ***Passage to the limit.***
Considering the approximate problem (P_n) one has:

$$-\int_Q u_n \frac{\partial \varphi}{\partial t} \, dx \, dt + \int_Q a(x,t,u_n,\nabla u_n)\nabla\varphi \, dx \, dt + \int_Q g_n(x,t,u_n,\nabla u_n)\varphi \, dx \, dt = \langle f,\varphi\rangle$$

for all $\varphi \in D(Q)$, in which, we can easily pass to the limit as $n \to \infty$, to get

$$(3.30) \quad -\int_Q u\frac{\partial \varphi}{\partial t} \, dx \, dt + \int_Q a(x,t,u,\nabla u)\nabla\varphi \, dx \, dt + \int_Q g(x,t,u,\nabla u)\varphi \, dx \, dt = \langle f,\varphi\rangle.$$

Let now $(\varphi \in C^1([0,T],L^2(\Omega,\sigma)) \cap L^\infty(Q) \cap L^p(0,T,W_0^{1,p}(\Omega,w))$,
there exists $(\varphi_j) \subset D(Q)$ such that $\varphi_j \to \varphi$ in $\prod_{i=1}^N L^p(Q,w_i)$ and weak in $L^\infty(Q)$. Taking

$\varphi = \varphi_j$ in (3.30), and letting $j \to \infty$, yields

$$\begin{aligned} -\int_Q u \frac{\partial \varphi}{\partial t}\, dx\, dt \quad &+ \quad \left[\int_\Omega u(t)\varphi(t)\, dx\right]_0^T + \int_Q a(x,t,u,\nabla u)\nabla\varphi\, dx\, dt \\ &+ \quad \int_Q g(x,t,u,\nabla u)\varphi\, dx\, dt = \langle f, \varphi\rangle \end{aligned}$$

for all $\varphi \in L^p(0,T,W_0^{1,p}(\Omega,w)) \cap L^\infty(Q) \cap C^1([0,T],L^2(\Omega,\sigma))$.
Step 5: ***Show that*** $u \in C^1([0,T],H)$**.**
As in [13], we have for all $\phi \in D(\Omega)$, by letting $\tilde{\phi}(x,t) = \phi(x) \ \in D(Q)$

$$\begin{aligned} \langle u_n', \phi\rangle + \int_Q a(x,t,u_n,\nabla u_n)\nabla\tilde{\phi}\, dx\, dt + \int_Q g(x,t,u_n,\nabla u_n)\tilde{\phi}\, dx\, dt \\ = \int_Q f\tilde{\phi}\, dx\, dt. \end{aligned}$$

This implies (see [13]), that $(u_n(t))$ is weakly convergent in H for all t and also that $u(t)$ is weakly continuous in H.
Let now $w_\mu^{i,k} = (T_k(u))_\mu - e^{-\mu t}T_k(\psi_i)$. On the one hand, we have for every $\tau \in [0,T]$

(3.31) $$\langle (w_\mu^{i,k})', u_n - w_\mu^{i,k}\rangle_{Q_\tau} \to \mu \int_{Q_\tau} (T_k(u) - w_\mu^{i,k})(u - w_\mu^{i,k})\, dx\, dt \geq 0 \ \text{ as } \ n \to \infty.$$

On the other hand, by using $(u_n - w_\mu^{i,k})$ as test function in (P_n), we can write,

$$\begin{aligned} \langle u_n', u_n - w_\mu^{i,k}\rangle \quad &= \quad \langle f, u_n - w_\mu^{i,k}\rangle + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla(w_\mu^{i,k} - u_n)\, dx\, dt \\ &+ \quad \int_{Q_\tau} g(x,t,u_n,\nabla u_n)(w_\mu^{i,k} - u_n)\, dx\, dt \end{aligned}$$

in which we can use Fatou's Lemma and Lebesgue theorem to pass to the limit sup first over n and μ, k, to get

(3.32) $$\langle u_n', u_n - w_\mu^{i,k}\rangle_{Q_\tau} \leq \epsilon(n,\mu,k).$$

Therefore, by writing

$$\begin{aligned} \frac{1}{2}\|u_n(\tau) - w_\mu^{i,k}(\tau)\|_H^2 \quad &= \quad \langle u_n' - (w_\mu^{i,k})', u_n - w_\mu^{i,k}\rangle_{Q_\tau} + \frac{1}{2}\|u_0 - T_k(\psi_i)\|_H^2 \\ &= \quad \langle u_n', u_n - w_\mu^{i,k}\rangle_{Q_\tau} - \langle (w_\mu^{i,k})', u_n - w_\mu^{i,k}\rangle_{Q_\tau} \\ &+ \quad \frac{1}{2}\|u_0 - T_k(\psi_i)\|_H^2 \end{aligned}$$

and observing

$$0 \leq \frac{1}{2}\|u(\tau) - w_\mu^{i,k}(\tau)\|_H^2 = \lim_{n\to\infty} \frac{1}{2}\|u_n(\tau) - w_\mu^{i,k}(\tau)\|_H^2$$

we deduce that, in view of (3.31) and (3.32),
$\|u(\tau) - w_\mu^{i,k}(\tau)\|_H^2 \leq \epsilon(\mu,k,i)$ not depending on $\tau \in [0,T]$.
Implying that $w_\mu^{i,k}$ is a Cauchy sequence in $C([0,T],H)$ converging to u and thus $u \in C([0,T],H)$. ∎

4. EXAMPLE

Let Ω be a bounded domain of $\mathbb{R}^N (N \geq 1)$, satisfying the cone condition. Let us consider the Carathéodory functions:

$$a_i(x,t,s,\epsilon) = w_i|\epsilon_i|^{p-1}\text{sgn}(\epsilon_i) \ \text{ for } \ i = 1,...,N$$

$$g(x,t,s,\epsilon) = \rho s|s|^r \sum_{i=1}^{N} w_i|\epsilon_i|^p, \qquad \rho > 0, \ \ r > 0$$

where $w_i(x)$ $(i = 0, 1, ..., N)$ are given weight functions, strictly positive almost everywhere in Ω. We shall assume that the weight functions satisfy, $w_i(x) = w(x), x \in \Omega$, for all $i = 0, ..., N$. Then, we can consider the Hardy inequality (2.6) in the from,

$$\left(\int_Q |u(x,t)|^p \sigma(x)\ dx\ dt\right)^{\frac{1}{p}} \leq c \left(\int_Q |\nabla u(x,t)|^p w\ dx\ dt\right)^{\frac{1}{p}}.$$

It is easy to show that the functions $a_i(x,t,s,\epsilon)$ are Carathéodory functions satisfying the growth condition (2.11) and the coercivity (2.12). Also the Carathéodory function $g(x,t,s,\epsilon)$ satisfies the conditions (2.14) and (2.15). On the other hand, the monotonicity condition is satisfied, in fact,

$$\begin{aligned}&\sum_{i=1}^{N}(a_i(x,t,s,\epsilon) - a_i(x,t,s,\hat{\epsilon}))(\epsilon_i - \hat{\epsilon}_i)\\ &= w(x)\sum_{i=1}^{N}(|\epsilon_i|^{p-1}\text{sgn}\epsilon_i - |\hat{\epsilon}_i|^{p-1}\text{sgn}\hat{\epsilon}_i)(\epsilon_i - \hat{\epsilon}_i) > 0\end{aligned}$$

for almost all $(x,t) \in Q$ and for all $\epsilon, \hat{\epsilon} \in \mathbb{R}^N$ with $\epsilon \neq \hat{\epsilon}$, since $w > 0$ a.e. in Ω. In particular, let us use the special weight functions w and σ expressed in terms of the distance to the boundary $\partial\Omega$. Denote $d(x) = \text{dist}(x, \partial\Omega)$ and set

$$w(x) = d^\lambda(x), \qquad \sigma(x) = d^\mu(x).$$

In this case, the Hardy inequality reads

$$\left(\int_Q |u(x,t)|^p d^\mu(x)\ dx\ dt\right)^{\frac{1}{p}} \leq \left(c\int_Q |\nabla u(x,t)|^p d^\lambda(x)\ dx\ dt\right)^{\frac{1}{p}}.$$

For

$$\lambda < p - 1, \ \ \frac{\mu - \lambda}{p} + 1 > 0.$$

(See for example [10])

Remark 4.1. The last condition is sufficient for the hypotheses $(A_1) - (A_3)$ are satisfied, therefore the problem (P) has at least one weak solution.

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