The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 5, Issue 1, Article 9, pp. 1-18, 2008

A SUM FORM FUNCTIONAL EQUATION AND ITS RELEVANCE IN INFORMATION THEORY

PREM NATH AND DHIRAJ KUMAR SINGH

Received 27 February, 2007; accepted 16 November, 2007; published 9 May, 2008.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI, DELHI - 110007, INDIA.
pnathmaths@gmail.com
dksingh@maths.du.ac.in
dhiraj426@rediffmail.com

ABSTRACT. The general solutions of a sum form functional equation containing four unknown mappings have been investigated. The importance of these solutions in relation to various entropies in information theory has been emphasised.

Key words and phrases: Sum form functional equation, Additive function, Multiplicative function, The Shannon entropy, The nonadditive entropies of degree α.

2000 *Mathematics Subject Classification.* 39B52, 39B82.

ISSN (electronic): 1449-5910

1. INTRODUCTION

For $n = 1, 2, \ldots$ let $\Gamma_n = \{(x_1, \ldots, x_n) : x_i \geq 0, i = 1, \ldots, n; \sum_{i=1}^n p_i = 1\}$ denote the set of all finite discrete n-component complete probability distributions with nonnegative elements. Throughout the sequel, $\mathbb{R}$ will denote the set of all real numbers and $I = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$.

For any probability distribution $(x_1, \ldots, x_n) \in \Gamma_n$, $n = 1, 2, \ldots$

$$H_n(x_1, \ldots, x_n) = -\sum_{i=1}^n x_i \log_2 x_i \tag{1.1}$$

with $H_n : \Gamma_n \to \mathbb{R}$, $n = 1, 2, \ldots$ and $0 \log_2 0 = 0$ are called the Shannon [8] entropies. It is obvious that

$$H_n(x_1, \ldots, x_n) = \sum_{i=1}^n \overline{h}(x_i)$$

for all $H_n : \Gamma_n \to \mathbb{R}$, $(x_1, \ldots, x_n) \in \Gamma_n$, $n = 1, 2, \ldots$ where $\overline{h} : I \to \mathbb{R}$ is a mapping defined as

$$\overline{h}(x) = -\, x \log_2 x \tag{1.2}$$

for all $x \in I$. The mapping $\overline{h} : I \to \mathbb{R}$, defined by (1.2), is known as the generating function of the sequence $H_n : \Gamma_n \to \mathbb{R}$, $n = 1, 2, \ldots$ of the Shannon entropies (1.1) which are additive. The expression $\sum_{i=1}^n \overline{h}(x_i)$ is known as the sum form representation of the Shannon entropies (1.1).

Havrda and Charvát [4] introduced axiomatically the nonadditive entropies of degree α defined as

$$H_n^\alpha(x_1, \ldots, x_n) = \left(1 - 2^{1-\alpha}\right)^{-1} \left(1 - \sum_{i=1}^n x_i^\alpha\right) \tag{1.3}$$

with $H_n^\alpha : \Gamma_n \to \mathbb{R}$, $n = 1, 2, \ldots$; $\alpha \in \mathbb{R}$, $\alpha > 0$, $\alpha \neq 1$ and $0^\alpha := 0$, $1^\alpha := 1$ for all $\alpha \neq 1$. They called α the characteristic parameter. Here, too, it is clear that

$$H_n^\alpha(x_1, \ldots, x_n) = \sum_{i=1}^n z_\alpha(x_i)$$

where the mappings $z_\alpha : I \to \mathbb{R}$, $\alpha \in \mathbb{R}, \alpha > 0, \alpha \neq 1$, are defined as

$$z_\alpha(x) = \left(1 - 2^{1-\alpha}\right)^{-1} (x - x^\alpha) \tag{1.4}$$

for all $x \in I$ with $0^\alpha := 0, 1^\alpha := 1$, for all $\alpha \neq 1, \alpha > 0, \alpha \in \mathbb{R}$. The mapping $z_\alpha : I \to \mathbb{R}$, defined by (1.4), is known as the generating function of the sequence $H_n^\alpha : \Gamma_n \to \mathbb{R}, n = 1, 2, \ldots$ of the nonadditive entropies. The expression $\sum_{i=1}^n z_\alpha(x_i)$ is known as the sum form representation for the entropies $H_n^\alpha : \Gamma_n \to \mathbb{R}, n = 1, 2, \ldots$.

With the purpose of giving a joint characterization of the entropies (1.1) and (1.3), Taneja [7] considered the functional equation

$$\sum_{i=1}^n \sum_{j=1}^m h(x_i y_j) = \sum_{i=1}^n f(x_i) \sum_{j=1}^m g(y_j) + \sum_{i=1}^n k(x_i) \tag{1.5}$$

where $h : I \to \mathbb{R}$, $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$, $k : I \to \mathbb{R}$, $(x_1, \ldots, x_n) \in \Gamma_n$, $(y_1, \ldots, y_m) \in \Gamma_m$ and $n = 1, 2, \ldots$; $m = 1, 2, \ldots$. By assuming each of the mappings h, f, g and k to be continuous on the interval I, he determined various solutions of (1.5) for all $(x_1, \ldots, x_n) \in \Gamma_n$,

$(y_1, \ldots, y_m) \in \Gamma_m$, $n = 1, 2, \ldots$; $m = 1, 2, \ldots$ and finally characterized the entropies (1.1) and (1.3) by assuming $h(1) = h(0)$ and $h(\frac{1}{2}) = \frac{1}{2}$.

Dial [3] assumed the mappings h, f, g and k to be measurable in the sense of Lebesgue; found the solutions of (1.5) for all $(x_1, \ldots, x_n) \in \Gamma_n, (y_1, \ldots, y_m) \in \Gamma_m$, $n, m = 2, 3$; and also finally characterized the entropies (1.1) and (1.3) by assuming $h(1) = h(0)$ and $h(\frac{1}{2}) = \frac{1}{2}$.

The object of this paper is to determine the general solutions of (1.5) for all $(x_1, \ldots, x_n) \in \Gamma_n$, $(y_1, \ldots, y_m) \in \Gamma_m$, $n \geq 3$, $m \geq 3$ arbitrary but fixed integers. Notice that no regularity condition has been imposed upon any of the mappings h, f, g and k.

During the process of finding the general solutions of (1.5) for all $(x_1, \ldots, x_n) \in \Gamma_n$, $(y_1, \ldots, y_m) \in \Gamma_m$, $n \geq 3$, $m \geq 3$ fixed integers, the authors have come across the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} G(x_i y_j) = \sum_{i=1}^{n} F(x_i) \sum_{j=1}^{m} G(y_j) + \sum_{i=1}^{n} G(x_i) + n(m-1)G(0) \tag{1.6}$$

in which $G : I \to \mathbb{R}$, $F : I \to \mathbb{R}$, $(x_1, \ldots, x_n) \in \Gamma_n$, $(y_1, \ldots, y_m) \in \Gamma_m$.

The general solutions of (1.6), for all $(x_1, \ldots, x_n) \in \Gamma_n$, $(y_1, \ldots, y_m) \in \Gamma_m$, $n \geq 3, m \geq 3$ fixed integers, have been investigated in section 3. Making use of these solutions, the corresponding general solutions of (1.5) have been investigated in section 4. To develop sections 3 and 4, some general definitions and results are needed and these have been stated in section 2. Finally, some general comments and observations have been mentioned in section 5.

2. SOME PRELIMINARY RESULTS

Let $\Delta = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq x + y \leq 1\}$ denote the unit closed triangle. A mapping $e : I \to \mathbb{R}$ is said to be additive on I if it satisfies the equation

$$e(x + y) = e(x) + e(y)$$

for all $(x, y) \in \Delta$. A mapping $\overline{E} : \mathbb{R} \to \mathbb{R}$ is said to be additive on $\mathbb{R}$ if it satisfies the equation

$$\overline{E}(x + y) = \overline{E}(x) + \overline{E}(y) \tag{2.1}$$

for all $x \in \mathbb{R}, y \in \mathbb{R}$. It is known [2] that if $e : I \to \mathbb{R}$ is additive on I, then it has a unique additive extension $\overline{E} : \mathbb{R} \to \mathbb{R}$ in the sense that $\overline{E} : \mathbb{R} \to \mathbb{R}$ satisfies (2.1) for all $x \in \mathbb{R}, y \in \mathbb{R}$ and $\overline{E}(x) = e(x)$ for all $x \in I$.

A mapping $M : I \to \mathbb{R}$ is said to be multiplicative on I if

$$M(0) = 0 \tag{2.2}$$

$$M(1) = 1 \tag{2.3}$$

and

$$M(xy) = M(x)M(y) \tag{2.4}$$

for all $x \in]0, 1[$, $y \in]0, 1[$ where $]0, 1[= \{x \in \mathbb{R} : 0 < x < 1\}$.

Notice that in the sense of this definition, the mappings $M_1 : I \to \mathbb{R}$ and $M_2 : I \to \mathbb{R}$, defined as

$$M_1(x) = 0$$

for all $x \in I$ and

$$M_2(x) = 1$$

for all $x \in I$, are not multiplicative mappings as $M_1(1) = 0 \neq 1$ and $M_2(0) = 1 \neq 0$.

Result 2.1 ([5])**.** Suppose c is a given constant and a mapping $\varphi : I \to \mathbb{R}$ satisfies the equation

$$\sum_{i=1}^{n} \varphi(x_i) = c$$

for all $(x_1, \ldots, x_n) \in \Gamma_n$, $n \geq 3$ a fixed integer. Then there exists an additive mapping $A : \mathbb{R} \to \mathbb{R}$ such that

$$\varphi(x) = A(x) - \frac{1}{n}A(1) + \frac{c}{n}$$

for all $x \in I$.

Result 2.2 ([5])**.** If a mapping $G_1 : I \to \mathbb{R}$ satisfies the functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} G_1(x_i y_j) = \sum_{i=1}^{n} G_1(x_i) + \sum_{j=1}^{m} G_1(y_j) \tag{2.5}$$

for all $(x_1, \ldots, x_n) \in \Gamma_n$, $(y_1, \ldots, y_m) \in \Gamma_m$, $n \geq 3$, $m \geq 3$ being fixed integers, then G_1 is of the form

$$G_1(x) = \begin{cases} c + c(nm - n - m)x + a(x) + D(x, x) & \text{if } 0 < x \leq 1 \\ c & \text{if } x = 0 \end{cases} \tag{2.6}$$

where $c = G_1(0)$ is an arbitrary real constant; $a : \mathbb{R} \to \mathbb{R}$ is an additive mapping; $D : \mathbb{R}\times\,]0, 1] \to \mathbb{R}$ is additive in the first variable and there exists a mapping $E : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, additive in both variables, such that $E(1, 1) = a(1)$, and

$$D(xy, xy) - D(xy, x) - D(xy, y) = E(x, y) \tag{2.7}$$

holds for all $x \in]0, 1]$, $y \in]0, 1]$ where $]0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$.

Notice that if we put $x = y = 1$ in (2.7) and use $E(1, 1) = a(1)$, we get

$$a(1) + D(1, 1) = 0. \tag{2.8}$$

Note. The functional equation (2.5) is due to Chaundy and Mcleod [1] who came across it while studying some problems in statistical thermodynamics.

Result 2.3 ([6])**.** Suppose $T : I \to \mathbb{R}$ is a mapping which satisfies the functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} T(x_i y_j) = \sum_{i=1}^{n} T(x_i) \sum_{j=1}^{m} T(y_j) + (m - n)T(0) \sum_{j=1}^{m} T(y_j) + m(n - 1)T(0) \tag{2.9}$$

for all $(x_1, \ldots, x_n) \in \Gamma_n$, $(y_1, \ldots, y_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers. Then, any general solution of *(2.9)* is of the form

$$T(x) = \overline{a}(x) + T(0) \tag{2.10}$$

where $\overline{a} : \mathbb{R} \to \mathbb{R}$ is an additive mapping with

$$\begin{cases} \overline{a}(1) = -\,mT(0) & \text{if } T(1) + (m - 1)T(0) \neq 1 \text{ or} \\ \overline{a}(1) = 1 - mT(0) & \text{if } T(1) + (m - 1)T(0) = 1 \end{cases} \tag{2.11}$$

or

$$T(x) = M(x) - b(x) + T(0) \tag{2.12}$$

where $b : \mathbb{R} \to \mathbb{R}$ is an additive mapping with

$$b(1) = mT(0) \tag{2.13}$$

and $M : I \to \mathbb{R}$ is a nonconstant nonadditive mapping which is multiplicative in the sense that it satisfies *(2.2)*, *(2.3)* and *(2.4)* for all $x \in]0,1[$, $y \in]0,1[$.

Below, we prove a result which is similar to Result 2.3.

Result 2.4. Suppose $F_1 : I \to \mathbb{R}$ is a mapping which satisfies the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} F_1(x_i y_j) = \sum_{i=1}^{n} F_1(x_i)\sum_{j=1}^{m} F_1(y_j) + (n-m)F_1(0)\sum_{i=1}^{n} F_1(x_i) + n(m-1)F_1(0) \tag{2.14}$$

for all $(x_1, \ldots, x_n) \in \Gamma_n$, $(y_1, \ldots, y_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers. Then, any general solution of *(2.14)* is of the form

$$F_1(x) = \overline{b}_1(x) + F_1(0)$$

where $\overline{b}_1 : \mathbb{R} \to \mathbb{R}$ is an additive mapping with

$$\begin{cases} \overline{b}_1(1) = -nF_1(0) & \text{if } F_1(1) + (n-1)F_1(0) \neq 1 \text{ or} \\ \overline{b}_1(1) = 1 - nF_1(0) & \text{if } F_1(1) + (n-1)F_1(0) = 1 \end{cases}$$

or

$$F_1(x) = M(x) - \overline{b}(x) + F_1(0)$$

where $\overline{b} : \mathbb{R} \to \mathbb{R}$ is an additive mapping with

$$\overline{b}(1) = nF_1(0)$$

and $M : I \to \mathbb{R}$ is a nonconstant nonadditive mapping which is multiplicative in the sense that it satisfies *(2.2)*, *(2.3)* and *(2.4)* for all $x \in]0,1[$, $y \in]0,1[$.

Proof. Let us write (2.14) in the form

$$\sum_{i=1}^{n}\sum_{j=1}^{m} F_1(x_i y_j) = \sum_{i=1}^{n} F_1(x_i)\left[\sum_{j=1}^{m} F_1(y_j) + (n-m)F_1(0)\right] + n(m-1)F_1(0). \tag{2.15}$$

Define the mapping $T : I \to \mathbb{R}$ as

$$T(x) = F_1(x) + (n-m)F_1(0)x \tag{2.16}$$

for all $x \in I$. Then equation (2.15) reduces to equation (2.9). From (2.16), it follows that

$$T(0) = F_1(0)$$

and

$$T(1) = F_1(1) + (n-m)F_1(0).$$

Now, from Result 2.3 and the above observations, the required solutions of (2.14) follow by defining the mappings $\overline{b}_1 : \mathbb{R} \to \mathbb{R}$ and $\overline{b} : \mathbb{R} \to \mathbb{R}$ as

$$\overline{b}_1(x) = \overline{a}(x) - (n-m)F_1(0)x$$

and

$$\overline{b}(x) = b(x) + (n-m)F_1(0)x$$

for all $x \in \mathbb{R}$. Notice that $\overline{b}_1 : \mathbb{R} \to \mathbb{R}$ and $\overline{b} : \mathbb{R} \to \mathbb{R}$ are additive. ∎

3. THE FUNCTIONAL EQUATION (1.6)

The main result of this section is:

Theorem 3.1. *Let* $F : I \to \mathbb{R}$, $G : I \to \mathbb{R}$ *be mappings which satisfy the functional equation* (1.6) *for all* $(x_1, \dots, x_n) \in \Gamma_n$, $(y_1, \dots, y_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ *being fixed integers. Then, any general solution of* (1.6) *is of the form* (*for all* $x \in I$) *:*

$$\text{(3.1)} \qquad \begin{cases} F(x) = b_1(x) - \dfrac{1}{n} b_1(1) + \dfrac{1}{n} \\ G(x) = \begin{cases} G(0) - mG(0)x + a(x) + D(x,x) & \text{if } 0 < x \leq 1 \\ G(0) & \text{if } x = 0 \end{cases} \end{cases}$$

or

$$\text{(3.2)} \qquad \begin{cases} F(x) = \overline{b}_1(x) - \dfrac{1}{n} \overline{b}_1(1) \\ G(x) = \overline{b}_2(x) + G(0) \end{cases}$$

or

$$\text{(3.3)} \qquad \begin{cases} F \text{ is an arbitrary real-valued mapping} \\ G(x) = b_2(x) - \dfrac{1}{m} b_2(1) \end{cases}$$

with $b_2(1) = -\,mG(0)$ *or*

$$\text{(3.4)} \qquad \begin{cases} F(x) = M(x) + b_4(x) - \dfrac{1}{n} b_4(1) \\ G(x) = -\,[b_3(1) + mG(0)]\, M(x) + b_3(x) + G(0), \ b_3(1) + mG(0) \neq 0 \end{cases}$$

where $b_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, 3, 4)$, $\overline{b}_j : \mathbb{R} \to \mathbb{R}$ $(j = 1, 2)$ *are additive mappings,* $M : I \to \mathbb{R}$ *is a nonconstant nonadditive mapping which is multiplicative in the sense that it satisfies* (2.2), (2.3) *and* (2.4) *for all* $x \in \,]0,1[$, $y \in \,]0,1[$; *and* $a : \mathbb{R} \to \mathbb{R}$, $D : \mathbb{R} \times \,]0,1] \to \mathbb{R}$ *are as mentioned in Result* 2.2.

Proof. Let us write (1.6) in the form

$$\sum_{j=1}^{m} \left\{ \sum_{i=1}^{n} G(x_i y_j) - G(y_j) \sum_{i=1}^{n} F(x_i) - y_j \sum_{i=1}^{n} G(x_i) \right\} = n(m-1)G(0).$$

By Result 2.1, there exists a mapping $\overline{A} : \Gamma_n \times \mathbb{R} \to \mathbb{R}$, additive in the second variable, such that

$$\text{(3.5)} \qquad \begin{aligned} &\sum_{i=1}^{n} G(x_i y) - G(y) \sum_{i=1}^{n} F(x_i) - y \sum_{i=1}^{n} G(x_i) \\ &\quad = \overline{A}(x_1, \dots, x_n; y) - \frac{1}{m} \overline{A}(x_1, \dots, x_n; 1) + \frac{n}{m}(m-1)G(0). \end{aligned}$$

Putting $y = 0$ in (3.5) and making use of the fact that $\overline{A}(x_1, \dots, x_n; 0) = 0$, we get

$$\text{(3.6)} \qquad \overline{A}(x_1, \dots, x_n; 1) = G(0) \left[m \sum_{i=1}^{n} F(x_i) - n \right]$$

for all $(x_1, \ldots, x_n) \in \Gamma_n$. Equations (3.5) and (3.6) give

$$\sum_{i=1}^{n} G(x_i y) - [G(y) - G(0)] \sum_{i=1}^{n} F(x_i) - y \sum_{i=1}^{n} G(x_i) = \overline{A}(x_1, \ldots, x_n; y) + nG(0). \tag{3.7}$$

Let $(r_1, \ldots, r_n) \in \Gamma_n$ be any probability distribution. Putting $y = r_1, \ldots, r_n$ successively in (3.7), adding the resulting n equations and using the additivity of $\overline{A}$ in the second variable, we obtain

$$\sum_{i=1}^{n} \sum_{t=1}^{n} G(x_i r_t) - \left[\sum_{t=1}^{n} G(r_t) - nG(0)\right] \sum_{i=1}^{n} F(x_i) - \sum_{i=1}^{n} G(x_i) = \overline{A}(x_1, \ldots, x_n; 1) + n^2 G(0). \tag{3.8}$$

From (3.6) and (3.8), it follows that

$$\sum_{i=1}^{n} \sum_{t=1}^{n} G(x_i r_t) - n(n-1)G(0) = \left[\sum_{t=1}^{n} G(r_t) + (m-n)G(0)\right] \sum_{i=1}^{n} F(x_i) + \sum_{i=1}^{n} G(x_i). \tag{3.9}$$

The left hand side of (3.9) is symmetric in x_i and r_t. So, the right hand side of (3.9) must also be symmetric in x_i and r_t. This gives the equation

$$\left[\sum_{t=1}^{n} G(r_t) + (m-n)G(0)\right] \sum_{i=1}^{n} F(x_i) + \sum_{i=1}^{n} G(x_i) = \left[\sum_{i=1}^{n} G(x_i) + (m-n)G(0)\right] \sum_{t=1}^{n} F(r_t) + \sum_{t=1}^{n} G(r_t)$$

which can be written in the form

$$\left[\sum_{t=1}^{n} G(r_t) + (m-n)G(0)\right] \left[\sum_{i=1}^{n} F(x_i) - 1\right] = \left[\sum_{i=1}^{n} G(x_i) + (m-n)G(0)\right] \left[\sum_{t=1}^{n} F(r_t) - 1\right]. \tag{3.10}$$

Case 1. $\sum_{i=1}^{n} F(x_i) - 1$ vanishes identically on Γ_n. This means that

$$\sum_{i=1}^{n} F(x_i) = 1 \tag{3.11}$$

holds for all $(x_1, \ldots, x_n) \in \Gamma_n$. By Result 2.1, there exists an additive mapping $b_1 : \mathbb{R} \to \mathbb{R}$ such that

$$F(x) = b_1(x) - \frac{1}{n} b_1(1) + \frac{1}{n} \tag{3.12}$$

for all $x \in I$. The substitution $x = 0$ in (3.12) and the use of the fact that $b_1(0) = 0$ gives

$$b_1(1) = 1 - nF(0).$$

From (1.6) and (3.11), we obtain the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} G(x_i y_j) = \sum_{j=1}^{m} G(y_j) + \sum_{i=1}^{n} G(x_i) + n(m-1)G(0). \tag{3.13}$$

Define the mapping $G_1 : I \to \mathbb{R}$ as

$$G_1(p) = G(p) + n(m-1)G(0)p \tag{3.14}$$

for all $p \in I$. Then

$$G_1(0) = G(0). \tag{3.15}$$

Moreover, G_1 satisfies the functional equation (2.5). Making use of Result 2.2, and equations (3.14) and (3.15), it follows that

$$G(x) = \begin{cases} G(0) - mG(0)x + a(x) + D(x,x) & \text{if } 0 < x \le 1 \\ G(0) & \text{if } x = 0. \end{cases} \tag{3.16}$$

Equations (3.12) and (3.16) constitute the solution (3.1) of equation (1.6) with a and D as mentioned in Result 2.2.

Case 2. $\sum_{i=1}^{n} F(x_i) - 1$ does not vanish identically on Γ_n. Then there exists a probability distribution $(x_1^*, \ldots, x_n^*) \in \Gamma_n$ such that

$$\sum_{i=1}^{n} F(x_i^*) - 1 \neq 0. \tag{3.17}$$

Choosing $x_i = x_i^*$, $i = 1, \ldots, n$ in (3.10), making use of (3.17) and performing the necessary calculations, it follows that

$$\sum_{t=1}^{n} G(r_t) = c\left[\sum_{t=1}^{n} F(r_t) - 1\right] - (m-n)G(0) \tag{3.18}$$

where

$$c = \left[\sum_{i=1}^{n} F(x_i^*) - 1\right]^{-1}\left[\sum_{i=1}^{n} G(x_i^*) + (m-n)G(0)\right].$$

Case 2.1. $c = 0$.

In this case, (3.18) reduces to

$$\sum_{t=1}^{n} G(r_t) = (n-m)G(0). \tag{3.19}$$

Proceeding as in the case of (3.11), it follows that there exists an additive mapping $b_2 : \mathbb{R} \to \mathbb{R}$ with

$$b_2(1) = -\,mG(0) \tag{3.20}$$

such that

$$G(x) = b_2(x) - \frac{1}{m} b_2(1) \tag{3.21}$$

for all $x \in I$. From (3.21) and (3.20), it is easy to conclude that

$$\sum_{j=1}^{m} G(y_j) = 0 \tag{3.22}$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{m} G(x_i y_j) = m(n-1)G(0) \tag{3.23}$$

hold for all $(x_1, \ldots, x_n) \in \Gamma_n$ and $(y_1, \ldots, y_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers. From (1.6), (3.19), (3.22) and (3.23), it follows that F can be any arbitrary real-valued mapping. This statement, together with (3.20) and (3.21), constitute the solution (3.3) of (1.6).

Case 2.2. $c \neq 0$.
Let us write (3.18) in the form

$$\sum_{t=1}^{n}[G(r_t) - cF(r_t)] = (n-m)G(0) - c.$$

By Result 2.1, there exists an additive mapping $A : \mathbb{R} \to \mathbb{R}$ such that

$$G(x) - cF(x) = A(x) - \frac{1}{n}A(1) + \frac{1}{n}\{(n-m)G(0) - c\} \tag{3.24}$$

for all $x \in I$. The substitution $x = 0$ in (3.24) and the use of $A(0) = 0$ gives

$$A(1) = c\,nF(0) - mG(0) - c. \tag{3.25}$$

From (3.24) and (3.25), it follows that

$$G(x) = c\,[F(x) - F(0)] + A(x) + G(0) \tag{3.26}$$

for all $x \in I$. From (3.25) and (3.26), the following three equations can be derived:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} G(x_i y_j) = c\sum_{i=1}^{n}\sum_{j=1}^{m} F(x_i y_j) - cn(m-1)F(0) + m(n-1)G(0) - c \tag{3.27}$$

$$\sum_{i=1}^{n} G(x_i) = c\sum_{i=1}^{n} F(x_i) + (n-m)G(0) - c \tag{3.28}$$

$$\sum_{j=1}^{m} G(y_j) = c\sum_{j=1}^{m} F(y_j) + c(n-m)F(0) - c. \tag{3.29}$$

From (1.6), (3.27), (3.28), (3.29) and the fact that $c \neq 0$, it follows that

$$\sum_{i=1}^{n}\sum_{j=1}^{m} F(x_i y_j) = \sum_{i=1}^{n} F(x_i)\sum_{j=1}^{m} F(y_j) + (n-m)F(0)\sum_{i=1}^{n} F(x_i) + n(m-1)F(0). \tag{3.30}$$

Thus F satisfies (2.14) for all $(x_1, \ldots, x_n) \in \Gamma_n$, $(y_1, \ldots, y_m) \in \Gamma_m$, $n \geq 3$, $m \geq 3$ being fixed integers. But we need only those solutions for which $\sum_{i=1}^{n} F(x_i) - 1$ does not vanish identically on Γ_n. Making use of Result 2.4, there are only two possibilities which we discuss below:

The first possibility is that F is of the form

$$F(x) = \bar{b}_1(x) + F(0)$$

for all $x \in I$ where $\bar{b}_1 : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $\bar{b}_1(1) = -nF(0)$ if $F(1) + (n-1)F(0) - 1 \neq 0$, as the condition $F(1) + (n-1)F(0) - 1 \neq 0$ ensures that $\sum_{i=1}^{n} F(x_i) - 1$ does not vanish identically on Γ_n. Hence

$$F(x) = \bar{b}_1(x) - \frac{1}{n}\bar{b}_1(1). \tag{3.31}$$

From (3.31), (3.26) and the fact $\bar{b}_1(1) = -\,nF(0)$, it follows that

$$G(x) = c\,\bar{b}_1(x) + A(x) + G(0), \quad c \neq 0$$

for all $x \in I$. Define $\bar{b}_2 : \mathbb{R} \to \mathbb{R}$ as

$$\bar{b}_2(x) = c\,\bar{b}_1(x) + A(x) \tag{3.32}$$

for all $x \in \mathbb{R}$. Then b_2 is additive and

$$G(x) = \bar{b}_2(x) + G(0) \tag{3.33}$$

for all $x \in I$. Making use of (3.32), (3.25) and the fact that $\bar{b}_1(1) = -\,nF(0)$, it follows that $\bar{b}_2(1) = -\,mG(0) - c$. Equations (3.31) and (3.33) constitute the solution (3.2) of (1.6).

The second possibility is that

$$F(x) = M(x) - \bar{b}(x) + F(0) \tag{3.34}$$

where $\bar{b} : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $\bar{b}(1) = nF(0)$ and $M : I \to \mathbb{R}$ is a nonconstant nonadditive multiplicative mapping.

We claim that, in this case also, $\sum_{i=1}^{n} F(x_i) - 1$ does not vanish identically on Γ_n. Suppose our claim is false. This means that

$$\sum_{i=1}^{n} F(x_i) = 1$$

for all $(x_1, \ldots, x_n) \in \Gamma_n$. Also, from (3.34), using the fact that $\bar{b}(1) = nF(0)$, we have

$$\sum_{i=1}^{n} F(x_i) = \sum_{i=1}^{n} M(x_i)$$

for all $(x_1, \ldots, x_n) \in \Gamma_n$. So

$$\sum_{i=1}^{n} M(x_i) = 1$$

for all $(x_1, \ldots, x_n) \in \Gamma_n$. Then by Result 2.1, there exists an additive mapping $B : \mathbb{R} \to \mathbb{R}$ such that

$$M(x) = B(x) - \frac{1}{n}B(1) + \frac{1}{n}$$

for all $x \in I$. Putting $x = 0$ in the above equation and using the fact that $M(0) = B(0) = 0$, we obtain $B(1) = 1$. So,

$$M(x) = B(x)$$

for all $x \in I$. This means that M is additive, thereby, contradicting the fact that M is nonadditive.

From (3.26) and (3.34), we have

$$G(x) = cM(x) - c\,\bar{b}(x) + A(x) + G(0) \tag{3.35}$$

for all $x \in I$. Let us define a mapping $b_3 : \mathbb{R} \to \mathbb{R}$ as

$$b_3(x) = -\,c\,\bar{b}(x) + A(x) \tag{3.36}$$

for all $x \in \mathbb{R}$. Then b_3 is an additive mapping. Also, from (3.35) and (3.36), it follows that

$$G(x) = cM(x) + b_3(x) + G(0) \tag{3.37}$$

for all $x \in I$. Putting $p = 1$ in (3.36), making use (3.25) and $\bar{b}(1) = nF(0)$, it is easy to conclude that

$$c = -\,[b_3(1) + mG(0)]. \tag{3.38}$$

From (3.37) and (3.38), we have

$$G(x) = -[b_3(1) + mG(0)]M(x) + b_3(x) + G(0),\ [b_3(1) + mG(0)] \neq 0. \tag{3.39}$$

Let us define a mapping $b_4 : \mathbb{R} \to \mathbb{R}$ as

$$b_4(x) = -\overline{b}(x)$$

for all $x \in \mathbb{R}$. Then b_4 is additive. Equation (3.34) can now be written as

$$F(x) = M(x) + b_4(x) - \frac{1}{n}b_4(1) \tag{3.40}$$

for all $x \in I$, with $b_4(1) = -nF(0)$. Equations (3.39) and (3.40) constitute the solution (3.4) of (1.6). This completes the proof of Theorem 3.1. ∎

4. THE FUNCTIONAL EQUATION (1.5)

The main result of this paper is the following:

Theorem 4.1. *Let $h : I \to \mathbb{R}$, $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ and $k : I \to \mathbb{R}$ be mappings which satisfy equation (1.5) for all $(x_1, \ldots, x_n) \in \Gamma_n$, $(y_1, \ldots, y_m) \in \Gamma_m$, $n \geq 3$, $m \geq 3$ being fixed integers. Then, any general solution of (1.5) is of the form*

$$(\mathrm{S}_1)\qquad \begin{cases} h(x) = A_1(x) + h(0) \\ f(x) = A_3(x) - \dfrac{1}{n}A_3(1) \\ g \ \textit{ is an arbitrary real-valued mapping} \\ k(x) = A_2(x) - \dfrac{1}{n}[A_2(1) - A_1(1) - nm\, h(0)] \end{cases}$$

or

$$(\mathrm{S}_2)\qquad \begin{cases} h(x) = [f(1) + (n-1)f(0)]\,\{b_2(x) + [g(1) + (m-1)g(0)]x\} + \overline{B}(x) + h(0) \\ f \ \textit{ is an arbitrary real-valued mapping} \\ g(x) = b_2(x) + [g(1) + (m-1)g(0)]x - \dfrac{1}{m}b_2(1) \\ k(x) = [f(1) + (n-1)f(0)]\,\{b_2(x) + [g(1) + (m-1)g(0)]x\} \\ \qquad\qquad - [g(1) + (m-1)g(0)]f(x) + B^*(x) + \dfrac{1}{n}[\overline{B}(1) - B^*(1)] + m\, h(0) \end{cases}$$

or

$$(\mathrm{S}_3)\qquad \begin{cases} h(x) = [f(1){+}(n{-}1)f(0)]\,\{[g(1){-}g(0)]x{+}a(x){+}D(x,x)\}{+}\overline{B}(x){+}h(0) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{if } 0 < x \leq 1 \\ \qquad = h(0) \qquad \textit{if } x = 0 \\ f(x) = [f(1) + (n-1)f(0)]\left\{b_1(x) - \dfrac{1}{n}b_1(1) + \dfrac{1}{n}\right\} \\ g(x) = [g(1) - g(0)]x + a(x) + D(x,x) + g(0) \qquad \textit{if } 0 < x \leq 1 \\ \qquad = g(0) \qquad \textit{if } x = 0 \\ k(x) = [f(1) + (n-1)f(0)]\,\{[g(1) - g(0)]x + a(x) + D(x,x)\} \\ \qquad\qquad - [f(1) + (n-1)f(0)][g(1) + (m-1)g(0)]\left\{b_1(x) - \dfrac{1}{n}b_1(1) + \dfrac{1}{n}\right\} \\ \qquad\qquad + B^*(x) + \dfrac{1}{n}[\overline{B}(1) - B^*(1)] + m\, h(0) \qquad \textit{if } 0 < x \leq 1 \\ \qquad = \dfrac{1}{n}[\overline{B}(1){-}B^*(1)]{-}\dfrac{1}{n}[f(1){+}(n{-}1)f(0)][g(1){+}(m{-}1)g(0)][1{-}b_1(1)] \\ \qquad\qquad + m\, h(0) \qquad \textit{if } x = 0 \end{cases}$$

with $[f(1)+(n-1)f(0)]\neq 0$ *or*

$$(\mathrm{S}_4)\quad \begin{cases} h(x)=[f(1)+(n-1)f(0)]\{-[b_3(1)+mg(0)]M(x) \\ \qquad\qquad +[g(1)+(m-1)g(0)]x+b_3(x)\}+\overline{B}(x)+h(0) \\ f(x)=[f(1)+(n-1)f(0)]\left\{M(x)+b_4(x)-\dfrac{1}{n}b_4(1)\right\} \\ g(x)=-[b_3(1)+mg(0)]M(x)+[g(1)+(m-1)g(0)]x+b_3(x)+g(0) \\ k(x)=[f(1)+(n-1)f(0)]\{-[b_3(1)+mg(0)]M(x) \\ \qquad\qquad +[g(1)+(m-1)g(0)]x+b_3(x)\}-[g(1)+(m-1)g(0)] \\ \qquad\qquad \times[f(1)+(n-1)f(0)]\left\{M(x)+b_4(x)-\dfrac{1}{n}b_4(1)\right\} \\ \qquad\qquad +B^*(x)+\dfrac{1}{n}[\overline{B}(1)-B^*(1)]+mh(0), \\ (b_3(1)+mg(0)\neq 0,[f(1)+(n-1)f(0)]\neq 0) \end{cases}$$

where $\overline{B}:\mathbb{R}\to\mathbb{R}$, $B^*:\mathbb{R}\to\mathbb{R}$, $A_i:\mathbb{R}\to\mathbb{R}$ $(i=1,2,3)$, $b_j:\mathbb{R}\to\mathbb{R}$ $(j=1,2,3,4)$ *are additive mappings,* $M:I\to\mathbb{R}$ *is a nonconstant nonadditive mapping which is multiplicative in the sense that it satisfies* (2.2), (2.3) *and* (2.4) *for all* $x\in\,]0,1[$, $y\in\,]0,1[$*; and* $a:\mathbb{R}\to\mathbb{R}$, $D:\mathbb{R}\times\,]0,1]\to\mathbb{R}$ *are as mentioned in Result 2.2.*

Before giving the proof of this theorem, we need to prove some lemmas.

Lemma 4.2. *If* $h:I\to\mathbb{R}$, $k:I\to\mathbb{R}$ *are mappings which satisfy the functional equation*

$$\sum_{i=1}^{n}\sum_{j=1}^{m}h(x_iy_j)=\sum_{i=1}^{n}k(x_i) \tag{4.1}$$

for all $(x_1,\ldots,x_n)\in\Gamma_n$, $(y_1,\ldots,y_m)\in\Gamma_m$*;* $n\geq 3, m\geq 3$ *being fixed integers, then* h *and* k *are of the form*

$$h(x)=A_1(x)+h(0) \tag{4.2}$$

$$k(x)=A_2(x)-\frac{1}{n}[A_2(1)-A_1(1)-nmh(0)] \tag{4.3}$$

where $A_i:\mathbb{R}\to\mathbb{R}$ $(i=1,2)$ *are additive mappings.*

Proof of Lemma 4.2. Choose $x_1=1, x_2=\ldots=x_n=0$ in (4.1). We obtain the equation

$$\sum_{j=1}^{m}h(y_j)=k(1)+(n-1)k(0)-m(n-1)h(0)$$

for all $(y_1,\ldots,y_m)\in\Gamma_m$. By Result 2.1, there exists an additive mapping $A_1:\mathbb{R}\to\mathbb{R}$ such that

$$h(x)=A_1(x)-\frac{1}{m}A_1(1)+\frac{1}{m}\{k(1)+(n-1)k(0)-m(n-1)h(0)\} \tag{4.4}$$

for all $x\in I$. The substitution $x=0$ in (4.4) gives

$$A_1(1)=k(1)+(n-1)k(0)-nmh(0) \tag{4.5}$$

as $A_1(0)=0$. On putting this value of $A_1(1)$ in (4.4), (4.2) follows. Now, from (4.2), (4.5) and (4.1), the equation

$$\sum_{i=1}^{n}k(x_i)=k(1)+(n-1)k(0)$$

follows for all $(x_1, \ldots, x_n) \in \Gamma_n$. By Result 2.1, there exists an additive mapping $A_2 : \mathbb{R} \to \mathbb{R}$ such that

$$k(x) = A_2(x) - \frac{1}{n} A_2(1) + \frac{1}{n}\{k(1) + (n-1)k(0)\} \tag{4.6}$$

for all $x \in I$, with $A_2(1) = k(1) - k(0)$. From (4.5) and (4.6), (4.3) follows. This completes the proof of Lemma 4.2. ∎

Lemma 4.3. *If the mappings $h : I \to \mathbb{R}$, $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ and $k : I \to \mathbb{R}$ satisfy the functional equation (1.5) for all $(x_1, \ldots, x_n) \in \Gamma_n$, $(y_1, \ldots, y_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers, then the mappings f and g also satisfy the functional equation*

$$\begin{aligned} &[f(1) + (n-1)f(0)] \sum_{i=1}^{n} \sum_{j=1}^{m} g(x_i y_j) \\ &= \sum_{i=1}^{n} f(x_i) \sum_{j=1}^{m} g(y_j) + [f(1) + (n-1)f(0)] \sum_{i=1}^{n} g(x_i) - [g(1) + (m-1)g(0)] \\ &\quad \times \sum_{i=1}^{n} f(x_i) + n(m-1)g(0)[f(1) + (n-1)f(0)]. \end{aligned} \tag{4.7}$$

Proof of Lemma 4.3. Putting $x_1 = 1, x_2 = \ldots = x_n = 0$ in (1.5), we obtain the equation

$$\sum_{j=1}^{m} \{h(y_j) - [f(1) + (n-1)f(0)]g(y_j)\} = k(1) + (n-1)k(0) - m(n-1)h(0)$$

for all $(y_1, \ldots, y_m) \in \Gamma_m$. By Result 2.1, there exists an additive mapping $\overline{B} : \mathbb{R} \to \mathbb{R}$ such that

$$h(x) = [f(1) + (n-1)f(0)][g(x) - g(0)] + \overline{B}(x) + h(0) \tag{4.8}$$

for all $x \in I$ with

$$\overline{B}(1) = m[f(1) + (n-1)f(0)]g(0) + k(1) + (n-1)k(0) - mnh(0). \tag{4.9}$$

Making use of the equations (4.8) and (4.9) in (1.5), we obtain the equation

$$\begin{aligned} &\sum_{i=1}^{n} f(x_i) \sum_{j=1}^{m} g(y_j) + \sum_{i=1}^{n} k(x_i) \\ &= [f(1)+(n-1)f(0)] \left[\sum_{i=1}^{n} \sum_{j=1}^{m} g(x_i y_j) - m(n-1)g(0) \right] + k(1) + (n-1)k(0) \end{aligned} \tag{4.10}$$

for all $(x_1, \ldots, x_n) \in \Gamma_n$, $(y_1, \ldots, y_m) \in \Gamma_m$. The substitution $y_1 = 1, y_2 = \ldots = y_m = 0$ in (4.10) gives

$$\begin{aligned} &\sum_{i=1}^{n} \{[g(1) + (m-1)g(0)]f(x_i) + k(x_i) - [f(1) + (n-1)f(0)]g(x_i)\} \\ &\qquad = (m-n)g(0)[f(1) + (n-1)f(0)] + k(1) + (n-1)k(0) \end{aligned} \tag{4.11}$$

for all $(x_1, \ldots, x_n) \in \Gamma_n$. By Result 2.1, there exists an additive mapping $B^* : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} k(x) = &[f(1) + (n-1)f(0)][g(x) - g(0)] - [g(1) + (m-1)g(0)][f(x) - f(0)] \\ &+ B^*(x) + k(0) \end{aligned} \tag{4.12}$$

for all $x \in I$ with

$$B^*(1) = mg(0)[f(1) + (n-1)f(0)] - nf(0)[g(1) + (m-1)g(0)] + k(1) - k(0). \tag{4.13}$$

Elimination of $\sum_{i=1}^{n} k(x_i)$ from (4.10) and (4.11) gives equation (4.7). This completes the proof of Lemma 4.3. ∎

Proof of Theorem 4.1. We divide the discussion into two cases:

Case 1. $f(1) + (n-1)f(0) = 0$.
In this case, equation (4.7) reduces to the equation

$$\left\{ \sum_{j=1}^{m} g(y_j) - [g(1) + (m-1)g(0)] \right\} \sum_{i=1}^{n} f(x_i) = 0 \tag{4.14}$$

valid for all $(x_1, \ldots, x_n) \in \Gamma_n$, $(y_1, \ldots, y_m) \in \Gamma_m$ such that $f(1) + (n-1)f(0) = 0$. Hence, either

$$\sum_{i=1}^{n} f(x_i) = 0 \tag{4.15}$$

for all $(x_1, \ldots, x_n) \in \Gamma_n$ or

$$\sum_{j=1}^{m} g(y_j) - [g(1) + (m-1)g(0)] = 0 \tag{4.16}$$

for all $(y_1, \ldots, y_m) \in \Gamma_m$. In the former case, by Result 2.1, there exists an additive mapping $A_3 : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = A_3(x) - \frac{1}{n} A_3(1) \tag{4.17}$$

for all $x \in I$. Keeping in view (4.15); from (4.14), it follows that "g is an arbitrary real-valued mapping". On the other hand, using (4.15) in (1.5), we obtain the equation (4.1) whose solutions are given by (4.2) and (4.3). Thus, equations (4.2), (4.3), (4.17) and the statement "g is an arbitrary real-valued mapping", constitute the solution (S_1) of (1.5).

Now in the later case, by Result 2.1, there exists an additive mapping $A_4 : \mathbb{R} \to \mathbb{R}$ such that

$$g(x) = A_4(x) - \frac{1}{m} A_4(1) + \frac{1}{m}[g(1) + (m-1)g(0)] \tag{4.18}$$

for all $x \in I$. The substitution $x = 0$ in (4.18) gives $A_4(1) = g(1) - g(0)$. Making use of this value of $A_4(1)$ in (4.18), equation (4.18) reduces to

$$g(x) = A_4(x) + g(0) \tag{4.19}$$

for all $x \in I$. Also, from (4.14) and (4.16), one is led to the conclusion "f is an arbitrary real-valued mapping". Since $f(1) + (n-1)f(0) = 0$, equations (4.8) and (4.12) give

$$h(x) = \overline{B}(x) + h(0) \tag{4.20}$$

and

$$k(x) = B^*(x) - [g(1) + (m-1)g(0)][f(x) - f(0)] + k(0) \tag{4.21}$$

for all $x \in I$. But, from (4.9) and (4.13), we have

$$k(0) = \frac{1}{n}[\overline{B}(1) - B^*(1)] + mh(0) - f(0)[g(1) + (m-1)g(0)]. \tag{4.22}$$

Making use of this value of $k(0)$ in (4.21), we obtain

$$k(x) = B^*(x) - [g(1) + (m-1)g(0)]f(x) + \frac{1}{n}[\overline{B}(1) - B^*(1)] + mh(0) \tag{4.23}$$

for all $x \in I$. The solution, of (1.5), consisting of (4.19), (4.20), (4.23) and the statement "f is an arbitrary real-valued mapping" is included in (S_2) if we set $A_4(x) = b_2(x) + [g(1) + (m-1)g(0)]x$ and use the fact that $A_4(1) = g(1) - g(0)$ where $b_2 : \mathbb{R} \to \mathbb{R}$ is an additive mapping.

Case 2. $f(1) + (n-1)f(0) \neq 0$.
In this case, let us write (4.7) in the form

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{j=1}^{m} \{g(x_i y_j) - [g(1) + (m-1)g(0)]x_i y_j\} \\ &= \sum_{i=1}^{n} \left\{[f(1) + (n-1)f(0)]^{-1} f(x_i)\right\} \sum_{j=1}^{m} \{g(y_j) - [g(1) + (m-1)g(0)]y_j\} \\ &+ \sum_{i=1}^{n} \{g(x_i) - [g(1) + (m-1)g(0)]x_i\} + n(m-1)g(0). \end{aligned} \tag{4.24}$$

Define the mappings $F : I \to \mathbb{R}$ and $G : I \to \mathbb{R}$ as

$$F(x) = [f(1) + (n-1)f(0)]^{-1} f(x) \tag{4.25}$$

$$G(x) = g(x) - [g(1) + (m-1)g(0)]x \tag{4.26}$$

for all $x \in I$. Then (4.24) reduces to the equation (1.6) as

$$G(0) = g(0). \tag{4.27}$$

From (4.25), (4.26) and (4.27), it can be easily verified that

$$F(1) + (n-1)F(0) = 1 \tag{4.28}$$

and

$$G(1) + (m-1)G(0) = 0. \tag{4.29}$$

In Theorem 3.1, we have already obtained the required general solutions (3.1) to (3.4) of (1.6). We reject (3.2) because, in this case, the mapping F does not satisfy (4.28). The solutions (3.1), (3.3) and (3.4) do satisfy (4.28) and (4.29). The required solutions (S_2), (S_3) and (S_4) can be obtained from the equations (4.25), (4.26), (4.27), (4.8), (4.12), (4.9), (4.13), (3.1), (3.3) and (3.4). The calculation work is omitted for the sake of brevity. This completes the proof of Theorem 4.1. ∎

5. COMMENTS

In this section, we point out the importance of various solutions of equation (1.5) in information theory. The solution (S_1) is not of any relevance in information theory as the mapping $g : I \to \mathbb{R}$ is arbitrary and the summands $\sum_{i=1}^{n} h(x_i)$, $\sum_{i=1}^{n} f(x_i)$ and $\sum_{i=1}^{n} k(x_i)$ are independent of probabilities $x_1, \ldots, x_n$. The solution (S_2) is also not of any relevance in information theory for the similar reason. Before we discuss the importance of solution (S_3) in information theory, we mention the following:

Let $(x_1, \ldots, x_n) \in \Gamma_n$, $n \geq 3$ being a fixed integer. Let $S = \{i : 0 < x_i \leq 1, 1 \leq i \leq n\}$. Then S is nonempty. Let n_0 be the number of elements in S.

From solution (S_3) and (2.8), we obtain

$$\sum_{i=1}^{n} h(x_i) = n_0 h(0) + [f(1) + (n-1)f(0)] \left\{ [g(1) - g(0)] + a(1) + \sum_{i \in S} D(x_i, x_i) \right\} + \overline{B}(1) + (n - n_0)h(0)$$

or

$$\sum_{i=1}^{n} h(x_i) = nh(0) + [f(1) + (n-1)f(0)] \times \left\{ [g(1) - g(0)] - D(1,1) + \sum_{i \in S} D(x_i, x_i) \right\} + \overline{B}(1). \tag{5.1}$$

Similarly

$$\sum_{i=1}^{n} f(x_i) = f(1) + (n-1)f(0) \tag{5.2}$$

$$\sum_{i=1}^{n} g(x_i) = [g(1) + (n-1)g(0)] - D(1,1) + \sum_{i \in S} D(x_i, x_i) \tag{5.3}$$

$$\sum_{i=1}^{n} k(x_i) = [f(1) + (n-1)f(0)] \left\{ - mg(0) - D(1,1) + \sum_{i \in S} D(x_i, x_i) \right\} + \overline{B}(1) + nmh(0)\,. \tag{5.4}$$

Keeping in view the form of the Shannon entropy given by (1.1), it seems appropriate to choose the mapping $D : \mathbb{R} \times\,]0,1] \to \mathbb{R}$ defined as

$$D(x,y) = dx \log_2 y$$

for all $x \in \mathbb{R}$, $y \in\,]0,1]$, d an arbitrary real constant. The case $d = 0$ is not of much importance. So we restrict to $d \neq 0$. Now

$$D(x,x) = dx \log_2 x \tag{5.5}$$

for all $x \in\,]0,1]$ and $D(1,1) = 0$. To accommodate the 0-probabilities, we assume

$$\lim_{x \to 0^+} D(x,x) = 0 \tag{5.6}$$

or equivalently $0 \log_2 0 = 0$ as $d \neq 0$. Making use of (5.5), (5.6), (1.1) and the fact that $D(1,1) = 0$ in equations (5.1), (5.3), (5.4), we get

$$\sum_{i=1}^{n} h(x_i) = nh(0) + [f(1) + (n-1)f(0)]\{g(1) - g(0) - dH_n(x_1, \ldots, x_n)\} + \overline{B}(1)$$

$$\sum_{i=1}^{n} g(x_i) = [g(1) + (n-1)g(0)] - dH_n(x_1, \ldots, x_n)$$

$$\sum_{i=1}^{n} k(x_i) = [f(1) + (n-1)f(0)]\{- mg(0) - dH_n(x_1, \ldots, x_n)\} + \overline{B}(1) + nmh(0).$$

Thus we see that out of the four unknown mappings appearing in equation (1.5), the three mappings h, g and k are very closely connected to the Shannon entropy.

Now, we discuss the solution (S_4)

$$\sum_{i=1}^{n} h(x_i) = [f(1)+(n-1)f(0)][b_3(1)+mg(0)]\left[1-\sum_{i=1}^{n} M(x_i)\right]$$
$$+[f(1)+(n-1)f(0)][g(1)-g(0)]+\overline{B}(1)+nh(0)$$
$$\sum_{i=1}^{n} f(x_i) = -[f(1)+(n-1)f(0)]\left[1-\sum_{i=1}^{n} M(x_i)\right]+[f(1)+(n-1)f(0)]$$
$$\sum_{i=1}^{n} g(x_i) = [b_3(1)+mg(0)]\left[1-\sum_{i=1}^{n} M(x_i)\right]+[g(1)+(n-1)g(0)]$$
$$\sum_{i=1}^{n} k(x_i) = [f(1)+(n-1)f(0)][b_3(1)+g(1)+(2m-1)g(0)]\left[1-\sum_{i=1}^{n} M(x_i)\right]$$
$$-m[f(1)+(n-1)f(0)]g(0)+\overline{B}(1)+nmh(0).$$

Taking into consideration the form of the nonadditive entropy of degree α defined by (1.3) it seems appropriate to choose the mapping $M : I \to \mathbb{R}$ defined as $M(x) = x^\alpha$ for all $x \in I, \alpha \in \mathbb{R}, \alpha > 0, \alpha \neq 1, 0^\alpha := 0$ and $1^\alpha := 1$. Then using (1.3), the above equations give

$$\sum_{i=1}^{n} h(x_i) = [f(1)+(n-1)f(0)][b_3(1)+mg(0)](1-2^{1-\alpha})H_n^\alpha(x_1,\ldots,x_n)$$
$$+[f(1)+(n-1)f(0)][g(1)-g(0)]+\overline{B}(1)+nh(0)$$
$$\sum_{i=1}^{n} f(x_i) = -[f(1)+(n-1)f(0)](1-2^{1-\alpha})H_n^\alpha(x_1,\ldots,x_n)+[f(1)+(n-1)f(0)]$$
$$\sum_{i=1}^{n} g(x_i) = [b_3(1)+mg(0)](1-2^{1-\alpha})H_n^\alpha(x_1,\ldots,x_n)+[g(1)+(n-1)g(0)]$$
$$\sum_{i=1}^{n} k(x_i) = [f(1)+(n-1)f(0)][b_3(1)+g(1)+(2m-1)g(0)](1-2^{1-\alpha})H_n^\alpha(x_1,\ldots,x_n)$$
$$-\ m[f(1)+(n-1)f(0)]g(0)+\overline{B}(1)+nm\,h(0).$$

Thus we see that all the four mappings h, f, g and k appearing in (1.5) are related to the nonadditive entropy of degree α due to Havrda and Charvát [4].

REFERENCES

[1] T.W. CHAUDNDY and J.B. MCLEOD, On a functional equation. *Edinburgh Math. Notes,* **43** (1960), pp. 7–8.

[2] Z. DARÓCZY and L. LOSONCZI, Über die Erweiterung der auf einer Punktmenge additiven Funktionen. *Publ. Math.,* **14** (1967), pp. 239–245. (In German)

[3] G. DIAL, On measurable solutions of a fuctional equation and their application to information theory. *Aplikace Matematiky,* **28** (1983), pp. 103–107.

[4] J. HAVRDA and F. CHARVÁT, Quantification method of classification process. Concept of structural α-entropy. *Kybernetika,* **3** (1967), pp. 30–35.

[5] L. LOSONCZI and GY. MAKSA, On some functional equations of information theory. *Acta Math. Acad. Sci. Hungar.,* **39** (1982), pp. 73–82.

[6] P. NATH and D. K. SINGH, On a multiplicative type sum form functional equation and its role in information theory. *Applications of Mathematics,* **51** (5) (2006), pp. 495–516.

[7] I. J. TANEJA, A joint characterization of Shannon's entropy and entropy of type β through a functional equation. *Journal of Mathematical Sciences,* **10** (1975), pp. 69–74.

[8] C. E. SHANNON, A mathematical theory of communication. *Bell Syst. Tech. Jour.,* **27** (1948), 378–423, pp. 623–656.