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GENERALIZED STIELTJES TRANSFORM AND ITS FRACTIONAL INTEGRALS FOR INTEGRABLE BOEHMIAN

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ABSTRACT. This paper investigates generalized Stieltjes transform for integrable Boehmian and further, generalized Stieltjes transform of fractional integrals are defined for integrable Boehmian.

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1. **INTRODUCTION**

The transform studied by Stieltjes [8], in connection with his investigations of infinite continued fractions, is defined by [9, p. 25],

$$f(s)=\int_0^\infty \frac{\varphi(t)}{s+t}dt, \tag{1.1}$$

provided the integral involved is convergent. *The Stieltjes transform arises naturally from the iteration of the Laplace transform*

$$\int_0^\infty e^{-sy}dy\int_0^\infty e^{-ty}\varphi(t)dt=\int_0^\infty \varphi(t)dt\int_0^\infty e^{-(s+t)y}dy=\int_0^\infty \frac{\varphi(t)}{s+t}dt, \tag{1.2}$$

where $\varphi(t)\in L$ on $(0,\mathbf{R})$ for every $\mathbf{R}>0$. Real or complex inversion formulae and other results for Stieltjes transform may be seen in Widder [9]. In other words, Stiletljes transform (1.1) and (1.2) can be defined as

$$\mathcal{G}\{f(x);y\}=\int_0^\infty f(x)(x+y)^{-1}dx, \tag{1.3}$$

and

$$\mathcal{G}\{f(x);y\}=\mathcal{L}\{\mathcal{L}\{f(x);t\};y\}, \tag{1.4}$$

where $\mathcal{L}$ denotes the Laplace transform.

Pathak [5, p. 151-152] considered *p* to be any complex number except zero and the negative integers. Then for all *s* in the region Ω, where Ω is the *s* – plane cut from the origin along the negative real axis, the *Stieltjes transform* (*in general form*) is defined by

$$F(s)=(\mathcal{G}_pf)(s)=\int_0^\infty \frac{f(t)}{(s+t)^p}dt. \tag{1.5}$$

The real inversion formula is given as

$$\mathbf{L}^p_{k,t}[F(t)]=\frac{(-1)^{k-1}2^{p-1}(2k-1)!\Gamma(p)}{k!(k-2)!\Gamma(2k+p+1)}[t^{2k+p-2}F^{(k-1)}(t)]^{(k)}, \tag{1.6}$$

i.e.

$$\mathbf{L}^p_{k,t}[F(t)]=f(t). \tag{1.7}$$

If $Rep>1$, *f* is locally integrable in $[0,\infty)$, the improper Lebesgue integral (1.5) converges, and $\lambda>0$, then for each positive *x* for which the Lebesgue limits $f(x\pm 0)$ exist,

$$\frac{1}{2}\{f(x+0)+f(x-0)\}=\lim_{\eta\to 0}\frac{p-1}{2\pi i}\int_{-x}^{\lambda}(x+t)^{p-2}\{F(t-i\eta)-F(t++i\eta)\}dt. \tag{1.8}$$

The **distributional Stieltjes transform** $F(s)$ of an arbitrary element $f\in S'_\alpha$ is defined by

$$F(s)=\left\langle f(x),(s+x)^{-p}\right\rangle, \tag{1.9}$$

where s belongs to the complex plane cut along the negative real axis including the origin. $S'_\alpha(I)$ is the dual of the vector space $S_\alpha(I)$ over the field of complex numbers. The **Parseval relation** for the generalized Stieltjes transform is given by

$$\left\langle \mathcal{G}'_p f, \varphi \right\rangle = \left\langle f, \mathcal{G}_p \varphi \right\rangle, \tag{1.10}$$

where $f \in S'_\alpha$, $\varphi \in S_\alpha$.

Considering L_1 as the space of complex valued Lebesgue integrable functions on real line **R**, the norm of the function is defined as

$$\|f\| = \int_{\mathbf{R}} |f(x)|\, dx.$$

and the convolution product is

$$(f * g)(x) = \int_{\mathbf{R}} f(u)g(x-u)du, f, g \in L_1.$$

A delta sequence of continuous real functions $\delta_n \in L_1$satisfies the following properties :

(i) $\int_{\mathbf{R}} \delta_n(x)dx = 1, \forall n \in \mathbb{N}$,
(ii) $\|\delta_n\| < M$, for some $M \in$ **R** and all $n \in \mathbb{N}$,
(iii) $\lim\limits_{n\to\infty} \int_{|x|>\varepsilon} |\delta_n(x)|\, dx = 0$, for each $\varepsilon > 0$.

If (φ_n) and (ψ_n) are delta sequences, then $(\varphi_n * \psi_n)$is also a delta sequence. If $f \in L_1$ and $(\delta_n)$is a delta sequence, then $\|f * \delta_n - f\| \to 0$ as $n \to \infty$. A pair of sequence (f_n,φ_n) is called a **quotient of the sequence**, denoted by f_n/φ_n, $f_n \in L_1(n = 1, 2, \ldots)$, (φ_n) is a delta sequence and $f_m * \varphi_n = f_n * \varphi_m$, $\forall m, n \in \mathbb{N}$. Two quotients of sequences f_n/φ_n and g_n/ψ_n are equivalent if $f_n * \psi_n = g_n * \varphi_n$, $\forall n \in \mathbb{N}$. The equivalence class of quotient of sequence is called the **integrable Boehmian** and denoted as B_{L_1}.If $F = [f_n/\delta_n]$, then $f * \delta_n = f_n$ and therefore, $f * \delta_n \in L_1, \forall n \in \mathbb{N}$. Convergence of Boehmians is defined by Mikusinski [4]. If

$$\Delta - \lim_{n\to\infty} F_n = F$$

and

$$\Delta - \lim_{n\to\infty} G_n = G,$$

then

$$\Delta - \lim_{n\to\infty} F_n * G_n = F * G,$$

where Δ is a class of sequence $(\delta_n)(n = 1, 2, \ldots)$.

If $F = [f_n/\delta_n] \in B_{L_1}$, then $\forall n \in \mathbb{N}$, $f_1 * \delta_n = f_n * \delta_1$. Since (i) holds true,

$$\begin{aligned} \int_{\mathbf{R}} f(x)dx &= \int_{\mathbf{R}} (f_1 * \delta_n)(x)dx \\ &= \int_R (f_n * \delta_1)(x)dx = \int_R f_n(x)dx. \end{aligned} \tag{1.11}$$

This property defines the *integral of a Boehmian* as follows :if $F = [f_n/\delta_n] \in B_{L_1}$ then

$$\int_{\mathbf{R}} F(x)dx = \int_{\mathbf{R}} f_1(x)dx \tag{1.12}$$

The function defined in (1.12) is analogous to the Lebesgue integral. However, there are functions which are integrable in the sense of Boehmians, but not so as an ordinary function. The authors in [2] investigated the Laplace transform for integrable Boehmians and in [3] defined the Laplace and Fourier transforms of fractional integrals for integrable Boehmians. In this paper we define the generalized Stieltjes transform and then investigate the generalized Stieltjes transform of fractional integrals for integrable Boehmians.

2. GENERALIZED STIELTJES TRANSFORM FOR INTEGRABLE BOEHMIANS

Lemma 2.1. *If* $[f_n/\delta_n] \in B_{L_1}$*, then the sequence*

$$F(s) = (\mathcal{G}_p f)(s) = \int_0^\infty \frac{f(t)}{(s+t)^p} dt, \tag{2.1}$$

converges uniformly on each compact set in $\mathbf{R}$*.*

Proof. If (δ_n) is a delta sequence, then $\mathcal{G}_p(\delta_n)$ converges uniformly on each compact set to a constant function 1. Therefore, for each compact set K, $\mathcal{G}_p(\delta_k) > 0$ on *K*, and for almost all $k \in K$, we have

$$\mathcal{G}_p(f_n) = \mathcal{G}_p(f_n)\frac{\mathcal{G}_p(\delta_k)}{\mathcal{G}_p(\delta_k)} = \frac{\mathcal{G}_p(f_n * \delta_k)}{\mathcal{G}_p(\delta_k)}$$

i.e. $=\frac{\mathcal{G}_p(f_k*\delta_n)}{\mathcal{G}_p(\delta_k)} = \frac{\mathcal{G}_p(f_k)}{\mathcal{G}_p(\delta_k)}\mathcal{G}_p(\delta_n)$, on *K*.

In view of Lemma 2.1, the Stieltjes transform of an integrable Boehmian $F = [f_n/\delta_n]$ can be defined as the limit of $\mathcal{G}_p(f_n)$ in the space of continuous functions on **R.** Hence, this proves that the Stieltjes transform of an integrable Boehmian is a continuous function and, thereby, the lemma is proved. ∎

Theorem 2.2. *Let* $F, G \in B_{L_1}$*. Then*

(i) $\mathcal{G}_p(\lambda F) = \lambda \mathcal{G}_p(F)$ *, (for any complex number* λ*),*
and $\mathcal{G}_p(F+G) = \mathcal{G}_p(F) + \mathcal{G}_p(G)$ *.*
(ii) $\mathcal{G}_p(F * G) = \mathcal{G}_p(F)\mathcal{G}_p(G)$.
(iii) $\mathcal{G}_p[f(x-b)](s) = \mathcal{G}_p(s+b)$.
(iv) $\mathcal{G}_p[D^k f](x) = (p)_k F_{p+k}(s)$.
(v) *If* $\Delta - \lim\limits_{n\to\infty} F_n = F$*, then* $\mathcal{G}_p(F_n) \to \mathcal{G}_p(F)$ *uniformly on each compact set.*

Proof. By virtue of the properties of Stieltjes transform [9, p. 142-143], the proof of (i) is obvious. By the definition of the convolution transform [9, p. 170] and by [9, Ex. 9, p. 170], the property (ii) can easily be proved. The proofs of properties (iii) and (iv) are same as in [1, p. 378].
Proof of the property (v) is as follows :
We have

$$\delta - \lim_{n\to\infty} F_n - F \Rightarrow \mathcal{G}_p(F_n) \to \mathcal{G}_p(F)$$

uniformly on each compact set. Let (δ_n) be a delta sequence such that

$$F_n * \delta_k, F * \delta_k \in L_1, \forall n, k \in \mathbb{N}$$

and

$$\|(F_n - F) * \delta_k\| \to 0, \text{ as } n \to \infty, \forall k \in \mathbb{N},$$

where *k* is well defined. Then $\mathcal{G}_p(\delta_k) > 0$ on *K* for some $k \in \mathbb{N}$. Since $\mathcal{G}_p(\delta_k)$ is a continuous function, it is enough to prove that

$$\mathcal{G}_p(F_n) \cdot \mathcal{G}_p(\delta_k) \to \mathcal{G}_p(F)\mathcal{G}_p(\delta_k)$$

uniformly on *K*. But we have

$$\mathcal{G}_p(F_n)\mathcal{G}_p(\delta_k) - \mathcal{G}_p(F) \cdot (\delta_k) = \mathcal{G}_p((F_n - F) * \delta_k)$$

and

$$\|(F_n - F) * \delta_k\| \to 0, \text{ as } n \to \infty$$

This, explicitly, proves the property (v). The theorem is thus, completely proved. ∎

Remark 2.1. Let $f \in L_1$ and by (1.7)

$$\mathbf{L}^p_{k,t}[F_n(t)] = f_n(t). \tag{2.2}$$

Then (f_n) converges to *f* in L_1 - norm.

3. GENERALIZED STIELTJES TRANSFORM OF FRACTIONAL INTEGRALS FOR INTEGRABLE BOEHMIANS

The generalized fractional integral operators introduced by [6, p. 85], $I^{\alpha,\beta,\eta}$ and $J^{\alpha,\beta,\eta}$, respectively, for $\alpha, \beta, \eta \in C$ (set of complex numbers) and $Re(\alpha) > 0$ are

$$I^{\alpha,\beta,\eta}\varphi = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1 - \frac{t}{x}\right)\varphi(t)dt, \tag{3.1}$$

and

$$J^{\alpha,\beta,\eta}\varphi = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1 - \frac{x}{t}\right)\varphi(t)dt, \tag{3.2}$$

where ${}_2F_1\,(a, b; c; z)$ is Gauss hypergeometric function. One of the special cases of the integral operator $I^{\alpha,\beta,\eta}$ is

$$I^{\alpha,-\alpha,\eta}\varphi(x) = \mathrm{R}^\alpha\varphi(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\varphi(t)dt, \text{ (Riemann-Liouville) }. \tag{3.3}$$

Definition 3.1. Let $f \in F'_{p,\mu}$ (space of continuous linear functionals). Then

$$\left\langle I^{\alpha,\beta,\eta} f, \varphi \right\rangle = \left\langle f, J^{\alpha,\beta,\eta}\varphi \right\rangle, \tag{3.4}$$

and

$$\left\langle J^{\alpha,\beta,\eta} f, \varphi \right\rangle = \left\langle f, I^{\alpha,\beta,\eta}\varphi \right\rangle, \tag{3.5}$$

for $\varphi \in F_{p,\mu+\beta}$ (the testing function space) and $\varphi \in L_p$ (the Lebesgue space) cf. [6, Definition 2.1, p. 84].

The **generalized transform of fractional integrals** are defined cf. [1, p.380] as

$$\begin{aligned} \mathcal{G}_p[I^{\alpha,\beta,\eta} f](s) &= \langle f(x), \varphi_1(x,s) \rangle \\ &= \left\langle f, [J^{\alpha,\beta,\eta}(s+x)^{-p}](x) \right\rangle, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\mathcal{G}_p[J^{\alpha,\beta,\eta}f](s) &= \langle f(x), \varphi_2(x,s)\rangle \\ &= \left\langle f, [I^{\alpha,\beta,\eta}(s+x)^{-p}](x)\right\rangle,\end{aligned} \tag{3.7}$$

where $f \in F'_{p,\mu}$.

If $\alpha + \beta = 0$, then cf. [1, p.382]

$$\mathcal{G}_p[\mathrm{R}^\alpha f](s) = \frac{\Gamma(p-\alpha)}{\Gamma(p)}\mathcal{G}_{p-\alpha}[f](s). \tag{3.8}$$

The Riemann – Liouville fractional integral (3.3) has alternate representations cf. [7, p. 33]

$$(I^\alpha_{a+}\varphi)(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\varphi(t)dt, x > a, \alpha > 0, \varphi(x) \in L_1(a,b) \tag{3.9}$$

$$(I^\alpha_{b-}\varphi)(x) := \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\varphi(t)dt, x < a, \alpha > 0, \varphi(x) \in L_1(a,b). \tag{3.10}$$

They are fractional integrals of the order α and are known as *left – sided* and *right – sided* fractional integrals. For the half axis $[a, b]$ they are written as

$$(I^\alpha_{0+}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\varphi(t)dt, 0 < x < \infty, \tag{3.11}$$

while for the whole axis [7, p. 94]

$$(I^\alpha_{+}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-t)^{\alpha-1}\varphi(t)dt, -\infty < x < \infty \tag{3.12}$$

and

$$(I^\alpha_{-}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}\varphi(t)dt, -\infty < x < \infty. \tag{3.13}$$

The convolution of functions *t* and φ , appearing in (3.12) and (3.13), is stated by

$$(I^\alpha_{\pm}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^\infty t_\pm^{\alpha-1}\varphi(x-t)dt$$

i. e.

$$= \frac{1}{\Gamma(\alpha)}\int_0^t t^{\alpha-1}\varphi(x-t)dt, \tag{3.14}$$

where

$$t_+^{\alpha-1} = \begin{cases} t^{\alpha-1}, t > 0 \\ 0, t < 0 \end{cases} \tag{3.15}$$

and

$$t_-^{\alpha-1} = \begin{cases} 0, t > 0 \\ |t|^{\alpha-1}, t < 0 \end{cases} . \tag{3.16}$$

The fractional integrals and derivatives of generalized functions are defined as [7, p. 146]. The *first* is based on *convolution* as

$$\frac{1}{\Gamma(\alpha)}(x_{\pm})^{\alpha-1} * f \tag{3.17}$$

of the function $\frac{1}{\Gamma(\alpha)}x_{\pm}^{\alpha-1}$, with the generalized function f , whereas the *second* is based on the use *of adjoint operatotrs*. By employing fractional integration by parts, (3.9) and (3.10) are written as

$$(I^{\alpha}_{a+}f)(\varphi) = (f, I^{\alpha}_{b-})(\varphi). \tag{3.18}$$

The function f , involving in (3.18), may indeed, be defined as the generalized function if I^{α}_{b-} continuously maps the space of test functions X into itself. When f and $I^{\alpha}_{a+}(f)$ are considered to be generalized functions on different test function spaces X and Y such that $f \in X'$(the dual of the test function space X), $I^{\alpha}_{a+}(f) \in Y'$ (the dual of test function space Y), then I^{α}_{b-} must map continuously Y into X. The fractional integral operator $I^{\alpha}_{\pm}$ of a generalized function $f \in \Phi'$ (the dual of Φ) is given by

$$(I^{\alpha}_{\pm}f, \varphi) = (f, I^{\alpha}_{\mp}\varphi), \varphi \in \Phi \tag{3.19}$$

whereas the operator $I^{\alpha}_{\pm}$ for the Laplace transform for ceratin generalized function $f \in X'$, is expressed as

$$(I^{\alpha}_{0+}f, \varphi) = (I^{\alpha}_{+}f, \varphi) = \left(\frac{1}{\Gamma(\alpha)}x_{\pm}^{\alpha-1} * f, \varphi\right), \tag{3.20}$$

i.e.

$$(f, I^{\alpha}_{\mp}\varphi) = (f, \mathcal{L}(I^{\alpha}_{\mp}\varphi)) = (f, (x^{-p}\mathcal{L}\varphi)), \tag{3.21}$$

where the fractional integral operator $(I^{\alpha}_{0}\varphi)(x)$, $Re(\alpha) > 0$, is the Laplace convolution cf. [7, p. 140]:

$$(I^{\alpha}_{0}\varphi)(x) = \left[\varphi(x) * \frac{x_{+}^{\alpha-1}}{\Gamma(\alpha)}\right], \varphi \in L_1(a,b), Re(\alpha) > 0, \tag{3.22}$$

and

$$(\mathcal{L}I^{\alpha}_{0+}\varphi_n)(p) = p^{-\alpha}(\mathcal{L}\varphi_n)(p), \varphi \in L_1(a,b). \tag{3.23}$$

Similarly, Stieltjes transform of fractional integral $(I^{\alpha}_{0}\varphi)$(which is given by (3.8)) may indeed, possess following expression for f_n

$$(\mathcal{G}_p I^{\alpha}_{0+}f_n)(s) = \frac{\Gamma(p-\alpha)}{\Gamma(p)}\mathcal{G}_{p-\alpha}[f_n](s). \tag{3.24}$$

In what follows is the investigation of (3.24) for integrable Boehmians.

Theorem 3.1. *If* $[f_n/\delta_n] \in B_{L_1}$, *then the sequence*

$$\mathcal{G}_p(I^{\alpha}_{0+}f_n)(s) = \frac{\Gamma(p-\alpha)}{\Gamma(p)}\mathcal{G}_{p-\alpha}[f_n](s), \tag{3.25}$$

converges uniformly on each compact set in $\mathbf{R}$*.*

Proof. If (δ_n) is a delta sequence, then $(\mathcal{G}_p\delta_n)$ converges uniformly on each compact set to the constant function 1. Therefore, for each compact set *K*, $\mathcal{G}_p(\delta_k) > 0$ on *K*. For almost all $k \in K$,, owing to the left hand side of (3.25) and by using (3.19), we have

$$\begin{aligned}
\mathcal{G}_p(I^{\alpha}_{0+}f_n) &= \mathcal{G}_p(I^{\alpha}_{0+}f_n)\frac{\mathcal{G}_p(\delta_k)}{\mathcal{G}_p(\delta_k)} \\
&= \frac{\mathcal{G}_p(I^{\alpha}_{0+}f_n * \delta_k)}{\mathcal{G}_p(\delta_k)} = \frac{\mathcal{G}_p(I^{\alpha}_{0+}f_k * \delta_n)}{\mathcal{G}_p(\delta_k)} \\
&= \frac{\mathcal{G}_p(I^{\alpha}_{0+}f_k)\mathcal{G}_p(\delta_n)}{\mathcal{G}_p(\delta_k)}, \text{ on } K \\
&= \frac{\Gamma(p-\alpha)}{\Gamma(p)}\mathcal{G}_{p-\alpha}(f_n)\frac{\mathcal{G}_p(\delta_n)}{\mathcal{G}_p(\delta_k)}, \text{[cf. Eqn. (3.25)]}.
\end{aligned}$$

This shows that the Stieltjes transform of fractional integral for an integrable Boehmian $F = [f_n/\delta_n]$ can be defined as the limit of $(\mathcal{G}_p I^{\alpha}_{0+} f_n)$, which is the space of continuous functions on **R.** Hence, in the process we have proved that the Stieltjes transform of fractional integral for an integrable Boehmian is a continuous function, and, thereby, the theorem is proved. ∎

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