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ISOPERIMETRIC INEQUALITIES FOR DUAL HARMONIC QUERMASSINTEGRALS

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ABSTRACT. In this paper, some isoperimetric inequalities for the dual harmonic quermassintegrals are established.

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1. INTRODUCTION

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) and $\mathcal{K}_o^n$ denote the subset of $\mathcal{K}^n$ that consists of convex bodies with the origin in their interiors. Denote by $\mathrm{vol}_i(K|\xi)$ the i-dimensional volume of the orthogonal projection of K onto an i-dimensional subspace $\xi \subset \mathbb{R}^n$. The important geometric invariants related to the projection of convex body K are the quermassintegrals defined by

$$W_{n-i}(K) = k_n \int_{G(n,i)} \frac{\mathrm{vol}_i(K|\xi)}{k_i}\, d\mu_i(\xi),\ 0 \le i \le n, \tag{1.1}$$

where the Grassmann manifold $G(n,i)$ is endowed with the normalized Haar measure, and k_n is the volume of the unit ball B_n in $\mathbb{R}^n$. The quermassintegrals are generalizations of the surface area and the volume. Indeed, $nW_1(K)$ is the surface area of K, and $W_0(K)$ is the volume of K.

The dual quermassintegrals of a star body L, $\widetilde{W}_i(L)$, were introduced by Lutwak [5], which are defined by letting $\widetilde{W}_0(L) = V(L)$, $\widetilde{W}_n(L) = k_n$ and for $0 < i < n$,

$$\widetilde{W}_{n-i}(L) = k_n \int_{G(n,i)} \frac{\mathrm{vol}_i(L \cap \xi)}{k_i} d\mu_i(\xi), \tag{1.2}$$

where $\mathrm{vol}_i(L \cap \xi)$ denotes the i-dimensional volume of intersection of L with an i-dimensional subspace $\xi \subset \mathbb{R}^n$.

Also associated with a convex body K are its harmonic quermassintegrals. These quermassintegrals were introduced by Hadwiger ([3], sect.6.4.8), and can be defined by letting $\hat{W}_0(K) = V(K),\ \hat{W}_n(K) = k_n$, and for $0 < i < n$,

$$\hat{W}_{n-i}(K) = k_n \left(\int_{G(n,i)} \left[\frac{\mathrm{vol}_i(K|\xi)}{k_i} \right]^{-1} d\mu_i(\xi) \right)^{-1}. \tag{1.3}$$

Following Hadwiger, in [8], we introduced the dual harmonic quermassintegrals of a star body L, $\breve{W}_{n-i}(L)$, which can be defined by letting $\breve{W}_0(L) = V(L)$, $\breve{W}_n(L) = k_n$, and for $0 < i < n$,

$$\breve{W}_{n-i}(L) = k_n \left(\int_{G(n,i)} \left[\frac{\mathrm{vol}_i(L \cap \xi)}{k_i} \right]^{-1} d\mu_i(\xi) \right)^{-1}. \tag{1.4}$$

And the Brunn-Minkowski inequality and the Blaschke-Santaló inequality for the dual harmonic quermassintegrals were established in [8].

Let $S(K)$ denote the surface area of a convex body K. The classical isoperimetric inequality [6] states that: if $K \in \mathcal{K}^n$, then

$$\left(\frac{S(K)}{S(B_n)} \right)^n \ge \left(\frac{V(K)}{V(B_n)} \right)^{n-1},$$

with equality if and only if K is a ball. That is to say among convex bodies of given volume, precisely the ball has the minimal surface area.

The aim of this paper is to study the dual harmonic quermassintegrals further. We prove that among convex bodies of given volume, precisely the ball attains the maximal value of $\breve{W}_i(K)$.

Theorem 1.1. *Let $K \in \mathcal{K}_o^n$ and $V(K) = V(B_n)$. If $1 \le i \le n-1$, then*

$$\breve{W}_i(K) \le \breve{W}_i(B_n).$$

with equality if and only if $K = B_n$*.*

At the same time, by applying the well-known theorem of John, we establish the reverse of the isoperimetric inequality for the dual harmonic quermassintegrals.

Theorem 1.2. *Let* K *be a symmetric convex body in* $\mathbb{R}^n$ *and* $1 \le i \le n-1$*. Then there exists an affine image* $\widetilde{K}$ *of* K*, such that*

$$\breve{W}_i(\widetilde{K}) \ge n^{-\frac{n-i}{2}} \breve{W}_i(B_n).$$

Let K be a convex body of constant width and K^* be the polar body of K. The other aim of this paper is to prove that among convex bodies of constant width, precisely the ball attains the minimal value of $\breve{W}_{n-1}(K^*)$.

Theorem 1.3. *Let* $K \in \mathcal{K}_o^n$ *and* $\xi \in G(n,1)$*. If*

$$vol_1(K|\xi) = vol_1(B_n|\xi), \tag{1.5}$$

then

$$\breve{W}_{n-1}(K^*) \ge \breve{W}_{n-1}(B_n^*),$$

with equality if and only if $K = B_n$*.*

For quick reference we recall some basic results from the Brunn-Minkowski theory. Good references are Gardner [2] and Schneider [7].

Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$. If K is a convex body in $\mathbb{R}^n$, then its support function h_K is defined by

$$h_K(u) = max\{\langle u, x\rangle : x \in K\},\ u \in S^{n-1},$$

where $\langle u, x\rangle$ denotes the usual inner product of u and x in $\mathbb{R}^n$.

If K is a convex body that contains the origin in its interior, the polar body K^* of K, with respect to the origin, is define by

$$K^* = \{x \in \mathbb{R}^n | x \cdot y \le 1, y \in K\}.$$

For a compact subset L of $\mathbb{R}^n$, which is star-shaped with respect to the origin, we shall use $\rho(L, \cdot)$ to denote its radial function; i.e., for $u \in S^{n-1}$,

$$\rho(L, u) = max\{\lambda > 0 : \lambda u \in L\}.$$

If $\rho(L, \cdot)$ is continuous and positive, L will be called a star body.

If $K \in \mathcal{K}_o^n$, then ([2], p.44)

$$\rho(K, \cdot) = \frac{1}{h(K^*, \cdot)}. \tag{1.6}$$

Let K be a convex body in $\mathbb{R}^n$. If $\mathrm{vol}_1(L|\xi)$ has the same value for each $\xi \in G(n,1)$, we say K is of constant width.

If L is a star body in $\mathbb{R}^n$ such that for some i with $1 \le i \le n-1$, $\mathrm{vol}_i(L \cap \xi)$ has the same value for each $\xi \in G(n,i)$, we say L is of constant i-section.

2. PROOFS OF THEOREMS

Now we give the proofs of theorems.

Proof of Theorem 1.1. From (1.2), (1.4) and Hölder inequality, we have

$$\breve{W}_i(L) \le \widetilde{W}_i(L),\ 1 \le i \le n-1, \tag{2.1}$$

with equality if and only if L is of constant $(n-i)$-section.

Let $K \in \mathcal{K}_o^n$ and $1 \leq i \leq n-1$. Then the classical inequality between the volume of K and its dual quermassintegral is

$$\widetilde{W}_i(K) \leq k_n^{i/n} V(K)^{n-i/n}, \tag{2.2}$$

with equality if and only if K is a ball centered at the origin.

Consider $K \in \mathcal{K}_0^n$ such that $V(K) = V(B_n)$. Then from (2.1) and (2.2) we deduce:

$$\breve{W}_i(K) \leq \widetilde{W}_i(K) \leq k_n^{i/n} V(K)^{n-i/n} = k_n^{i/n} V(B_n)^{n-i/n} = \breve{W}_i(B_n),$$

with equality if and only if $K = B_n$. ∎

To prove Theorem 1.2, we shall use the well-known theorem of John, which characterizes ellipsoids of minimal volume containing convex bodies.

Lemma 2.1. *(John [4])Let K be a symmetric convex body in $\mathbb{R}^n$. The ellipsoid of minimal volume containing K is the unit ball B_n, if and only if K is contained in B_n and there is a sequence $\{u_i\}_1^m$ on the boundary K and a sequence $\{c_i\}_1^m$ of positive numbers satisfying*

$$\sum_{i=1}^m c_i u_i \otimes u_i = I_n, \tag{2.3}$$

where $u_i \otimes u_i$ is the rank-one orthogonal projection onto the span of u_i and I_n is the identity on $\mathbb{R}^n$.

The condition (2.3) shows that the u_i behave like an orthonormal basis to the extent that, for each $x \in \mathbb{R}^n$,

$$\|x\|^2 = \sum_{i=1}^m c_i \langle u_i, x\rangle^2. \tag{2.4}$$

The equality of the traces in (2.4) shows that

$$\sum_{i=1}^m c_i = n. \tag{2.5}$$

Obviously, for every convex body in $\mathbb{R}^n$, there is an affine transformation ϕ such that the minimal (in volume) ellipsoid containing ϕK is the unit ball in $\mathbb{R}^n$.

Lemma 2.2. *If K is a symmetric convex body in $\mathbb{R}^n$, then there exists an affine image $\widetilde{K}$ of K, such that for every i-dimensional subspace ξ of $\mathbb{R}^n$ $(1 \leq i \leq n-1)$,*

$$\mathit{vol}_i(\widetilde{K} \cap \xi) \geq \frac{1}{n^{i/2}}\, \mathit{vol}_i(B_n \cap \xi). \tag{2.6}$$

Proof. Let $\widetilde{K}$ be an affine image of K so that the minimal ellipsoid containing $\tilde{K}$ is the unit ball in $\mathbb{R}^n$. From Lemma 2.1, there are a sequence unit vectors $\{u_i\}_1^m$ on the boundary of $\tilde{K}$ and a sequence positive numbers $\{c_i\}_1^m$ such that

$$\sum_{i=1}^m c_i u_i \otimes u_i = I_n. \tag{2.7}$$

By symmetry, $\widetilde{K}$ contains the symmetric convex hull S of vectors $u_1, \cdots, u_m$. Therefore, for all x ([1])

$$\rho_{\widetilde{K}}(x) \geq \rho_S(x) = \frac{1}{\inf\{\sum_{i=1}^m |a_i|; x = \sum_{i=1}^m a_i u_i\}}. \tag{2.8}$$

From (2.7), one obtains

$$x=\sum_{i=1}^{m}c_i\langle x,u_i\rangle u_i,$$

for every $x\in\mathbb{R}^n$. So

$$\rho_S(u)\geq\frac{1}{\sum\limits_{i=1}^{m}c_i|\langle u,u_i\rangle|}. \tag{2.9}$$

Applying Cauchy inequality, by (2.4) and (2.5), we have

$$\frac{1}{\sum\limits_{i=1}^{m}c_i|\langle u,u_i\rangle|}\geq\frac{1}{\Big(\sum\limits_{i=1}^{m}c_i\Big)^{1/2}\Big(\sum\limits_{i=1}^{m}c_i\langle u,u_i\rangle^2\Big)^{1/2}}=\frac{1}{n^{1/2}\|u\|^{1/2}}=\frac{1}{n^{1/2}}. \tag{2.10}$$

By (2.8), (2.9) and (2.10), we obtain

$$\rho_{\widetilde{K}}(u)\geq\frac{1}{n^{1/2}}. \tag{2.11}$$

Noticing the obvious fact $\rho_{B_n}(u)=1$, from (2.11), for $\xi\in G(n,i)$, we have

$$\int_{S^{n-1}\cap\xi}\rho^i_{\widetilde{K}}(u)\,d(u)\geq\frac{1}{n^{i/2}}\int_{S^{n-1}\cap\xi}\rho^i_{B_n}(u)\,d(u). \tag{2.12}$$

On the other hand, for $K\in\mathbb{R}^n$, from the polar coordinate formula for volume we have

$$\int_{S^{n-1}\cap\xi}\rho^i_K(u)\,d(u)=i\mathrm{vol}_i(K\cap\xi).$$

Thus (2.12) yields

$$\mathrm{vol}_i(\widetilde{K}\cap\xi)\geq\frac{1}{n^{i/2}}\,\mathrm{vol}_i(B_n\cap\xi).$$

The proof of lemma 2.2 is completed. ∎

The proof of Theorem 1.2. According to (1.4) and Lemma 2.2, we have

$$\begin{aligned}\breve{W}_i(\widetilde{K})&=k_n\left(\int_{G(n,n-i)}\left[\frac{V_{n-i}(\widetilde{K}\cap\xi)}{k_{n-i}}\right]^{-1}d\mu_{n-i}(\xi)\right)^{-1}\\&\geq n^{-\frac{n-i}{2}}k_n\left(\int_{G(n,n-i)}\left[\frac{V_{n-i}(B_n\cap\xi)}{k_{n-i}}\right]^{-1}d\mu_{n-i}(\xi)\right)^{-1}\\&=n^{-\frac{n-i}{2}}\breve{W}_i(B_n).\end{aligned}$$

∎

Proof of Theorem 1.3. For any $u\in S^{n-1}$, (1.5) is equivalent to

$$h(K,u)+h(K,-u)=2.$$

According to (1.6), the chord length of K^* in direction u satisfies

$$\rho(K^*,u)+\rho(K^*,-u)\geq\frac{4}{h(K,u)+h(K,-u)}=2, \tag{2.13}$$

where we have used the inequality between arthmetic and harmonic means.

Notice that if $\xi\in G(n,1)$, then $\mathrm{vol}_1(K^*\cap\xi)$ is just the chord length of K^* along ξ. Associated with (1.4) and (2.13), we have

$$\begin{aligned}\breve{W}_{n-1}(K^*) &= k_n\left(\int_{G(n,1)}\left[\frac{\mathrm{vol}_1(K^*\cap\xi)}{2}\right]^{-1}d\mu_1(\xi)\right)^{-1}\\ &= k_n\left(\frac{1}{nk_n}\int_{S^{n-1}}\left[\frac{\rho(K^*,u)+\rho(K^*,-u)}{2}\right]^{-1}du\right)^{-1}\\ &\geq k_n = \breve{W}_{n-1}(B_n^*).\end{aligned}$$

Equality holds if and only if $h(K,u) = h(K,-u) = 1$, which implies K is the unit ball B_n. ∎

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