The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 5, Issue 1, Article 4, pp. 1-7, 2008

THE INVARIANT SUBSPACE PROBLEM FOR LINEAR RELATIONS ON HILBERT SPACES

DANIEL GRIXTI-CHENG

Received 16 October, 2007; accepted 13 December, 2007; published 25 March, 2008.

DEPARTMENT OF MATHEMATICS AND STATISTICS, THE UNIVERSITY OF MELBOURNE, MELBOURNE, VIC, 3010, AUSTRALIA
D.Grixti@ms.unimelb.edu.au

ABSTRACT. We consider the invariant subspace problem for linear relations on Hilbert spaces with the aim of promoting interest in the problem as viewed from the theory of linear relations. We present an equivalence between the single valued and multivalued invariant subspace problems and give some new theorems pertaining to the invariant subspace problem for linear relations on a Hilbert space.

Key words and phrases: Linear relations, Multivalued linear operators, Invariant subspace problem, Invariance, Weak-invariance.

2000 *Mathematics Subject Classification.* Primary 47A06, 47A15.

ISSN (electronic): 1449-5910

1. INTRODUCTION

In the single valued theory, the invariant subspace problem is posed as the simple question: 'Does every $T \in \mathcal{B}(X)$ have a nontrivial invariant subspace?' Although this problem is simply stated, it has proved to be extraordinarily rich and difficult to resolve. In 1976, it was announced by P. Enflo that he had obtained a counterexample showing that there exists a bounded linear operator on a Banach space X for which there is no nontrivial invariant subspace. This counterexample was eventually published in 1987 in [2]. Following Enflo's insight, C. J. Read constructed some shorter and easier to follow counterexamples published (curiously before Enflo's paper) in [6], [7] and [8]. These results of Enflo and Read constitute solutions to the invariant subspace problem on Banach spaces in the negative (a negative result to the invariant subspace problem in the single valued theory implies a negative result to the invariant subspace problem in the multivalued theory, given that an operator is also a linear relation). However, the problem remains open on Hilbert spaces.

In the theory of linear relations, the invariant subspace problem translates to the question: 'Does every $T \in \mathcal{B}LR(X)$ have a nontrivial T-weakly-invariant subspace?' In this paper, we will concern ourselves with the invariant subspace problem for bounded linear relations on Hilbert spaces.

2. PRELIMINARIES

We use notation and terminology consistent with that used in [1]. Much of the background to this paper is also to be found in [4]. All spaces X, Y etc. will be assumed to be Hilbert spaces unless otherwise stated. Following [1], a *relation* T from a set U to a set V is any mapping having domain $D(T)$, a nonempty subset of U, and taking values in $2^V \backslash \emptyset$ (the collection of nonempty subsets of V). If U and V are vector spaces over the field $\mathbb{K}$ then T is said to be a *linear relation* if for all $x, z \in D(T)$ and nonzero scalars α we have

$$Tx + Tz = T(x+z)$$
$$\alpha Tx = T(\alpha x).$$

The set of all linear relations from X to Y will be denoted by $LR(X,Y)$. As a shorthand we write $LR(X,X) = LR(X)$.

We call T a *bounded linear relation* if $T \in LR(X,Y)$, $D(T) = X$ and $\|T\| < \infty$. We will denote the set of all bounded linear relations from X to Y by $\mathcal{B}LR(X,Y)$. As a shorthand, we define $\mathcal{B}LR(X,X) = \mathcal{B}LR(X)$. We further denote by $\mathcal{B}LR_0(X)$, the set $\{T \in \mathcal{B}LR(X) : \dim T(0) < \infty\}$.

A relation T is called *closed* if its graph $G(T)$ is closed in $X \times Y$. A relation T is called *open* if for every neighbourhood $U \subseteq D(T)$, the image $T(U)$ is a neighbourhood in $R(T)$ (the range of T).

A single valued linear operator A is called a *linear selection* of T if $T = A + T - T$ and $D(A) = D(T)$.

The quotient mapping $Q_T : X \longrightarrow X/\overline{T(0)}$ is defined by

$$Q_T x = x + \overline{T(0)}$$

where the image of x is a *coset* in $X/\overline{T(0)}$.

Definition 2.1. Let $T \in LR(X)$. We define $\Phi_T : X/\overline{T(0)} \longrightarrow X$ by

$$\Phi_T\left(x + \overline{T(0)}\right) = \Phi_T\left(u + v + \overline{T(0)}\right) = v$$

where $u \in \overline{T(0)}$ and $v \in T(0)^{\perp}$ such that $x = u + v$.

We have the following theorem [4, Theorem 2.3].

Theorem 2.1. *Let $T \in LR(X)$. Then $\Phi_T Q_T T : X \longrightarrow X$ is a single valued linear operator. Moreover, if T is a bounded linear relation, then $\Phi_T Q_T T$ is a single valued bounded linear operator on X.*

Remark 2.1. The operator $\Phi_T Q_T T$ will be an important tool for us in linking single valued invariant subspace problem with the multivalued invariant subspace problem.

Turning more specifically to invariant subspaces, we have the following definition.

Definition 2.2. [9, Section 1] For $T \in LR(X)$, a closed linear subspace M of X, where $\{0\} \neq M \neq X$, is said to be T-*weakly-invariant* if, for all $x \in M$, $Tx \cap M \neq \emptyset$.

Remark 2.2. We draw attention to the fact that when T is single valued in Definition 2.2, T-weakly-invariant is identical to T-invariant. That is to say, for single valued linear operators, T-weak-invariance is the same as T-invariance.

3. THE STATE OF THE PROBLEM

Exploring the invariant subspace problem in the theory of linear relations serves a twofold purpose. The first is for it own sake and intrinsic value; the second is that such an exploration may shed light on the long standing invariant subspace problem in the single valued theory. For both purposes, it is important to study the nexus between the single valued problem and the multivalued. We have already seen some results related to the invariant subspace problem for linear relations in [4, Theorem 6.6], [4, Theorem 6.7], [4, Theorem 6.8], and also in [9, Theorem 6.4]. These results touch the nexus between the invariant subspace problem for linear relations and the invariant subspace problem for the single valued theory to varying degrees. In this section we continue to explore the connection between the single valued and multivalued invariant subspace problems in order to establish an equivalence between the invariant subspace problem for linear relations and the single valued invariant subspace problem. The following begins this exploration with the use of the quotient mapping.

Lemma 3.1. [4, Lemma 6.4] *Let X be a Hilbert space and let $T \in \mathcal{B}LR(X)$ be such that $T(0)$ is closed. If $\Phi_T Q_T T$ has an invariant subspace M, then M is a T-weakly-invariant subspace of X.*

Theorem 3.2. *Let X be a Hilbert space and let $T \in \mathcal{B}LR(X)$ be such that $T(0)$ is closed. A closed linear subspace M of X such that $M \subseteq T(0)^{\perp}$ is an invariant subspace for $\Phi_T Q_T T$ if and only if M is T-weakly-invariant.*

Proof. By Lemma 3.1, if $\Phi_T Q_T T$ has an invariant subspace M, then M is T-weakly-invariant.

For the converse, if M is T-weakly-invariant then for each $x \in M$,

$$Tx \cap M \neq \emptyset.$$

Now, $Tx = y + T(0)$ for each $y \in Tx$. But $X = T(0) \oplus T(0)^{\perp}$ (as $T(0)$ is closed). Hence $y = u + v$, where $u \in T(0)$ and $v \in T(0)^{\perp}$. Therefore, $Tx = v + T(0)$ and $\Phi_T Q_T Tx = v$. Now, as $M \subseteq T(0)^{\perp}$, we have that

$$Tx \cap M = (v + T(0)) \cap M \subseteq (v + T(0)) \cap T(0)^{\perp} \neq \emptyset.$$

But, if $x \in (v + T(0)) \cap T(0)^{\perp}$ then $x \in v + T(0)$ and $x \in T(0)^{\perp}$. Hence it follows that $x = v$. Therefore we have that

$$(v + T(0)) \cap M = v.$$

Thus, $\Phi_T Q_T Tx = v \in M$ for each $x \in M$. So, M is an invariant subspace for $\Phi_T Q_T T$. ∎

In Theorem 3.2, the limitations $T(0)$ closed and $M \subseteq T(0)^{\perp}$ are too much for us to obtain an equivalence between the single valued and multivalued invariant subspace problems. But, we notice that for $T(0)$ closed, the mapping $\Phi_T Q_T T$ is a linear selection of T. So, we look more generally at linear selections in order to gain greater insight with regard to the connection between the single valued and multivalued invariant subspace problems. To this end, we have the following theorem.

Theorem 3.3. *M is a T-weakly-invariant subspace if any only if there exists a linear selection A of T such that M is an invariant subspace for A.*

Proof. Let A be a linear selection of T such that M is an invariant subspace for A. Then,

$$Tx = Ax + T(0)$$

for each $x \in X$. Now, as M is an invariant subspace for A we have that $Am \in M$ for each $m \in M$. Therefore, for each $m \in M$,

$$Tm \cap M = (Am + T(0)) \cap M \neq \emptyset$$

as $0 \in T(0)$. Thus, if A is a linear selection of T such that M is an invariant subspace for A, then M is T-weakly-invariant.

Conversely, If M is a T-weakly-invariant subspace, then for each $x \in M$,

$$Tx \cap M \neq \emptyset.$$

Thus, for each $x \in M$, $Tx = m + T(0)$ for some $m \in M$ (by [1, Proposition I.2.8]). Now, by [1, Proposition I.5.2], any single valued linear projection P with domain $R(T)$ and kernel $T(0)$ determines a linear selection A of T given by

$$A = PT.$$

Therefore, for any such P we have that

$$PTx = P(m + T(0)) = Pm$$

for each $x \in M$ and some $m \in M$. Now, as M is a subspace, we can choose P such that $P(M \cap R(T)) \subseteq M$. So, making such a choice we find that if M is a T-weakly-invariant subspace, then there exists a linear selection $A = PT$ of T such that M is an invariant subspace for A. ∎

Remark 3.1. We remark that Theorem 3.3 is algebraic in its approach. The existence of a projection P with domain $R(T)$ and kernel $T(0)$ is guaranteed as $T(0)$ is always algebraically complemented in $R(T)$. The existence of a choice of such P so that $P(M \cap R(T)) \subseteq M$ is obvious as $M \cap R(T)$ is a subspace of $R(T)$.

From here, we are able to obtain some equivalence between the single valued and multivalued invariant subspace problems.

Theorem 3.4. *Let X be a Hilbert space. Every $A \in \mathcal{B}(X)$ has a nontrivial invariant subspace M if and only if every $T \in \mathcal{B}LR(X)$ such that $T(0)$ is closed has a nontrivial T-weakly-invariant subspace N.*

Proof. Let us first suppose that every $A \in \mathcal{B}(X)$ has a nontrivial invariant subspace M. We have, from [1, Corollary II.4.6], that every $T \in \mathcal{B}LR(X)$ such that $T(0)$ is closed has a continuous linear selection. Let T be any element of $\mathcal{B}LR(X)$ such that $T(0)$ is closed. Then T has a continuous linear selection A. Now, by assumption, A has a nontrivial invariant subspace M. But, it then follows from Theorem 3.3 that M is a nontrivial T-weakly-invariant subspace. Therefore, if every $A \in \mathcal{B}(X)$ has a nontrivial invariant subspace M, then every $T \in \mathcal{B}LR(X)$

such that $T(0)$ is closed has a nontrivial T-weakly-invariant subspace N (corresponding to a nontrivial invariant subspace associated with a continuous linear selection of T).

Conversely, if every $T \in \mathcal{BLR}(X)$ such that $T(0)$ is closed has a nontrivial T-weakly-invariant subspace N, then as $\mathcal{B}(X) \subset \mathcal{BLR}(X)$, it immediately follows that every $A \in \mathcal{B}(X)$ has a nontrivial A-weakly-invariant subspace N. Thus, for each $x \in N$,

$$Ax \cap N \neq \emptyset.$$

But A is single valued and so it follows that for each $x \in N$, $Ax \in N$. Thus, N is an invariant subspace for A. Therefore, if every $T \in \mathcal{BLR}(X)$ such that $T(0)$ is closed has a nontrivial T-weakly-invariant subspace N, then every $A \in \mathcal{B}(X)$ has a nontrivial invariant subspace $M = N$. ∎

Theorem 3.4 is a significant statement as it provides a rather important link between the single valued and multivalued invariant subspace problems. It is a consequence of Theorem 3.4 that the single valued invariant subspace problem can now be solved by using linear relations – whether this solution comes about by taking a counter-example from $\mathcal{BLR}(X)$ (a much larger set than $\mathcal{B}(X)$ and so more likely to yield such a counter-example) or by treating the problem positively and dealing with weakly-invariant subspaces. This latter approach may prove to be easier to deal with than the exclusively single valued approach of the past as weak-invariance is a simpler object than invariance.

We are able to restrict the set of all $T \in \mathcal{BLR}(X)$ such that $T(0)$ is closed in order to obtain less general (possibly more useful) equivalences between the single valued and multivalued invariant subspace problems. We can do this provided that $\mathcal{B}(X)$ remains a subset of the new set in $\mathcal{BLR}(X)$. We provide one such restriction in the following theorem.

Theorem 3.5. *Let X be a Hilbert space. Every $A \in \mathcal{B}(X)$ has a nontrivial invariant subspace M if and only if every $T \in \mathcal{BLR}_0(X)$ has a nontrivial T-weakly-invariant subspace N.*

Proof. As $T \in \mathcal{BLR}_0(X)$, we have that $T(0)$ is closed. Hence, it follows from Theorem 3.4 that if every $A \in \mathcal{B}(X)$ has a nontrivial invariant subspace M, then every $T \in \mathcal{BLR}_0(X)$ has a nontrivial T-weakly-invariant subspace N.

Conversely, if every $T \in \mathcal{BLR}_0(X)$ has a nontrivial T-weakly-invariant subspace N, then as $\mathcal{B}(X) \subset \mathcal{BLR}_0(X)$, it immediately follows that every $A \in \mathcal{B}(X)$ has a nontrivial A-weakly-invariant subspace N. Thus, for each $x \in N$,

$$Ax \cap N \neq \emptyset.$$

But A is single valued and so it follows that for each $x \in N$, $Ax \in N$. Thus, N is an invariant subspace for A. Therefore, if every $T \in \mathcal{BLR}_0(X)$ has a nontrivial T-weakly-invariant subspace N, then every $A \in \mathcal{B}(X)$ has a nontrivial invariant subspace $M = N$. ∎

We note here, that the equivalences obtained in Theorems 3.4 and 3.5 do not cover the whole of the invariant subspace problem in the theory of linear relations. The problem, in the multivalued case, takes on its own character and interest when we consider relations T such that $T(0)$ is not closed.

4. THE MULTIVALUED INVARIANT SUBSPACE PROBLEM

In the previous section, we began our study of the invariant subspace problem for linear relations and have already seen some significant links between this problem and the single valued invariant subspace problem. In this section, our focus is to draw out some of the richness of the multivalued invariant subspace problem as distinct from the single valued invariant subspace problem.

We begin with the following lemma.

Lemma 4.1. *Let $T \in \mathcal{B}LR(X)$. If there exists a linear selection A of T such that $\{0\} \neq \overline{R(A)} \neq X$, then $M = \overline{R(A)}$ is a nontrivial T-weakly-invariant subspace.*

Proof. It follows from the axiom of choice that every $T \in \mathcal{B}LR(X)$ has a linear selection (refer to [5, p.113] for a complete discussion of this fact). Now, if there exists a linear selection A of T such that $\{0\} \neq \overline{R(A)} \neq X$, then it immediately follows that $M = \overline{R(A)}$ is an invariant subspace for A. Therefore, by Theorem 3.3, M is T-weakly-invariant. ∎

Under the impetus of Lemma 4.1, we obtain the following theorem.

Theorem 4.2. *Let X be a Hilbert space. If $T \in \mathcal{B}LR(X)$ is both a closed and open linear relation such that $T(0) \neq \{0\}$, then there exists a nontrivial T-weakly-invariant subspace M.*

Proof. As $T \in \mathcal{B}LR(X)$ is both open and closed, we have that $R(T)$ is closed in X ([1, Exercise II.5.15]). Hence, as X is a Hilbert space, we have that $R(T)$ is also a Hilbert space. Furthermore, by [1, Definition II.5.1(5)], we have that $T(0)$ is closed in X and so closed in $R(T)$. Thus it follows, that $T(0)$ is topologically complemented in $R(T)$. Therefore, there exists a single valued linear projection $P : R(T) \longrightarrow R(T)$ with kernel $T(0)$. So, by [1, Proposition I.5.2] we have that $A = PT$ is a linear selection of T, and so $T = A + T - T$. Hence,

$$R(T) = R(A + T - T) = R(A) + T(0).$$

But $R(A) = R(P)$ and so it follows that $R(A) \cap T(0) = \{0\}$. Therefore,

$$R(T) = R(A) \oplus T(0).$$

Now, as $R(T)$ is closed in X, we have that $R(A) \oplus T(0)$ is closed in X. Furthermore, as $T(0)$ is closed in X, it follows from [3, Theorem XI.2.2] that $R(A)$ is closed in X.

So, we have shown that if $T \in \mathcal{B}LR(X)$ is both a closed and open linear relation, then T has a linear selection A such that $R(A)$ is closed in X. To conclude the proof, there are three cases to consider:

(1) If $0 \neq R(A) \neq X$ then, by Lemma 4.1, there exists a nontrivial T-weakly-invariant subspace $M = R(A)$.
(2) If $R(A) = \{0\}$, then it follows that $Tx = T(0)$ for each $x \in X$. So, if $T(0) \neq X$ then $T(0)$ is a nontrivial T-weakly-invariant subspace. On the other hand, if $T(0) = X$ then $Tx = X$ for each $x \in X$ and so any closed proper subspace of X is a T-weakly-invariant subspace (it is clear that any X of dimension greater than 1 – i.e. any space under consideration in the invariant subspace problem – has at least one closed proper subspace in the form of some finite dimensional subspace of dimension less than the dimension of X). Therefore, if $R(A) = \{0\}$ then there exists a nontrivial T-weakly-invariant subspace M.
(3) Finally, if $R(A) = X$, then it is immediate that $T(0) = \{0\}$. But, by assumption, $T(0) \neq \{0\}$ and so this case is not applicable to the theorem.

∎

From Lemma 4.1, we also obtain the following proposition.

Proposition 4.3. *Let $T \in \mathcal{B}LR(X)$. If $\overline{R(T)} \neq X$, then there exists a nontrivial T-weakly-invariant subspace M.*

Proof. It follows from the axiom of choice that every $T \in \mathcal{B}LR(X)$ has a linear selection A. Therefore we have that $T = A + T - T$ for some single valued linear operator A. Hence $R(T) = R(A) + T(0)$. Therefore,

$$\overline{R(A)} \subseteq \overline{R(A) + T(0)} = \overline{R(T)}.$$

Hence, as $\overline{R(T)} \neq X$, we have that $\overline{R(A)} \neq X$. Now, if $\overline{R(T)} \neq \{0\}$ then there are two cases to consider:

(1) If $\overline{R(A)} \neq \{0\}$ then by Lemma 4.1, we have that there exists a nontrivial T-weakly-invariant subspace $M = \overline{R(A)}$.

(2) If $\overline{R(A)} = \{0\}$ then $R(A) = \{0\}$. Thus, we have that $R(T) = T(0)$. But this means that $\{0\} \neq \overline{T(0)} \neq X$ and so it immediately follows that $\overline{T(0)}$ is a nontrivial T-weakly-invariant subspace.

If, on the other hand, $\overline{R(T)} = \{0\}$ then $R(T) = \{0\}$ and so $Tx = \{0\}$ for each $x \in X$. Therefore any closed proper subspace of X is a T-weakly-invariant subspace. ∎

Proposition 4.3 simplifies the consideration needed in addressing the invariant subspace problem in both the single valued and multivalued theories. Clearly, it follows from Proposition 4.3 that we need only examine linear relations with ranges dense in X – given that all other linear relations $T \in \mathcal{B}LR(X)$ admit a T-weakly-invariant subspace.

We conclude by noting that it remains an open problem to resolve the invariant subspace problem on Hilbert spaces in either the single valued or multivalued theory.

ACKNOWLEDGEMENTS

I am grateful to my PhD supervisor, Jerry Koliha, for his guidance and support.

REFERENCES

[1] R. CROSS, *Multivalued Linear Operators*, Marcel Dekker, Pure and Applied Mathematics vol. 213, New York, 1998.

[2] P. ENFLO, On the invariant subspace problem for Banach spaces, *Acta Math.*, **158** (1987), 213–313.

[3] I. GOHBERG, S. GOLDBERG and M. A. KAASHOEK, *Classes of Linear Operators, Vol. I*, Operator Theory Advances and Applications, Vol. 49, Birkhäuser Verlag, Berlin, 1990.

[4] D. GRIXTI-CHENG, Calkin algebras for bounded linear relations on Hilbert spaces, *Submitted to Mathematica Bohemica*.

[5] S. J. LEE and M. Z. NASHED, Algebraic and topological selections of multi-valued linear relations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **17** (1990), pp. 111–126.

[6] C. J. READ, A short proof concerning the invariant subspace problem, *J. London Math. Soc.*, **34** (1986), pp. 335–348.

[7] C. J. READ, A solution to the invariant subspace problem, *Bull. London Math. Soc.*, **16** (1984), pp. 337–401.

[8] C. J. READ, A solution to the invariant subspace problem on the space ℓ_1, *Bull. London Math. Soc.*, **17** (1985), pp. 305–317.

[9] P. SAVELIEV, Lomonosov's invariant subspace theorem for multivalued linear operators, *Proc. Amer. Math. Soc.*, **131** (2003), No. 3, pp. 825–834.