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ON A SUBCLASS OF UNIFORMLY CONVEX FUNCTIONS DEFINED BY THE DZIOK-SRIVASTAVA OPERATOR

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ABSTRACT. Making use of the Dziok-Srivastava operator, we define a new subclass $T^l_m([\alpha_1]; \alpha, \beta)$ of uniformly convex function with negative coefficients. In this paper, we obtain coefficient estimates, distortion theorems, locate extreme points and obtain radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $T^l_m([\alpha_1]; \alpha, \beta)$. We consider integral operators associated with functions belonging to the class $H^l_m([\alpha_1]; \alpha, \beta)$ defined via the Dziok-Srivastava operator. We also obtain several results for the modified Hadamard products of functions belonging to the class $T^l_m([\alpha_1]; \alpha, \beta)$ and we obtain properties associated with generalized fractional calculus operators.

Key words and phrases: Dziok-Srivastava operator, Analytic, Uniformly convex, Extreme points, Modified Hadamard products, Fractional calculus.

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1. INTRODUCTION

Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

that are analytic and univalent in the open unit disc $U = \{z : |z| < 1\}$. Let $K(\alpha)$ and $S^*(\alpha)$ denote the subclasses of S that are, respectively, convex and starlike functions of order $\alpha, 0 \leq \alpha < 1$. For convenience, we write $K(0) = K$ and $S^*(0) = S^*$ (e.g., [24]). Goodman ([6] and [7]) defined the following subclasses of K and S^*.

Definition 1.1. A function $f(z)$ is uniformly convex (starlike) in U if $f(z)$ is in $K(S^*)$ and has the property that for every circular are γ contained in U, with center ζ also in U, the arc $f(\gamma)$ is convex (starlike) with respect to $f(\zeta)$.

Goodman ([6] and [7]) then gave the following two-variable analytic characterizations of these classes, denoted, respectively, by UCV and UST.

Theorem 1.1 (A). *A function $f(z)$ of the form (1.1) is in UCV if and only if*

$$\operatorname{Re}\left\{1 + (z-\zeta)\frac{f''(z)}{f'(z)}\right\} \geq 0, (z,\zeta) \in U \times U\,, \tag{1.2}$$

and is in UST if and only if

$$\operatorname{Re}\left\{\frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)}\right\} \geq 0, (z,\zeta) \in U \times U\,. \tag{1.3}$$

Ma and Minda [14] *and Ronning* [19] *independently found a more applicable one-variable characterization for UCV.*

Theorem 1.2 (B). *A function $f(z)$ of the form (1.1) is in UCV if and only if*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right| \ ,\ z \in U\ . \tag{1.4}$$

We note that [6] that the classical Alexander's result $f(z) \in K \Leftrightarrow zf^{'}(z) \in S^*$ does not hold between the classes UCV and UST. Later on, Ronning [20] introduced a new class S_p of starlike functions related to UCV defined as

$$f(z) \in S_p \Leftrightarrow \operatorname{Re}\left\{\frac{zf^{'}(z)}{f(z)}\right\} \geq \left|\frac{zf^{'}(z)}{f(z)} - 1\right|\ ,\ z \in U\,. \tag{1.5}$$

Note that

$$f(z) \in UCV \Leftrightarrow zf^{'}(z) \in S_p\ . \tag{1.6}$$

Also in [19], Ronning generalized the classes UCV and S_p by introducing a parameter α in the following way.

Definition 1.2. A function $f(z)$ of the form (1.1) is in $S_p(\alpha)$, if it satisfies the analytic characterization:

$$\operatorname{Re}\left\{\frac{zf^{'}(z)}{f(z)} - \alpha\right\} \geq \left|\frac{zf^{'}(z)}{f(z)} - 1\right|, \alpha \in R; z \in U\,, \tag{1.7}$$

and $f(z) \in UCV(\alpha)$, the class of uniformly convex functions of order α, if and only if $zf'(z) \in S_p(\alpha)$.

For the class $S_p(\alpha)$, we get a domain whose boundary is a parabola with vertex $w = \frac{1+\alpha}{2}$. Also, we note that $S_p(\alpha) \subset S^*$ for all $-1 \leq \alpha < 1, S_p(\alpha) \nsubseteq S$ for $\alpha < -1$ and $UCV(\alpha) \subset K$ for $\alpha \geq -1$.

By $\beta - UCV, 0 \leq \beta < \infty$, we denote the class of all $\beta-$ uniformly convex functions introduced by Kanas and Wisniowska [8]. Recall that a function $f(z) \in S$ is said to be β-uniformly convex in U, if the image of every circular arc contained in U with center at ζ, where $|\zeta| \leq \beta$, is convex. Note that the class $1 - UCV$ coincides with the class UCV. Moreover, for $\beta = 0$ we get the class K. From [8] it is known that $f(z) \in \beta - UCV$ if and only if it satisfies the following condition

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta \left|\frac{zf''(z)}{f'(z)}\right|, z \in U, 0 \leq \beta < \infty \, . \tag{1.8}$$

We consider the class $\beta - S^*, 0 \leq \beta < \infty$, of $\beta-$ starlike functions (see [9]) which are associated with $\beta-$ uniformly convex functions by the relation

$$f(z) \in \beta - UCV \Leftrightarrow zf'(z) \in \beta - S^* \ . \tag{1.9}$$

Thus, the class $\beta - S^*, 0 \leq \beta < \infty$, is the subclass of S, consisting of functions that satisfy the analytic condition

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \left|\frac{zf'(z)}{f(z)} - 1\right|, z \in U \, . \tag{1.10}$$

Let $(f * g)(z)$ denotes the Hadamard product (or convolution) of the functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \ , \tag{1.11}$$

then

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) \ . \tag{1.12}$$

For $\alpha_j \in C(j = 1, 2, ..., l)$ and $\beta_j \in C\backslash\{0, -1, -2, ...\}(j = 1, 2, ..., m)$, the generalized hypergeometric function ${}_lF_m(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z)$ is defined by the infinite series

$${}_lF_m(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z) = \sum_{n=2}^{\infty} \frac{(\alpha_1)_n ... (\alpha_l)_n}{(\beta_1) ... (\beta_m)_n} \frac{z^n}{n!}$$

$$(l \leq m + 1; l, m \in N_0 = N \cup \{0\}; z \in U) \ , \tag{1.13}$$

where $(\lambda)_n$ is the Pochhamer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0)\,; \\ \lambda(\lambda + 1)...(\lambda + n - 1) & (n \in N)\,. \end{cases} \tag{1.14}$$

Corresponding to the function

$$h(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z) = z \, {}_lF_m(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z) \, ,$$

the Dziok-Srivastava operator ([4], [5], [11] and [12]) $H_m^l(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m)$ is defined by the Hadamard product

$$h(\alpha_1,...,\alpha_l;\beta_1,...,\beta_m)f(z) = h(\alpha_1,...,\alpha_l;\beta_1,...,\beta_m;z)*f(z)$$

$$= z+\sum_{n=2}^{\infty}\Gamma_n a_n z^n \tag{1.15}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-1}...(\alpha_l)_{n-1}}{(\beta_1)_{n-1}...(\beta_m)_{n-1}}\frac{1}{(n-1)!}\ . \tag{1.16}$$

For brevity, we write

$$H_m^l[\alpha_1]f(z) = H_m^l(\alpha_1,...,\alpha_l;\beta_1,...,\beta_m)f(z)\,.$$

It is easy to see from (1.15) that

$$z(H_m^l[\alpha_1]f(z))' = \alpha_1 H_m^l[\alpha_1+1]f(z)-(\alpha_1-1)H_m^l[\alpha_1]f(z)\,. \tag{1.17}$$

For $\beta \geq 0$ and $-1 \leq \alpha < 1$, we let $S_m^l([\alpha_1];\alpha,\beta)$ denote the subclass of S consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion

$$\mathrm{Re}\left\{\frac{z(H_m^l[\alpha_1]f(z))'}{H_m^l[\alpha_1]f(z)}-\alpha\right\} > \beta\left|\frac{z(H_m^l[\alpha_1]f(z))'}{H_m^l[\alpha_1]f(z)}-1\right|, z\in U\,. \tag{1.18}$$

We denote by T the subclass of S consisting of functions of the form

$$f(z) = z-\sum_{n=2}^{\infty}a_n z^n\,(a_n\geq 0)\ . \tag{1.19}$$

Further, we define the class $T_m^l([\alpha_1];\alpha,\beta)$ by

$$T_m^l([\alpha_1];\alpha,\beta) = S_m^l([\alpha_1];\alpha,\beta)\cap T\ .$$

We note that

(I) $T_0^1([1];\alpha,\beta) = S_pT(\alpha,\beta)$

$$= \left\{f(z)\in T : \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}-\alpha\right\} > \beta\left|\frac{zf'(z)}{f(z)}-1\right|, -1\leq\alpha<1,\right.$$

$$\left.\beta\geq 0,\ z\in U\right\} \tag{1.20}$$

The class $S_pT(\alpha,1) = S_pT(\alpha)$ was studied by Bharati et al. [1] .

(II) $T_1^2([a,1;c];\alpha,\beta) = S_pT(a,c;\alpha,\beta)$ (Murugusungaramoorthy and Magesh [15])

$$= \left\{f(z)\in T : \mathrm{Re}\left\{\frac{z(L(a,c)f(z))'}{L(a,c)f(z)}-\alpha\right\} > \beta\left|\frac{z(L(a,c)f(z))'}{L(a,c)f(z)}-1\right|,\right.$$

$$\left.-1\leq\alpha<1, \beta\geq 0, a>0, c>0,\ z\in U\right\}; \tag{1.21}$$

where $L(a,c)$ is the Carlson - Shaffer operator [3] .

(III) $H_1^2([\lambda+1,1;1];\alpha,\beta) = S_pT(\lambda;\alpha,\beta)$ (Shams and Kulkarni [23])

$$= \left\{f(z)\in T : \mathrm{Re}\left\{\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}-\alpha\right\} > \beta\left|\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}-1\right|,\right.$$

$$\left.0\leq\alpha<1, \beta\geq 0, \lambda>-1,\ z\in U\right\}\ , \tag{1.22}$$

where $D^\lambda(\lambda>-1)$ is the Ruscheweyh derivative operator [21] .

(IV) $H_1^2([\nu+1,1;\nu+2];\alpha,\beta) = S_pT(\nu;\alpha,\beta)$

$$(1.23)\quad = \left\{ f(z)\in T : \mathrm{Re}\left\{ \frac{z(J_\nu f(z))'}{J_\nu f(z)} - \alpha \right\} > \beta \left| \frac{z(J_\nu f(z))'}{J_\nu f(z)} - 1 \right|, \; -1\le \alpha < 1, \beta \ge 0, \nu > -1,\ z\in U \right\},$$

where $J_\nu (\nu > -1)$ is the generalized Bernardi-Libera-Livingston operator ([2], [10] and [13]).

(V) $H_1^2([2,1;2-\mu];\alpha,\beta) = S_pT(\mu;\alpha,\beta)$

$$= \left\{ f(z)\in T : \mathrm{Re}\left\{ \frac{z(\Omega_z^\mu f(z))'}{\Omega_z^\mu f(z)} - \alpha \right\} > \beta \left| \frac{z(\Omega_z^\mu f(z))'}{\Omega_z^\mu f(z)} - 1 \right|, \; -1\le \alpha < 1, \beta \ge 0, 0 \le \mu < 1,\ z\in U \right\},$$

where

$$(1.24)\quad \Omega_z^\mu f(z) = \Gamma(2-\mu) z^\mu D_z^\mu f(z) (0 \le \mu < 1)\ ,$$

where Ω_z^μ is the Srivastava-Owa fractional derivative operator ([16] and [18]).

2. COEFFICIENT ESTIMATES

Theorem 2.1. *A function $f(z)$ of the form (1.1) is in the class $S_m^l([\alpha_1];\alpha,\beta)$ if*

$$(2.1)\quad \sum_{n=2}^{\infty} C_n |a_n| \le 1-\alpha\ ,$$

where

$$(2.2)\quad C_n = [n(1+\beta) - (\alpha+\beta)]\Gamma_n$$

and Γ_n is defined by (1.16).

Proof. It is suffices to show that

$$\beta \left| \frac{z(H_m^l[\alpha_1]f(z))'}{H_m^l[\alpha_1]f(z)} - 1 \right| - \mathrm{Re}\left\{ \frac{z(H_m^l[\alpha_1]f(z))'}{H_m^l[\alpha_1]f(z)} - 1 \right\} \le 1-\alpha\, .$$

We have

$$\begin{aligned} &\beta \left| \frac{z(H_m^l[\alpha_1]f(z))'}{H_m^l[\alpha_1]f(z)} - 1 \right| - \mathrm{Re}\left\{ \frac{z(H_m^l[\alpha_1]f(z))'}{H_m^l[\alpha_1]f(z)} - 1 \right\} \\ \le\ & (1+\beta)\left| \frac{z(H_m^l[\alpha_1]f(z))'}{H_m^l[\alpha_1]f(z)} - 1 \right| \\ \le\ & \frac{(1+\beta)\sum_{n=2}^{\infty}(n-1)\Gamma_n|a_n|}{1-\sum_{n=2}^{\infty}\Gamma_n|a_n|}\, . \end{aligned}$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{n=2}^{\infty} [n(1+\beta) - (\alpha+\beta)]\Gamma_n |a_n| \le 1-\alpha\ ,$$

and the proof is complete. ∎

Theorem 2.2. *A necessary and sufficient condition for $f(z)$ of the form (1.19) to be in the class $T_m^l([\alpha_1];\alpha,\beta)$ is that*

$$\sum_{n=2}^{\infty} C_n a_n \le 1-\alpha\ , \tag{2.3}$$

where C_n is given by (2.2).

Proof. In view of Theorem 2.1, we need only to prove the necessity. If $f(z) \in T_m^l([\alpha_1];\alpha,\beta)$ and z is real, then (1.18) yields

$$\frac{1-\sum\limits_{n=2}^{\infty} n\Gamma_n a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty} \Gamma_n a_n z^{n-1}} - \alpha \ge \beta \left| \frac{\sum\limits_{n=2}^{\infty} (n-1)\Gamma_n a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty} \Gamma_n a_n z^{n-1}} \right| .$$

Letting $z \to 1^-$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty} [n(1+\beta)-(\alpha+\beta)]\Gamma_n a_n \le 1-\alpha, -1 \le \alpha < 1, \beta \ge 0\,.$$

∎

Corollary 2.3. *Let the function $f(z)$ defined by (1.19) be in the class $T_m^l([\alpha_1];\alpha,\beta)$. Then*

$$a_n \le \frac{(1-\alpha)}{C_n}(n \ge 2)\,. \tag{2.4}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{C_n} z^n\ (n \ge 2)\,, \tag{2.5}$$

where C_n is defined by (2.2).

3. Growth and Distortion Theorem

Theorem 3.1. *Let the function $f(z)$ defined by (1.19) be in the class $T_m^l([\alpha_1];\alpha,\beta)$. If the sequence $\{C_n\}$ is nondecreasing, then*

$$|z| - \frac{1-\alpha}{(2-\alpha+\beta)\Gamma_2}\,|z|^2 \le |f(z)| \le |z| + \frac{1-\alpha}{(2-\alpha+\beta)\Gamma_2}\,|z|^2\ (z \in U)\,. \tag{3.1}$$

If the sequence $\left\{\frac{C_n}{n}\right\}$ is nondecreasing, then

$$1 - \frac{2(1-\alpha)}{(2-\alpha+\beta)\Gamma_2}|z| \le |f'(z)| \le 1 + \frac{2(1-\alpha)}{(2-\alpha+\beta)\Gamma_2}|z|\ (z \in U), \tag{3.2}$$

where C_n is defined by (2.2). The result is sharp, with the extremal function $f(z)$ defined by

$$f(z) = z - \frac{1-\alpha}{(2-\alpha+\beta)\Gamma_2} z^2\,. \tag{3.3}$$

Proof. Let a function $f(z)$ of the form (1.19) belong to the class $T_m^l([\alpha_1];\alpha,\beta)$. If the sequence $\{C_n\}$ is nondecreasing and positive, by Theorem 2.2, we have

$$\sum_{n=2}^{\infty} a_n \le \frac{(1-\alpha)}{(2-\alpha+\beta)\Gamma_2}\ , \tag{3.4}$$

and if the sequence $\left\{\frac{C_n}{n}\right\}$ is nondecreasing and positive, by Theorem 2.2, we have

$$\sum_{n=2}^{\infty} na_n \leq \frac{2(1-\alpha)}{(2-\alpha+\beta)\Gamma_2} \ . \tag{3.5}$$

Making use of the conditions (3.4) and (3.5), in conjunction with the definition (1.19), we readily obtain the assertion (3.1) and (3.2) of 3.1. ∎

4. EXTREME POINTS

In view of the necessary and sufficient conditions of Theorem 2.2, the family $T_m^l([\alpha_1];\alpha,\beta)$ is closed under convex linear combinations. This leads to the determination of the extreme points for the family.

Theorem 4.1. *Let C_n be defined by (2.2) and let us put*

$$f_1(z) = z \tag{4.1}$$

and

$$f_n(z) = z - \frac{1-\alpha}{C_n} z^n \ (n \geq 2) \tag{4.2}$$

for $-1 \leq \alpha < 1$ and $\beta \geq 0$. Then $f(z)$ is in the class $T_m^l([\alpha_1];\alpha,\beta)$ if and only if it can be expressed in the form :

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \ , \tag{4.3}$$

where $\lambda_n \geq 0 (n \geq 1)$ and $\sum_{n=1}^{\infty} \lambda_n = 1$.

Proof. Assume that

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) = z - \sum_{n=2}^{\infty} \frac{1-\alpha}{C_n} \lambda_n z^n \ .$$

Then it follows that

$$\sum_{n=2}^{\infty} \frac{C_n}{1-\alpha} \cdot \frac{1-\alpha}{C_n} \lambda_n = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1 \ . \tag{4.4}$$

So by Theorem 2.2, $f(z) \in T_m^l([\alpha_1];\alpha,\beta)$.

Conversely, assume that the function $f(z)$ defined by (1.19) belongs to the class $T_m^l([\alpha_1];\alpha,\beta)$. Then

$$a_n \leq \frac{1-\alpha}{C_n} \ (n \geq 2) \ . \tag{4.5}$$

Setting

$$\lambda_n = \frac{C_n}{1-\alpha} a_n \ (n \geq 2) \tag{4.6}$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n \ , \tag{4.7}$$

we can see that $f(z)$ can be expressed in the form (4.3). This completes the proof of Theorem 4.1. ∎

Corollary 4.2. *The extreme points of the class* $T_m^l([\alpha_1];\alpha,\beta)$ *are the functions* $f_n(z)(n \geq 1)$ *given by Theorem 4.1.*

5. RADII OF CLOSE -TO- CONVEXITY, STARLIKENESS AND CONVEXITY

Theorem 5.1. *Let the function* $f(z)$ *defined by (1.19) be in the class* $T_m^l([\alpha_1];\alpha,\beta)$*. Then* $f(z)$ *is close - to - convex of order* $\rho\,(0 \leq \rho < 1)$ *in* $|z| < r_1$*, where*

$$r_1 = \inf_n \left\{ \frac{(1-\rho)C_n}{n(1-\alpha)} \right\}^{\dfrac{1}{n-1}} (n \geq 2)\,, \tag{5.1}$$

where C_n *is defined by (2.2). The result is sharp, with the extremal function* $f(z)$ *given by (2.5).*

Proof. We must show that

$$|f'(z) - 1| \leq 1 - \rho \;\text{ for }\; |z| < r_1\;,$$

where r_1 is given by (5.1). Indeed we find from the definition (1.19) that

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} n a_n \, |z|^{n-1}\,.$$

Thus

$$|f'(z) - 1| \leq 1 - \rho$$

if

$$\sum_{n=2}^{\infty} (\frac{n}{1-\rho}) a_n \, |z|^{n-1} \leq 1. \tag{5.2}$$

But, by Theorem 2.2, (5.2) will be true if

$$(\frac{n}{1-\rho})\, |z|^{n-1} \leq \frac{C_n}{1-\alpha}\,,$$

that is, if

$$|z| \leq \left\{ \frac{(1-\rho)C_n}{n(1-\alpha)} \right\}^{\dfrac{1}{n-1}} (n \geq 2)\,. \tag{5.3}$$

Theorem 5.1 follows easily from (5.3). ∎

Theorem 5.2. *Let the function* $f(z)$ *defined by (1.19) be in the class* $T_m^l([\alpha_1];\alpha,\beta)$*. Then the function* $f(z)$ *is starlike of order* $\rho\,(0 \leq \rho < 1)$ *in* $|z| < r_2$*, where*

$$r_2 = \inf_n \left\{ \frac{(1-\rho)C_n}{(n-\rho)(1-\alpha)} \right\}^{\dfrac{1}{n-1}} (n \geq 2), \tag{5.4}$$

where C_n *is defined by (2.2). The result is sharp, with the extremal function* $f(z)$ *given by (2.5).*

Proof. It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \;\text{ for }\; |z| < r_2\;,$$

where r_2 is given by (5.4). Indeed we find, again from the definition (1.19) that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n\,|z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_n\,|z|^{n-1}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \rho$$

if

$$\sum_{n=2}^{\infty}\left(\frac{n-\rho}{1-\rho}\right)a_n\,|z|^{n-1} \leq 1. \tag{5.5}$$

But, by Theorem 2.2, (5.5) will be true if

$$(\frac{n-\rho}{1-\rho})\,|z|^{n-1} \leq \frac{C_n}{1-\alpha},$$

that is, if

$$|z| \leq \left\{\frac{(1-\rho)C_n}{(n-\rho)(1-\alpha)}\right\}^{\dfrac{1}{n-1}} \; (n \geq 2). \tag{5.6}$$

Theorem 5.2 now follows easily from (5.6). ∎

Corollary 5.3. *Let the function $f(z)$ defined by (1.19) be in the class $T_m^l([\alpha_1];\alpha,\beta)$. Then $f(z)$ is convex of order $\rho\,(0 \leq \rho < 1)$ in $|z| < r_3$, where*

$$r_3 = \inf_n \left\{\frac{(1-\rho)C_n}{(n-\rho)(1-\alpha)}\right\}^{\dfrac{1}{n-1}} \; (n \geq 2), \tag{5.7}$$

where C_n is defined by (2.2). The result is sharp, with the extremal function $f(z)$ given by (2.5).

6. A FAMILY OF INTEGRAL OPERATORS

Theorem 6.1. *Let the function $f(z)$ defined by (1.19) be in the class $T_m^l([\alpha_1];\alpha,\beta)$ and let c be a real number such that $c > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt \;\; (c > -1) \tag{6.1}$$

also belongs to the class $T_m^l([\alpha_1];\alpha,\beta)$.

Proof. From the representation (6.1) of $F(z)$, it follows that

$$F(z) = z - \sum_{n=2}^{\infty} b_n z^n,$$

where

$$b_n = \left(\frac{c+1}{c+n}\right)a_n\ .$$

Therefore, we have

$$\begin{aligned}\sum_{n=2}^{\infty} C_n b_n &= \sum_{n=2}^{\infty} C_n (\frac{c+1}{c+n}) a_n \\ &\leq \sum_{n=2}^{\infty} C_n a_n \leq 1-\alpha \ ,\end{aligned}$$

since $f(z) \in T_m^l([\alpha_1];\alpha,\beta)$. Hence, by Theorem 2.2, $F(z) \in T_m^l([\alpha_1];\alpha,\beta)$. On the other hand, the converse is not true. This leads to a radius of univalence result. ∎

Theorem 6.2. *Let the function* $F(z) = z - \sum_{n=2}^{\infty} a_n z^n \ (a_n \geq 0)$ *be in the class* $T_m^l([\alpha_1];\alpha,\beta)$*, and let* c *be a real number such that* $c > -1$*. Then the function* $f(z)$ *given by (6.1) is univalent in* $|z| < R^*$*, where*

$$R^* = \inf_n \left\{ \frac{(c+1)C_n}{n(c+n(1-\alpha)} \right\}^{\dfrac{1}{n-1}} (n \geq 2) \,, \tag{6.2}$$

The result is sharp.

Proof. Form (6.1), we have

$$f(z) = \frac{z^{1-c}(z^c F(z))'}{c+1} = z - \sum_{n=2}^{\infty} \left(\frac{c+n}{c+1} \right) a_n z^n .$$

In order to obtain the required result, it suffices to show that

$$|f'(z) - 1| < 1 \ \text{ wherever } \ |z| < R^*,$$

where R^* is given by (6.2). Now

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} \frac{n(c+n)}{(c+1)} a_n \ |z|^{n-1} .$$

Thus $|f'(z) - 1| < 1$ if

$$\sum_{n=2}^{\infty} \frac{n(c+n)}{(c+1)} a_n \ |z|^{n-1} < 1 \,. \tag{6.3}$$

But Theorem 2.2 confirms that

$$\sum_{n=2}^{\infty} \frac{C_n}{1-\alpha} a_n \ \leq 1. \tag{6.4}$$

Hence (6.4) will be satisfied if

$$\frac{n(c+n)}{(c+1)} |z|^{n-1} < \frac{C_n}{1-\alpha} \,,$$

that is, if

$$|z| < \left\{ \frac{(c+1)C_n}{n(c+n)(1-\alpha)} \right\}^{\frac{1}{n-1}} (n \geq 2) \,. \tag{6.5}$$

Therefore, the function $f(z)$ given by (6.1) is univalent in $|z| < R^*$. Sharpness of the result follows if we take

$$f(z) = z - \frac{(c+n)(1-\alpha)}{(c+1)C_n} z^n (n \geq 2) \,. \tag{6.6}$$

∎

7. MODIFIED HADAMARD PRODUCTS

Let the functions $f_\nu(z)(\nu = 1, 2)$ be defined by

$$f_\nu(z) = z - \sum_{n=2}^{\infty} a_{n,\nu} z^n (a_{n,\nu} \geq 0; \nu = 1, 2) \ . \tag{7.1}$$

The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z - \sum_{n=2}^{\infty} a_{n.1} a_{n,2} z^n. \tag{7.2}$$

Theorem 7.1. *Let each of the functions $f_\nu(z)$ $(\nu = 1, 2)$defined by (7.1) be in the class $T^l_m([\alpha_1]; \alpha, \beta)$. If the sequence $\{C_n\}$ is nondecreasing, then $(f_1 * f_2)(z) \in T^l_m([\alpha_1]; \delta([\alpha_1], \alpha, \beta), \beta)$, where*

$$\delta([\alpha_1], \alpha, \beta) = 1 - \frac{(1+\beta)(1-\alpha)^2}{(2-\alpha+\beta)^2\Gamma_2 - (1-\alpha)^2}. \tag{7.3}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [22], we need to find the largest $\delta = \delta([\alpha_1], \alpha, \beta)$ such that

$$\sum_{n=2}^{\infty} \frac{[n(1+\beta) - (\delta+\beta)]\Gamma_n}{1-\delta} a_{n.1} a_{n,2} \leq 1. \tag{7.4}$$

Since

$$\sum_{n=2}^{\infty} \frac{[n(1+\beta) - (\alpha+\beta)]\Gamma_n}{1-\alpha} a_{n.1} \leq 1 \tag{7.5}$$

and

$$\sum_{n=2}^{\infty} \frac{[n(1+\beta) - (\alpha+\beta)]\Gamma_n}{1-\alpha} a_{n.2} \leq 1, \tag{7.6}$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{n=2}^{\infty} \frac{[n(1+\beta) - (\alpha+\beta)]\Gamma_n}{1-\alpha} \sqrt{a_{n.1} a_{n.2}} \leq 1. \tag{7.7}$$

Thus it is sufficient to show that

$$\frac{[n(1+\beta) - (\delta+\beta)]\Gamma_n}{1-\delta} a_{n.1} a_{n.2} \leq \frac{[n(1+\beta) - (\alpha+\beta)]\Gamma_n}{1-\alpha} \sqrt{a_{n.1} a_{n.2}} (n \geq 2), \tag{7.8}$$

that is, that

$$\sqrt{a_{n.1} a_{n.2}} \leq \frac{[n(1+\beta) - (\alpha+\beta)](1-\delta)}{[n(1+\beta) - (\delta+\beta)](1-\alpha)} (n \geq 2). \tag{7.9}$$

Note that

$$\sqrt{a_{n.1} a_{n.2}} \leq \frac{(1-\alpha)}{[n(1+\beta) - (\alpha+\beta)]\Gamma_n} (n \geq 2). \tag{7.10}$$

Consequently, we need only to prove that

$$\frac{1-\alpha}{[n(1+\beta)-(\alpha+\beta)]\Gamma_n} \leq \frac{[n(1+\beta)-(\alpha+\beta)](1-\delta)}{[n(1+\beta)-(\delta+\beta)](1-\alpha)}(n \geq 2), \tag{7.11}$$

or, equivalently, that

$$\delta \leq 1-\frac{(n-1)(1+\beta)(1-\alpha)^2}{[n(1+\beta)-(\alpha+\beta)]^2\Gamma_n-(1-\alpha)^2}(n \geq 2). \tag{7.12}$$

Since

$$\Phi(n) = 1-\frac{(n-1)(1+\beta)(1-\alpha)^2}{[n(1+\beta)-(\alpha+\beta)]^2\Gamma_n-(1-\alpha)^2} \tag{7.13}$$

is an increasing function of $n (n \geq 2)$, letting $n = 2$ in (7.13), we obtain

$$\delta \leq \Phi(2) = 1-\frac{(1+\beta)(1-\alpha)^2}{(2-\alpha+\beta)^2\Gamma_2-(1-\alpha)^2}\ , \tag{7.14}$$

which proves the main assertion of Theorem 7.1.

Finally, by taking the functions $f_\nu(z) (\nu = 1, 2)$ given by

$$f_\nu(z) = z-\frac{1-\alpha}{(2-\alpha+\beta)\Gamma_2}z^2 (\nu = 1, 2), \tag{7.15}$$

we can see that the result is sharp.

Proceeding as in the proof of Theorem 7.1, we get ∎

Theorem 7.2. *Let the function* $f_1(z)$ *defined by (7.1) be in the class* $T^l_m([\alpha_1]; \alpha, \beta)$ *and the function* $f_2(z)$ *defined by* (7.1) *be in the class* $T^l_m([\alpha_1]; \gamma, \beta)$. *If the sequence* $\{C_n\}$ *is nondecreasing, then* $(f_1 * f_2)(z) \in T^l_m([\alpha_1]; \zeta([\alpha_1], \alpha, \gamma, \beta), \beta)$, *where*

$$\zeta([\alpha_1], \alpha, \gamma, \beta) = 1-\frac{(1+\beta)(1-\alpha)(1-\gamma)}{(2-\alpha+\beta)(2-\gamma+\beta)\Gamma_2-(1-\alpha)(1-\gamma)}. \tag{7.16}$$

The result is the best possible for the functions

$$f_1(z) = z-\frac{1-\alpha}{(2-\alpha+\beta)\Gamma_2}z^2 \tag{7.17}$$

and

$$f_2(z) = z-\frac{1-\gamma}{(2-\gamma+\beta)\Gamma_2}z^2\ . \tag{7.18}$$

Theorem 7.3. *Let the functions* $f_\nu(z)$ $(\nu = 1, 2)$ *defined by (7.1) be in the class* $T^l_m([\alpha_1]; \alpha, \beta)$. *If the sequence* $\{C_n\}$ *is nondecreasing. Then the function*

$$h(z) = z-\sum_{n=2}^{\infty}(a_{n,1}^2+a_{n,2}^2)z^n \tag{7.19}$$

belongs to the class $T^l_m([\alpha_1]; \Im([\alpha_1], \alpha, \beta), \beta)$*, where*

$$\Im([\alpha_1], \alpha, \beta) = 1-\frac{2(1+\beta)(1-\alpha)^2}{(2+\beta-\alpha)^2\Gamma_2-2(1-\alpha)^2}\ . \tag{7.20}$$

The result is sharp for the functions $f_\nu(z)$ *defined by (7.15).*

Proof. By virture of Theorem 2.2, we obtain

$$\sum_{n=2}^{\infty}\left\{\frac{[n(1+\beta)-(\alpha+\beta)]\Gamma_n}{1-\alpha}\right\}^2 a_{n,1}^2 \leq \left\{\sum_{n=2}^{\infty}\frac{[n(1+\beta)-(\alpha+\beta)]\Gamma_n}{1-\alpha}a_{n,1}\right\}^2 \leq 1 \tag{7.21}$$

and

$$\sum_{n=2}^{\infty}\left\{\frac{[n(1+\beta)-(\alpha+\beta)]\Gamma_n}{1-\alpha}\right\}^2 a_{n,2}^2 \leq \left\{\sum_{n=2}^{\infty}\frac{[n(1+\beta)-(\alpha+\beta)]\Gamma_n}{1-\alpha}a_{n,2}\right\}^2 \leq 1\,. \tag{7.22}$$

It follows from (7.21) and (7.22) that

$$\sum_{n=2}^{\infty}\frac{1}{2}\left\{\frac{[n(1+\beta)-(\alpha+\beta)]\Gamma_n}{1-\alpha}\right\}^2 (a_{n,1}^2+a_{n,2}^2) \leq 1\,. \tag{7.23}$$

Therefore, we need to find the largest $\Im([\alpha_1],\alpha,\beta)$ such that

$$\frac{[n(1+\beta)-(\Im+\beta)]\Gamma_n}{1-\Im} \leq \frac{1}{2}\left\{\frac{[n(1+\beta)-(\alpha+\beta)]\Gamma_n}{1-\alpha}\right\}^2 (n\geq 2), \tag{7.24}$$

that is,

$$\Im \leq 1-\frac{2(n-1)(1+\beta)(1-\alpha)^2}{[n(1+\beta)-(\alpha+\beta)]^2\Gamma_n-2(1-\alpha)^2}(n\geq 2)\,. \tag{7.25}$$

Since

$$D(n) = 1-\frac{2(n-1)(1+\beta)(1-\alpha)^2}{[n(1+\beta)-(\alpha+\beta)]^2\Gamma_n-2(1-\alpha)^2} \tag{7.26}$$

is an increasing function of $n(n\geq 2)$, we readily have

$$\Im \leq D(2) = 1-\frac{2(1+\beta)(1-\alpha)^2}{(2+\beta-\alpha)^2\,\Gamma_2-2(1-\alpha)^2}\,, \tag{7.27}$$

and Theorem 7.3 follows at once. ∎

8. PROPERTIES ASSOCIATED WITH GENERALIZED FRACTIONAL CALCULUS OPERATORS

In terms of the Gauss hypergeometric function :

$$ {}_2F_1(\delta,\mu;\nu;z) = \sum_{n=0}^{\infty}\frac{(\delta)_n(\mu)_n}{(\nu)_n}\frac{z^n}{n!} \tag{8.1}$$

$$(z\in U;\delta,\mu,\nu\in C;\nu\neq 0,-1,-2,...)\ ,$$

where (again) $(\lambda)_n$ denotes the Pochhammer symbol defined in (1.14), the generalized fractional calculus operators $I_{0,z}^{\mu,\nu,\eta}$ and $J_{0,z}^{\mu,\nu,\eta}$ are defined below (cf., e.g., [17] and [25]).

Definition 8.1. (Generalized Fractional Integral operator). The generalized fractional integral of order μ is defined, for a function $f(z)$, by

$$I_{0,z}^{\mu,\nu,\eta}f(z) = \frac{z^{-\mu-\nu}}{\Gamma(\mu)}\int_0^z (z-\zeta)^{\mu-1}\,{}_2F_1(\mu+\nu;-\eta;\mu;1-\frac{\zeta}{z})\,.\,f(\zeta)d\zeta \tag{8.2}$$

$$(\mu>0;\epsilon>\max\{0,\nu-\eta\}-1)\ ,$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$, provided further that

$$f(z) = O(|z|^{\epsilon})\,(z \to 0)\ . \tag{8.3}$$

Definition 8.2. (Generalized Fractional Derivative Operator). The generalized fractional derivative of order μ is defined, for a function $f(z)$, by

$$J_{0,z}^{\mu,\nu,\eta} f(z) = \begin{cases} \dfrac{1}{\Gamma(1-\mu)}\dfrac{d}{dz}\Big\{ z^{\mu-\nu}\int\limits_0^z (z-\zeta)^{-\mu}\,{}_2F_1(\nu-\mu, 1-\eta; 1-\mu\,; \\ 1-\frac{\zeta}{z}) f(\zeta) d\zeta \Big\}\ (0 \le \mu < 1)\ , \\ \dfrac{d^n}{dz^n}\, J_{0,z}^{\mu-n,\nu,\eta} f(z)\, (n \le \mu < n+1; n \in N) \end{cases} \tag{8.4}$$

$$(\epsilon > \max\{0, \nu-\eta\} - 1)\ ,$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed, as in Definition 8.1, and ϵ is given by the order estimate (8.3).

$$f(z) = O(|z|^{\epsilon})\,(z \to 0)\ . \tag{8.5}$$

It follows from Definition 8.1 and Definition 8.2 that

$$I_{0,z}^{\mu,-\mu,\eta} f(z) = D_z^{-\mu} f(z)\ (\mu > 0)\ , \tag{8.6}$$

and

$$J_{0,z}^{\mu,\mu,\eta} f(z) = D_z^{\mu} f(z)\ (0 \le \mu < 1)\ , \tag{8.7}$$

where $D_z^{\mu} (\mu \in R)$ is the fractional operator considered by Owa [16] and (subsequently) by Owa and Srivastava [18] and Srivastava and Owa [24]. Furthermore, in terms of Gamma functions, Definitions 8.1 and 8.2 readily yield.

Lemma 8.1. [25] . *The generalized fractional integral and the generalized fractional derivative of a power function are given by*

$$I_{0,z}^{\mu,\nu,\eta} z^{\rho} = \frac{\Gamma(\rho+1)\Gamma(\rho-\nu+\eta+1)}{\Gamma(\rho-\nu+1)\Gamma(\rho+\mu+\eta+1)} z^{\rho-\nu} \tag{8.8}$$

$$(\mu > 0; \rho > \max\{0, \nu-\eta\} - 1)\ ,$$

and

$$J_{0,z}^{\mu,\nu,\eta} z^{\rho} = \frac{\Gamma(\rho+1)\Gamma(\rho-\nu+\eta+1)}{\Gamma(\rho-\nu+1)\Gamma(\rho-\mu+\eta+1)} z^{\rho-\nu} \tag{8.9}$$

$$(0 \le \mu < 1; \rho > \max\{0, \nu-\eta\} - 1)\ .$$

Theorem 8.2. *Let the function $f(z)$ defined by (1.19) be in the class $T_m^l([\alpha_1]; \alpha, \beta)$. If the sequence $\{C_n\}$ is nondecreasing, then*

$$\frac{\Gamma(2-\nu+\eta)}{\Gamma(2-\nu)\Gamma(2+\mu+\eta)} |z|^{1-\nu} \left\{ 1 - \frac{2(1-\alpha)(2-\nu+\eta)}{(2-\nu)(2+\mu+\eta)(2-\alpha+\beta)\Gamma_2} |z| \right\}$$
$$\le \ \left| I_{0,z}^{\mu,\nu,\eta} f(z) \right| \le$$

$$\frac{\Gamma(2-\nu+\eta)}{\Gamma(2-\nu)\Gamma(2+\mu+\eta)}|z|^{1-\nu}\left\{1+\frac{2(1-\alpha)(2-\nu+\eta)}{(2-\nu)(2+\mu+\eta)(2-\alpha+\beta)\Gamma_2}|z|\right\} \tag{8.10}$$
$$(z\in U_0;\mu>0;\max\{\nu,\nu-\eta,-\mu-\eta\}<2;\nu(\mu+\eta)\leq 3\mu)$$

and

$$\frac{\Gamma(2-\nu+\eta)}{\Gamma(2-\nu)\Gamma(2-\mu+\eta)}|z|^{1-\nu}\left\{1-\frac{2(1-\alpha)(2-\nu+\eta)}{(2-\nu)(2+\mu+\eta)(2-\alpha+\beta)\Gamma_2}|z|\right\}$$
$$\leq \left|J_{0,z}^{\mu,\nu,\eta}f(z)\right|$$
$$\frac{\Gamma(2-\nu+\eta)}{\Gamma(2-\nu)\Gamma(2-\mu+\eta)}|z|^{1-\nu}\left\{1+\frac{2(1-\alpha)(2-\nu+\eta)}{(2-\nu)(2+\mu+\eta)(2-\alpha+\beta)\Gamma_2}|z|\right\} \tag{8.11}$$
$$(z\in U_0;0\leq\mu<1;\max\{\nu,\nu-\eta,\mu-\eta\}<2;\nu(\mu-\eta)\geq 3\mu)\ ,$$

where

$$U_0=\begin{cases} U & (\nu\leq 1)\\ U\backslash\{0\} & (\nu>1)\ . \end{cases} \tag{8.12}$$

Each of these results is sharp for the function $f(z)$ defined by (3.3).

Proof. Making use of the assertion (8.8) of Lemma 8.1, we find from (1.19) that

$$\begin{aligned} F(z) &= \frac{\Gamma(2-\nu)\Gamma(2+\mu+\eta)}{\Gamma(2-\nu+\eta)}z^{\nu}\, I_{0,z}^{\mu,\nu,\eta}f(z)\\ &= z-\sum_{n=2}^{\infty}\Phi(n)a_nz^n\ , \end{aligned} \tag{8.13}$$

where, for convenience,

$$\Phi(n)=\frac{(1)_n(2-\nu+\eta)_{n-1}}{(2-\nu)_{n-1}(2+\mu+\eta)_{n-1}}\,(n\in N\backslash\{1\})\ . \tag{8.14}$$

The function $\Phi(n)$ defined by (8.14) can easily be seen to be nonincreasing under the parametric constraints stated already with (8.10), and we thus have

$$0<\Phi(n)\leq\Phi(2)=\frac{2(2-\nu+\eta)}{(2-\nu)(2+\mu+\eta)}\,(n\in N\backslash\{1\})\ . \tag{8.15}$$

Now the assertion (8.10) of Theorem 8.2 would follow readily from (3.4), (8.13) and (8.15).

The assertion (8.11) of Theorem 8.2 can be proven similarly by noting from (8.9) that

$$\begin{aligned} G(z) &= \frac{\Gamma(2-\nu)\Gamma(2-\mu+\eta)}{\Gamma(2-\nu+\eta)}z^{\nu}\, J_{0,z}^{\mu,\nu,\eta}f(z)\\ &= z-\sum_{k=2}^{\infty}\Psi(n)a_nz^n\ , \end{aligned} \tag{8.16}$$

where

$$\begin{aligned} 0 &< \Psi(n)=\frac{(1)_n(2-\nu+\eta)_{n-1}}{(2-\nu)_{n-1}(2-\mu+\eta)_{n-1}}\\ &\leq \Psi(2)=\frac{2(2-\nu+\eta)}{(2-\nu)(2-\mu+\eta)}\,(n\in N\backslash\{1\})\ , \end{aligned} \tag{8.17}$$

under the parametric constraints stated already with (8.11).

Finally, by observing that the equalities in each of the assertions (8.10) and (8.11) are attained by the function $f(z)$ given by (3.3), we complete the proof of Theorem 8.2.

In view of the relationships (8.6) and (8.7), by setting $\nu = -\mu$ and $\nu = \mu$ in our assertions (8.10) and (8.11), respectively, we obtain ∎

Corollary 8.3. *Let the function $f(z)$ defined by (1.19) be in the class $T_m^l([\alpha_1];\alpha,\beta)$. If the sequence $\{C_n\}$ is nondecreasing, then*

$$\frac{|z|^{1+\mu}}{\Gamma(2+\mu)}\left\{1-\frac{2(1-\alpha)}{(2+\mu)(2-\alpha+\beta)\Gamma_2}\,|z|\right\} \leq \left|D_z^{-\mu}f(z)\right| \leq$$

$$\frac{|z|^{1+\mu}}{\Gamma(2+\mu)}\left\{1+\frac{2(1-\alpha)}{(2+\mu)(2-\alpha+\beta)\Gamma_2}\,|z|\right\}(z\in U;\mu>0)\,. \tag{8.18}$$

and

$$\frac{|z|^{1-\mu}}{\Gamma(2-\mu)}\left\{1-\frac{2(1-\alpha)}{(2-\mu)(2-\alpha+\beta)\Gamma_2}\,|z|\right\} \leq \left|D_z^{-\mu}f(z)\right| \leq$$

$$\frac{|z|^{1-\mu}}{\Gamma(2-\mu)}\left\{1+\frac{2(1-\alpha)}{(2-\mu)(2-\alpha+\beta)\Gamma_2}\,|z|\right\}(z\in U;0\leq\mu<1)\,. \tag{8.19}$$

Each of these results is sharp for the function $f(z)$ given by (3.3).

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