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OSCILLATION AND BOUNDEDNESS OF SOLUTIONS TO FIRST AND SECOND ORDER FORCED DYNAMIC EQUATIONS WITH MIXED NONLINEARITIES

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ABSTRACT. Some oscillation and boundedness criteria for solutions to certain first and second order forced dynamic equations with mixed nonlinearities are established. The main tool in the proofs is an inequality due to Hardy, Littlewood and Pólya. The obtained results can be applied to differential equations, difference equations and q-difference equations. The results are illustrated with numerous examples.

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1. INTRODUCTION

Throughout, we let $\mathbb{T}$ be a time scale (i.e., a nonempty closed subset of the real numbers) which is unbounded above. For the theory of time scales, which unifies continuous and discrete analysis and extends those concepts to "in between" cases, we refer to the monographs [5, 6] and to [4, Section 7] for readers of this journal. Here we only mention that for a function $f : \mathbb{T} \to \mathbb{R}$, the

1. delta derivative f^Δ satisfies $f^\Delta(t) = f'(t)$ if $\mathbb{T} = \mathbb{R}$, $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$ if $\mathbb{T} = \mathbb{N}_0$ and $f^\Delta(t) = (f(qt) - f(t))/((q-1)t)$ if $\mathbb{T} = q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$ with $q > 1$;
2. forward shift $f^\sigma = f \circ \sigma$, where $\sigma : \mathbb{T} \to \mathbb{T}$ is the forward jump operator, satisfies $f^\sigma(t) = f(t)$ if $\mathbb{T} = \mathbb{R}$, $f^\sigma(t) = f(t+1)$ if $\mathbb{T} = \mathbb{N}_0$ and $f^\sigma(t) = f(qt)$ if $\mathbb{T} = q^{\mathbb{N}_0}$,

and that if $p : \mathbb{T} \to \mathbb{R}$ is an rd-continuous and regressive (see [5, 6]) function and $t_0 \in \mathbb{T}$, then the

3. exponential function $e_p(\cdot, t_0)$ is defined as the unique solution of the initial value problem $y^\Delta = p(t)y$, $y(t_0) = 1$.

Now, let us introduce the dynamic equations that will be discussed in this paper. In Section 2 below we will study oscillatory behavior and boundedness of solutions of first order dynamic equations of the form

$$x^\Delta - p(t)x^\sigma + q_1(t)(x^\sigma)^\alpha = f(t), \tag{1.1}$$

$$x^\Delta + p(t)x^\sigma - q_2(t)(x^\sigma)^\beta = f(t), \tag{1.2}$$

$$x^\Delta + p(t)x^\sigma + q_1(t)(x^\sigma)^\alpha - q_2(t)(x^\sigma)^\beta = f(t), \tag{1.3}$$

$$x^\Delta + q_1(t)(x^\sigma)^\alpha - q_2(t)(x^\sigma)^\beta = f(t), \tag{1.4}$$

while in Section 3 below we examine second order dynamic equations of the form

$$[r(t)(x^\Delta)^\gamma]^\Delta - p(t)x^\sigma + q_1(t)(x^\sigma)^\alpha = f(t), \tag{1.5}$$

$$[r(t)(x^\Delta)^\gamma]^\Delta + p(t)x^\sigma - q_2(t)(x^\sigma)^\beta = f(t), \tag{1.6}$$

$$[r(t)(x^\Delta)^\gamma]^\Delta + p(t)x^\sigma + q_1(t)(x^\sigma)^\alpha - q_2(t)(x^\sigma)^\beta = f(t), \tag{1.7}$$

$$[r(t)(x^\Delta)^\gamma]^\Delta + q_1(t)(x^\sigma)^\alpha - q_2(t)(x^\sigma)^\beta = f(t), \tag{1.8}$$

subject to the general assumptions

$$\begin{cases} r, p, q_1, q_2, f : \mathbb{T} \to \mathbb{R} \quad \text{are rd-continuous,} \\ r(t), p(t), q_1(t), q_2(t) > 0 \quad \text{for all} \quad t \in \mathbb{T}, \\ \alpha, \beta, \gamma \quad \text{are ratios of odd positive integers,} \\ \alpha > 1, \quad 0 < \beta < 1, \end{cases} \tag{1.9}$$

and each of the equations (1.1)–(1.8) is considered on a subset $[t_0, \infty)$ of $\mathbb{T}$, where $t_0 \in \mathbb{T}$ is fixed. Throughout, we will make use of the rd-continuous functions $g_1, g_2, g_3, g_4 : \mathbb{T} \to \mathbb{R}$ defined by

$$\begin{cases} g_1 = (\alpha-1)\alpha^{\alpha/(1-\alpha)}p^{\alpha/(\alpha-1)}q_1^{1/(1-\alpha)}, \quad g_2 = (1-\beta)\beta^{\beta/(1-\beta)}p^{\beta/(\beta-1)}q_2^{1/(1-\beta)}, \\ g_3 = \delta^{\alpha/(\alpha-1)}g_1 + (1+\delta)^{\beta/(\beta-1)}g_2 \text{ with } \delta > 0, \quad g_4 = g_1 + g_2. \end{cases} \tag{1.10}$$

A solution of any of the equations (1.1)–(1.8) is called *nonoscillatory* if it is eventually positive or eventually negative; otherwise it is called *oscillatory*. Any of the equations (1.1)–(1.8) is called *oscillatory* if all of its solutions are oscillatory. In this paper, we shall give criteria in terms of r, p, q_1, q_2, α, β, γ guaranteeing solutions to equations (1.1)–(1.8) to be oscillatory. We also give criteria that guarantee that all nonoscillatory solutions of such an equation are bounded. Such criteria are available in the literature for differential equations (see, e.g., [3]), difference equations (see [1] for directly corresponding results and [2, Section 1.15] for linear difference equations and [2, Section 6.5] for difference equations with deviating arguments) and, in the linear case, for dynamic equations (see [7]; see also [5, Section 4.3] and [6, Section 4.5]).

We conclude this introduction by stating the following well-known inequality due to Hardy, Littlewood and Pólya [8]. This inequality is the major tool in the proofs of our criteria given in Sections 2 and 3 below.

Lemma 1.1. *If a and b are nonnegative, then*

(i) $a^\alpha - \alpha a b^{\alpha-1} + (\alpha - 1)b^\alpha \geq 0$ *for all* $\alpha > 1$*;*

(ii) $a^\beta - \beta a b^{\beta-1} - (1 - \beta)b^\beta \leq 0$ *for all* $0 < \beta < 1$*.*

In the above inequalities, equality holds if and only if $a = b$.

2. FIRST ORDER DYNAMIC EQUATIONS

In this section, we give oscillation criteria for solutions to first order dynamic equations of the form (1.1)–(1.4). Throughout, we assume (1.9), use the notation (1.10), and fix $t_0 \in \mathbb{T}$.

Theorem 2.1. *If*

$$\liminf_{t\to\infty} \int_{t_0}^{t} (f(\tau) + g_1(\tau))\Delta\tau = -\infty \quad \textit{and} \quad \limsup_{t\to\infty} \int_{t_0}^{t} (f(\tau) - g_1(\tau))\Delta\tau = \infty, \tag{2.1}$$

then (1.1) *is oscillatory.*

Proof. By way of contradiction, assume that (1.1) is not oscillatory so that it has at least one eventually positive solution or at least one eventually negative solution. First we assume that x satisfies (1.1) and is eventually positive. Hence there exists $t_1 \in \mathbb{T}$, $t_1 \geq t_0$, such that $x(t) > 0$ for all $t \geq t_1$. Set

$$a = q_1^{1/\alpha} x^\sigma \quad \text{and} \quad b = \left(\frac{1}{\alpha} p q_1^{-1/\alpha}\right)^{1/(\alpha-1)}$$

and use Lemma 1.1 (i) to obtain from (1.1) for $t \geq t_1$

$$\begin{aligned} x^\Delta(t) &= f(t) + p(t)x^\sigma(t) - q_1(t)(x^\sigma(t))^\alpha \\ &= f(t) + \alpha a(t)(b(t))^{\alpha-1} - (a(t))^\alpha \\ &\leq f(t) + (\alpha - 1)(b(t))^\alpha \\ &= f(t) + g_1(t) \end{aligned}$$

and thus

$$\begin{aligned} x^\Delta(t) &= x(t_1) + \int_{t_1}^{t} x^\Delta(\tau)\Delta\tau \leq x(t_1) + \int_{t_1}^{t} (f(\tau) + g_1(\tau))\Delta\tau \\ &= c + \int_{t_0}^{t} (f(\tau) + g_1(\tau))\Delta\tau \end{aligned}$$

with

$$c = x(t_1) - \int_{t_0}^{t_1} (f(\tau) + g_1(\tau))\Delta\tau.$$

Employing the first condition in (2.1), we find

$$0 \leq \liminf_{t\to\infty} x(t) \leq -\infty,$$

which is a contradiction. Next, we assume that x satisfies (1.1) and is eventually negative. If we put $\tilde{x} = -x$, then $\tilde{x}$ is an eventually positive solution of (1.1), where f is replaced by $-f$. This case therefore leads to a contradictions as in the first case, provided

$$\liminf_{t\to\infty} \int_{t_0}^{t} (-f(\tau) + g_1(\tau))\Delta\tau = -\infty,$$

which indeed holds due to the second condition in (2.1). ∎

Corollary 2.2. *If*

$$\liminf_{t\to\infty} \int_{t_0}^{t} f(\tau)\Delta\tau = -\infty \quad \textit{and} \quad \limsup_{t\to\infty} \int_{t_0}^{t} f(\tau)\Delta\tau = \infty \tag{2.2}$$

and

$$\int_{t_0}^{\infty} (p(\tau))^{\alpha/(\alpha-1)} (q_1(\tau))^{1/(1-\alpha)} \Delta\tau < \infty, \tag{2.3}$$

then (1.1) *is oscillatory.*

Proof. As (2.2) and (2.3) imply (2.1), this follows from Theorem 2.1. ∎

Example 2.1. *Let* $\mathbb{T}$ *be a time scale satisfying*

$$\mu(t) = \sigma(t) - t \neq 0 \textit{ for all } t \in \mathbb{T}, \quad t_0 \in \mathbb{T}, \quad t_0 > 0 \tag{2.4}$$

and consider on $[t_0, \infty)$ *the equation*

$$x^\Delta - \frac{1}{t(\sigma(t))^2} x^\sigma + \frac{1}{t(\sigma(t))^4} (x^\sigma)^3 = \frac{t + \sigma(t)}{\mu(t)} e_{-2/\mu}(\sigma(t), t_0), \tag{2.5}$$

which is of the form (1.1) *with*

$$p(t) = \frac{1}{t(\sigma(t))^2}, \quad q_1(t) = \frac{1}{t(\sigma(t))^4}, \quad f(t) = \frac{t + \sigma(t)}{\mu(t)} e_{-2/\mu}(\sigma(t), t_0), \quad \alpha = 3.$$

Note now that $F(t) = te_{-2/\mu}(t, t_0)$ *satisfies*

$$\begin{aligned} F^\Delta(t) &= e_{-2/\mu}(\sigma(t), t_0) - \frac{2t}{\mu(t)} e_{-2/\mu}(t, t_0) \\ &= e_{-2/\mu}(\sigma(t), t_0) + \frac{2t}{\mu(t)} e_{-2/\mu}(\sigma(t), t_0) \\ &= \frac{\mu(t) + 2t}{\mu(t)} e_{-2/\mu}(\sigma(t), t_0) \ = \ f(t), \end{aligned}$$

where we used the product rule on time scales [5, Theorem 1.20 (iii)] *and*

$$e_{-2/\mu}(\sigma(t), t_0) = \left(1 - \mu(t)\frac{2}{\mu(t)}\right) e_{-2/\mu}(t, t_0) = -e_{-2/\mu}(t, t_0), \tag{2.6}$$

which follows from [5, Theorem 2.36 (ii)]. *Note also that* (2.6) *together with* $e_{-2/\mu}(t_0, t_0) = 1$ *implies that*

$$\int_{t_0}^{t} f(\tau)\Delta\tau = F(t) - F(t_0) = te_{-2/\mu}(t, t_0) - t_0$$

has $\liminf$ *equal to* $-\infty$ *and* $\limsup$ *equal to* ∞ *when* $t \to \infty$*, i.e.,* (2.2) *is satisfied. Moreover,*

$$g_1(t) = \frac{2}{3\sqrt{3}} \left(\frac{1}{t(\sigma(t))^2}\right)^{3/2} \left(\frac{1}{t(\sigma(t))^4}\right)^{-1/2} = \frac{2}{3\sqrt{3}t\sigma(t)}$$

implies that (see [5, Theorem 1.20 (iv)]*)*

$$\int_{t_0}^{t} g_1(\tau)\Delta\tau = \frac{2}{3\sqrt{3}}\left(\frac{1}{t_0} - \frac{1}{t}\right) \to \frac{2}{3\sqrt{3}t_0} \quad \textit{as} \quad t \to \infty,$$

i.e., (2.3) *is satisfied. By Corollary 2.2, each solution of* (2.5) *is oscillatory. One such oscillatory solution is* $x = F$*. The same arguments apply to the equation*

$$x^{\Delta} - q_1(t)(\sigma(t))^2 x^{\sigma} + q_1(t)(x^{\sigma})^3 = \frac{t+\sigma(t)}{\mu(t)} e_{-2/\mu}(\sigma(t), t_0),$$

where $q_1 : \mathbb{T} \to \mathbb{R}$ *is any rd-continuous function satisfying* $q_1(t) > 0$ *for all* $t \in \mathbb{T}$ *and*

$$\int_{t_0}^{\infty} q_1(\tau)(\sigma(\tau))^3 \Delta\tau < \infty.$$

Theorem 2.3. *If*

$$\liminf_{t\to\infty} \int_{t_0}^{t} (f(\tau) + g_2(\tau))\Delta\tau = -\infty \quad \textit{and} \quad \limsup_{t\to\infty} \int_{t_0}^{t} (f(\tau) - g_2(\tau))\Delta\tau = \infty, \tag{2.7}$$

then (1.2) *is oscillatory.*

Proof. We proceed exactly as in the proof of Theorem 2.1, this time setting

$$a = q_2^{1/\beta} x^{\sigma} \quad \text{and} \quad b = \left(\frac{1}{\beta} p q_2^{-1/\beta}\right)^{1/(\beta-1)}$$

and this time using Lemma 1.1 (ii) in the subsequent calculation

$$\begin{aligned} x^{\Delta}(t) &= f(t) + q_2(t)(x^{\sigma}(t))^{\beta} - p(t)x^{\sigma}(t) \\ &= f(t) + (a(t))^{\beta} - \beta a(t)(b(t))^{\beta-1} \\ &\le f(t) + (1-\beta)(b(t))^{\beta} \\ &= f(t) + g_2(t), \end{aligned}$$

and the rest of the proof is line by line the same as the proof of Theorem 2.1 with g_1 replaced by g_2. ∎

Corollary 2.4. *If* (2.2) *holds and*

$$\int_{t_0}^{\infty} (p(\tau))^{\beta/(\beta-1)} (q_2(\tau))^{1/(1-\beta)} \Delta\tau < \infty, \tag{2.8}$$

then (1.2) *is oscillatory.*

Proof. As (2.2) and (2.8) imply (2.7), this follows from Theorem 2.3. ∎

Example 2.2. *Let* $\mathbb{T}$ *be a time scale satisfying* (2.4) *and consider on* $[t_0, \infty)$ *the equation*

$$x^{\Delta} + \frac{1}{t(\sigma(t))^2} x^{\sigma} - \frac{1}{t(\sigma(t))^{4/3}} (x^{\sigma})^{1/3} = \frac{t+\sigma(t)}{\mu(t)} e_{-2/\mu}(\sigma(t), t_0), \tag{2.9}$$

which is of the form (1.2) *with*

$$p(t) = \frac{1}{t(\sigma(t))^2}, \quad q_2(t) = \frac{1}{t(\sigma(t))^{4/3}}, \quad f(t) = \frac{t+\sigma(t)}{\mu(t)} e_{-2/\mu}(\sigma(t), t_0), \quad \beta = \frac{1}{3}.$$

As in Example 2.1, (2.2) *is satisfied, and*

$$g_2(t) = \frac{2}{3\sqrt{3}} \left(\frac{1}{t(\sigma(t))^2}\right)^{-1/2} \left(\frac{1}{t(\sigma(t))^{4/3}}\right)^{3/2} = \frac{2}{3\sqrt{3}t\sigma(t)}$$

implies as in Example 2.1 that (2.8) *is satisfied. By Corollary 2.4, each solution of* (2.9) *is oscillatory. One such oscillatory solution is $x = F$, where F is given in Example 2.1. The same arguments apply to the equation*

$$x^{\Delta} + q_2(t)(\sigma(t))^{-2/3}x^{\sigma} - q_2(t)(x^{\sigma})^{1/3} = \frac{t+\sigma(t)}{\mu(t)}e_{-2/\mu}(\sigma(t), t_0),$$

where $q_2 : \mathbb{T} \to \mathbb{R}$ is any rd-continuous function satisfying $q_2(t) > 0$ for all $t \in \mathbb{T}$ and

$$\int_{t_0}^{\infty} q_2(\tau)(\sigma(\tau))^{1/3}\Delta\tau < \infty.$$

Now we supply examples where Corollaries 2.2 and 2.4 cannot be applied but Theorems 2.1 and 2.3 can.

Example 2.3. *Let $\mathbb{T}$ be a time scale satisfying* (2.4) *and consider on $[t_0, \infty)$ the equation*

$$x^{\Delta} - \frac{3\sqrt{3}}{4\sigma(t)}x^{\sigma} + \frac{3\sqrt{3}}{4(\sigma(t))^3}(x^{\sigma})^3 = \frac{t+\sigma(t)}{\mu(t)}e_{-2/\mu}(\sigma(t), t_0), \tag{2.10}$$

which is of the form (1.1) *with*

$$p(t) = \frac{3\sqrt{3}}{4\sigma(t)}, \quad q_1(t) = \frac{3\sqrt{3}}{4(\sigma(t))^3}, \quad f(t) = \frac{t+\sigma(t)}{\mu(t)}e_{-2/\mu}(\sigma(t), t_0), \quad \alpha = 3.$$

While, as in Example 2.1, (2.2) *is satisfied,*

$$(p(t))^{\alpha/(\alpha-1)}(q_1(t))^{1/(1-\alpha)} = \left(\frac{3\sqrt{3}}{4\sigma(t)}\right)^{3/2}\left(\frac{3\sqrt{3}}{4(\sigma(t))^3}\right)^{-1/2} = \frac{3\sqrt{3}}{4}$$

shows that (2.3) *is not satisfied. So we cannot employ Corollary 2.2. However,*

$$\begin{aligned}\int_{t_0}^{t}(f(\tau) + g_1(\tau))\Delta\tau &= te_{-2/\mu}(t, t_0) - t_0 + \frac{2}{3\sqrt{3}}\frac{3\sqrt{3}}{4}(t - t_0) \\ &= te_{-2/\mu}(t, t_0) + \frac{t}{2} - \frac{3t_0}{2}\end{aligned}$$

shows that the first condition in (2.1) *is satisfied. By a similar calculation, the second condition in* (2.1) *is seen to hold as well. Therefore, by Theorem 2.1, every solution of* (2.10) *is oscillatory. One such oscillatory solution is $x = F$, where F is given in Example 2.1. Similarly, we can use Theorem 2.3 but not Corollary 2.4 to find that all solutions of*

$$x^{\Delta} + \frac{3\sqrt{3}}{4\sigma(t)}x^{\sigma} - \frac{3\sqrt{3}}{4(\sigma(t))^{1/3}}(x^{\sigma})^{1/3} = \frac{t+\sigma(t)}{\mu(t)}e_{-2/\mu}(\sigma(t), t_0)$$

are oscillatory.

Theorem 2.5. *If there exists a constant $\delta > 0$ such that*

$$\liminf_{t\to\infty}\int_{t_0}^{t}(f(\tau) + g_3(\tau))\Delta\tau = -\infty \quad \textit{and} \quad \limsup_{t\to\infty}\int_{t_0}^{t}(f(\tau) - g_3(\tau))\Delta\tau = \infty, \tag{2.11}$$

then (1.3) *is oscillatory.*

Proof. We proceed exactly as in the proof of Theorem 2.1, this time setting

$$a_1 = q_1^{1/\alpha}x^{\sigma}, \quad b_1 = \left(\frac{\delta}{\alpha}pq_1^{-1/\alpha}\right)^{1/(\alpha-1)}, \quad a_2 = q_2^{1/\beta}x^{\sigma}, \quad b_2 = \left(\frac{1+\delta}{\beta}pq_2^{-1/\beta}\right)^{1/(\beta-1)}$$

and this time using Lemma 1.1 (i) and (ii) in the subsequent calculation

$$\begin{aligned} x^{\Delta}(t) &= f(t)+\delta p(t)x^{\sigma}(t)-q_1(t)(x^{\sigma}(t))^{\alpha}+q_2(t)(x^{\sigma}(t))^{\beta}-(1+\delta)p(t)x^{\sigma}(t) \\ &= f(t)+\alpha a_1(t)(b_1(t))^{\alpha-1}-(a_1(t))^{\alpha}+(a_2(t))^{\beta}-\beta a_2(t)(b_2(t))^{\beta-1} \\ &\le f(t)+(\alpha-1)(b_1(t))^{\alpha}+(1-\beta)(b_2(t))^{\beta} \\ &= f(t)+g_3(t), \end{aligned}$$

and the rest of the proof is line by line the same as the proof of Theorem 2.1 with g_1 replaced by g_3. ∎

Corollary 2.6. *If* (2.2), (2.3) *and* (2.8) *hold, then* (1.3) *is oscillatory.*

Proof. As (2.2), (2.3) and (2.8) imply (2.11), this follows from Theorem 2.5. ∎

Theorem 2.7. *If there exists an rd-continuous function $p : \mathbb{T} \to \mathbb{R}$ satisfying $p(t) > 0$ for all $t \in \mathbb{T}$ such that*

$$\liminf_{t\to\infty}\int_{t_0}^{t}(f(\tau)+g_4(\tau))\Delta\tau=-\infty \quad \text{and} \quad \limsup_{t\to\infty}\int_{t_0}^{t}(f(\tau)-g_4(\tau))\Delta\tau=\infty, \tag{2.12}$$

then (1.4) *is oscillatory.*

Proof. We proceed exactly as in the proof of Theorem 2.1, this time setting a_1, b_1 and a_2, b_2 as a, b from the proofs of Theorem 2.1 and Theorem 2.3, respectively, and this time using Lemma 1.1 (i) and (ii) in the subsequent calculation

$$\begin{aligned} x^{\Delta}(t) &= f(t)+p(t)x^{\sigma}(t)-q_1(t)(x^{\sigma}(t))^{\alpha}+q_2(t)(x^{\sigma}(t))^{\beta}-p(t)x^{\sigma}(t) \\ &= f(t)+\alpha a_1(t)(b_1(t))^{\alpha-1}-(a_1(t))^{\alpha}+(a_2(t))^{\beta}-\beta a_2(t)(b_2(t))^{\beta-1} \\ &\le f(t)+(\alpha-1)(b_1(t))^{\alpha}+(1-\beta)(b_2(t))^{\beta} \\ &= f(t)+g_4(t), \end{aligned}$$

and the rest of the proof is line by line the same as the proof of Theorem 2.1 with g_1 replaced by g_4. ∎

Corollary 2.8. *If* (2.2), (2.3) *and* (2.8) *hold, then* (1.4) *is oscillatory.*

Proof. As (2.2), (2.3) and (2.8) imply (2.12), this follows from Theorem 2.7. ∎

Example 2.4. *On a time scale $\mathbb{T}$ satisfying* (2.4) *we consider*

$$x^{\Delta}+\frac{1}{t(\sigma(t))^4}(x^{\sigma})^3-\frac{1}{t(\sigma(t))^{4/3}}(x^{\sigma})^{1/3}=\frac{t+\sigma(t)}{\mu(t)}e_{-2/\mu}(\sigma(t),t_0), \tag{2.13}$$

which is of the form (1.4) *with q_1, q_2, f, α, β as in Examples 2.1 and 2.2. We define p as in those two examples, i.e., $p(t) = 1/(t(\sigma(t))^2)$, and then it follows from Example 2.1 that* (2.2) *and* (2.3) *are satisfied, and it follows from Example 2.2 that* (2.8) *is satisfied. By Corollary 2.8, each solution of* (2.13) *is oscillatory. One such oscillatory solution is $x = F$, where F is given in Example 2.1. The same arguments apply to the equation*

$$x^{\Delta}+q(t)(x^{\sigma})^3-q(t)(\sigma(t))^{8/3}(x^{\sigma})^{1/3}=\frac{t+\sigma(t)}{\mu(t)}e_{-2/\mu}(\sigma(t),t_0),$$

where $q : \mathbb{T} \to \mathbb{R}$ is any rd-continuous function satisfying $q(t) > 0$ for all $t \in \mathbb{T}$ and

$$\int_{t_0}^{\infty}q(\tau)(\sigma(\tau))^3\Delta\tau<\infty.$$

We conclude this section by mentioning that the same arguments as above may be used to establish criteria that guarantee that all nonoscillatory solutions of any of the equations (1.1)–(1.4) are bounded. We state the following two such results and supply an example.

Theorem 2.9. *If* (2.3) *holds and*

$$\int_{t_0}^{\infty} |f(\tau)|\Delta\tau < \infty, \tag{2.14}$$

then all nonoscillatory solutions of (1.1) *are bounded.*

Theorem 2.10. *If there exists an rd-continuous function* $p : \mathbb{T} \to \mathbb{R}$ *satisfying* $p(t) > 0$ *for all* $t \in \mathbb{T}$ *such that* (2.3), (2.8) *and* (2.14) *hold, then all nonoscillatory solutions of* (1.1) *are bounded.*

Example 2.5. *Let* $\mathbb{T} \subset (0, \infty)$ *be any time scale and consider the equation*

$$x^\Delta + \frac{1}{(\sigma(t))^3}(x^\sigma)^3 - \frac{2}{(\sigma(t))^{1/3}}(x^\sigma)^{1/3} = 0, \tag{2.15}$$

which is of the form (1.4) *with*

$$q_1(t) = \frac{1}{(\sigma(t))^3}, \quad q_2(t) = \frac{1}{(\sigma(t))^{1/3}}, \quad f(t) \equiv 0, \quad \alpha = 3, \quad \beta = \frac{1}{3}.$$

Clearly, (2.14) *is satisfied. We choose* $p(t) = t^{-2/3}(\sigma(t))^{-5/3}$*. Then*

$$\begin{aligned}\int_{t_0}^{\infty} (p(\tau))^{3/2}(q_1(\tau))^{-1/2}\Delta\tau &= \int_{t_0}^{\infty} (\tau^{-2/3}(\sigma(\tau))^{-5/3})^{3/2}\left(\frac{1}{(\sigma(\tau))^3}\right)^{-1/2}\Delta\tau \\ &= \int_{t_0}^{\infty} \frac{1}{\tau\sigma(\tau)}\Delta\tau = \frac{1}{t_0}\end{aligned}$$

as in Example 2.1, i.e., (2.3) *is satisfied. However,*

$$\begin{aligned}\int_{t_0}^{\infty} (p(\tau))^{-1/2}(q_2(\tau))^{3/2}\Delta\tau &= \int_{t_0}^{\infty} (\tau^{-2/3}(\sigma(\tau))^{-5/3})^{-1/2}\left(\frac{2}{(\sigma(\tau))^{1/3}}\right)^{3/2}\Delta\tau \\ &= 2\sqrt{2}\int_{t_0}^{\infty} (\tau\sigma(t))^{1/3}\Delta\tau = \infty\end{aligned}$$

so that (2.8) *is not satisfied. So Theorem 2.10 is not applicable. In fact, if* $x(t) = t$ *for all* $t \in \mathbb{T}$*, then* x *is a nonoscillatory unbounded solution of* (2.15).

3. Second Order Dynamic Equations

In this section, we give oscillation criteria for second order dynamic equations of the form (1.5)–(1.8). Throughout, we assume (1.9), use the notation (1.10), and fix $t_0 \in \mathbb{T}$.

Theorem 3.1. *If*

$$\begin{cases} \liminf\limits_{t\to\infty} \displaystyle\int_T^t \left(\frac{c}{r(s)} + \frac{1}{r(s)}\int_T^s (f(\tau) + g_1(\tau))\Delta\tau\right)^{1/\gamma}\Delta s = -\infty, \\ \limsup\limits_{t\to\infty} \displaystyle\int_T^t \left(\frac{c}{r(s)} + \frac{1}{r(s)}\int_T^s (f(\tau) - g_1(\tau))\Delta\tau\right)^{1/\gamma}\Delta s = \infty \\ \quad \text{for all} \quad T \geq t_0 \quad \text{and all} \quad c \in \mathbb{R}, \end{cases} \tag{3.1}$$

then (1.5) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.1, we assume that x is an eventually positive solution of (1.5). Hence there exists $t_1 \in \mathbb{T}$, $t_1 \geq t_0$ such that $x(t) > 0$ for all $t \geq t_1$. As in the proof of Theorem 2.1, we may employ Lemma 1.1 (i) to arrive at

$$[r(x^\Delta)^\gamma]^\Delta(t) = f(t) + p(t)x^\sigma(t) - q_1(t)(x^\sigma(t))^\alpha \leq f(t) + g_1(t)$$

for all $t \geq t_1$. Through integration, we obtain for $t \geq t_1$

$$r(t)(x^\Delta(t))^\gamma \leq c + \int_{t_1}^t (f(\tau) + g_1(\tau))\Delta\tau,$$

where $c = r(t_1)(x^\Delta(t_1))^\gamma$. Therefore, for $t \geq t_1$,

$$x(t) \leq x(t_1) + \int_{t_1}^t \left(\frac{c}{r(s)} + \frac{1}{r(s)}\int_{t_1}^s (f(\tau) + g_1(\tau))\Delta\tau\right)^{1/\gamma} \Delta s.$$

Employing the first condition in (3.1), we find

$$0 \leq \liminf_{t\to\infty} x(t) \leq -\infty,$$

which is a contradiction. The case of an eventually negative solution of (1.5) can be dealt with as in the proof of Theorem 2.1, this time employing the second condition in (3.1). ∎

Corollary 3.2. *If*

$$(3.2)\qquad \begin{cases} \liminf\limits_{t\to\infty} \displaystyle\int_{t_0}^t \left(\frac{c}{r(s)} + \frac{1}{r(s)}\int_{t_0}^s (f(\tau) + g_1(\tau))\Delta\tau\right)^{1/\gamma} \Delta s = -\infty, \\ \limsup\limits_{t\to\infty} \displaystyle\int_{t_0}^t \left(\frac{c}{r(s)} + \frac{1}{r(s)}\int_{t_0}^s (f(\tau) - g_1(\tau))\Delta\tau\right)^{1/\gamma} \Delta s = \infty \\ \quad \textit{for all} \quad c \in \mathbb{R}, \end{cases}$$

then (1.5) *is oscillatory.*

Proof. We show that (3.2) and (3.1) are equivalent so that the claim follows from Theorem 3.1. Clearly, (3.1) implies (3.2). Now assume (3.2). Let $T \geq t_0$ and $c \in \mathbb{R}$. Then

$$(3.3)\qquad \int_T^t \left(\frac{c}{r(s)} + \frac{1}{r(s)}\int_T^s (f(\tau) + g_1(\tau))\Delta\tau\right)^{1/\gamma} \Delta s \\ = c_1 + \int_{t_0}^t \left(\frac{c_2}{r(s)} + \frac{1}{r(s)}\int_{t_0}^s (f(\tau) + g_1(\tau))\Delta\tau\right)^{1/\gamma} \Delta s,$$

where

$$c_2 = c - \int_{t_0}^T (f(\tau) + g_1(\tau))\Delta\tau$$

and

$$c_1 = -\int_{t_0}^T \left(\frac{c_2}{r(s)} + \frac{1}{r(s)}\int_{t_0}^s (f(\tau) + g_1(\tau))\Delta\tau\right)^{1/\gamma} \Delta s$$

so that the first condition in (3.1) follows. By a similar argument, the second condition in (3.1) holds as well. ∎

Corollary 3.3. *If* (2.2) *and* (2.3) *hold and*

$$\begin{cases} \liminf_{t\to\infty} \int_T^t \left(\frac{1}{r(s)} \int_T^s f(\tau)\Delta\tau \right)^{1/\gamma} \Delta s = -\infty, \\ \limsup_{t\to\infty} \int_T^t \left(\frac{1}{r(s)} \int_T^s f(\tau)\Delta\tau \right)^{1/\gamma} \Delta s = \infty, \\ \quad \text{for all} \quad T \ge t_0, \end{cases} \tag{3.4}$$

then (1.5) *is oscillatory.*

Proof. We show that (2.2), (2.3) and (3.4) imply (3.2) so that the claim follows from Corollary 3.2. Assume (2.2), (2.3) and (3.4). Let $c \in \mathbb{R}$. By (2.2) and (2.3), there exists $T \ge t_0$ such that

$$\int_{t_0}^T f(\tau)\Delta\tau \le -c - \int_{t_0}^\infty g_1(\tau)\Delta\tau.$$

Then

$$\begin{aligned} &\int_{t_0}^t \left(\frac{c}{r(s)} + \frac{1}{r(s)} \int_{t_0}^s (f(\tau) + g_1(\tau))\Delta\tau \right)^{1/\gamma} \Delta s \\ \le\ & \int_{t_0}^t \left(\frac{c}{r(s)} + \frac{1}{r(s)} \int_{t_0}^\infty g_1(\tau)\Delta\tau + \frac{1}{r(s)} \int_{t_0}^s f(\tau)\Delta\tau \right)^{1/\gamma} \Delta s \\ =\ & \int_{t_0}^t \left(\frac{1}{r(s)} \left[c + \int_{t_0}^\infty g_1(\tau)\Delta\tau + \int_{t_0}^T f(\tau)\Delta\tau \right] + \frac{1}{r(s)} \int_T^s f(\tau)\Delta\tau \right)^{1/\gamma} \Delta s \\ \le\ & \int_{t_0}^t \left(\frac{1}{r(s)} \int_T^s f(\tau)\Delta\tau \right)^{1/\gamma} \Delta s \\ =\ & \tilde{c} + \int_T^t \left(\frac{1}{r(s)} \int_T^s f(\tau)\Delta\tau \right)^{1/\gamma} \Delta s, \end{aligned}$$

where

$$\tilde{c} = \int_{t_0}^T \left(\frac{1}{r(s)} \int_T^s f(\tau)\Delta\tau \right)^{1/\gamma} \Delta s$$

so that the first condition in (3.2) follows. By a similar argument, the second condition in (3.2) holds as well. ∎

Example 3.1. *On a time scale* $\mathbb{T}$ *satisfying* (2.4) *we consider*

$$\left(\frac{t^2(\sigma(t))^2}{t + \sigma(t)} x^\Delta \right)^\Delta - \frac{1}{t(\sigma(t))^2} x^\sigma + \frac{1}{t(\sigma(t))^4} (x^\sigma)^3 = G^\Delta(t), \tag{3.5}$$

where $G(t) = t^2(\sigma(t))^2 e_{-2/\mu}(\sigma(t), t_0)/\mu(t)$, *which is of the form* (1.5) *with*

$$r(t) = \frac{t^2(\sigma(t))^2}{t + \sigma(t)}, \quad p(t) = \frac{1}{t(\sigma(t))^2}, \quad q_1(t) = \frac{1}{t(\sigma(t))^4}, \quad f = G^\Delta, \quad \alpha = 3, \quad \gamma = 1.$$

By Example 2.1, (2.3) *is satisfied. Furthermore, as*

$$\frac{t^2(\sigma(t))^2}{\mu(t)} \ge t^3 \to \infty, \quad t \to \infty \quad \text{since} \quad \mu(t) = \sigma(t) - t,$$

(2.2) *is satisfied as well. Finally, for $T \geq t_0$ we have*

$$\begin{aligned}\int_T^t \left(\frac{1}{r(s)}\int_T^s f(\tau)\Delta\tau\right)^{1/\gamma}\Delta s &= \int_T^t \frac{s+\sigma(s)}{s^2(\sigma(s))^2}(G(s)-G(T))\Delta s \\ &= \int_T^t \frac{s+\sigma(s)}{\mu(s)}e_{-2/\mu}(\sigma(s),t_0)\Delta s - G(T)\int_T^t \frac{s+\sigma(s)}{s^2(\sigma(s))^2}\Delta s \\ &= te_{-2/\mu}(t,t_0) - Te_{-2/\mu}(T,t_0) + G(T)\left(\frac{1}{t^2}-\frac{1}{T^2}\right)\end{aligned}$$

so that (3.4) *is satisfied as well. By Corollary 3.3, each solution of* (3.5) *is oscillatory. One such oscillatory solution is $x = F$, where F is given in Example 2.1.*

Without explicitly stating the results, we mention that the statement of Theorem 3.1 remains true if (1.5) is replaced by (1.6), (1.7) and (1.8) and g_1 is replaced by g_2, g_3 and g_4, respectively. The proofs of these three results are similar to the proof of Theorem 3.1. We give the following counterparts of Corollary 3.3 for equations (1.6)–(1.8).

Corollary 3.4. *If* (2.2), (2.8) *and* (3.4) *hold, then* (1.6) *is oscillatory.*

Corollary 3.5. *If* (2.2), (2.3), (2.8) *and* (3.4) *hold, then* (1.7) *is oscillatory.*

Corollary 3.6. *If* (2.2), (2.3), (2.8) *and* (3.4) *hold, then* (1.8) *is oscillatory.*

As in Theorems 2.9 and 2.10, we can obtain results about the boundedness of all nonoscillatory solutions of (1.5)–(1.8). In particular, we state the following criterion for (1.8).

Theorem 3.7. *If there exists an rd-continuous function $p : \mathbb{T} \to \mathbb{R}$ satisfying $p(t) > 0$ for all $t \in \mathbb{T}$ such that*

$$\int_{t_0}^{\infty}\left(\frac{1}{r(s)}+\frac{1}{r(s)}\int_{t_0}^{s}(|f(\tau)|+g_4(\tau))\Delta\tau\right)^{1/\gamma}\Delta s < \infty,$$

then all nonoscillatory solutions of (1.8) *are bounded.*

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