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THE BERNOULLI INEQUALITY IN UNIFORMLY COMPLETE f-ALGEBRAS WITH IDENTITY

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ABSTRACT. The main purpose of this paper is to establish with a constructive proof the Bernoulli inequality: let A be a uniformly complete f-algebra with e as unit element, let $1 < p < \infty$, then

$$(e + a)^p \geq e + pa$$

for all $a \in A_+$. As an application we prove the Hölder inequality for positive linear functionals on a uniformly complete f-algebra with identity by using the Minkowski inequality.

Key words and phrases: Hölder inequality, Cauchy-Schwartz inequality, f-algebras, Minkowski inequality, Uniformly complete f-algebra with identity.

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1. INTRODUCTION

Let A be a uniformly complete Φ-algebra, that is a uniformly complete f-algebra with an identity element e. The Bernoulli inequality states that :
If $1 < p < \infty$, then

$$(e + f)^p \geq e + pf$$

for all $f \in A_+$.In this paper we prove that the Bernoulli inequality is true in any uniformly complete Φ-algebra.

The Hölder inequality on A states that:
If T is a positive linear functional and if $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$T\left(|fg|\right) \leq \left(T\left(|f|^p\right)\right)^{\frac{1}{p}} \left(T\left(|g|^q\right)\right)^{\frac{1}{q}}$$

for all $f, g \in A$. The Minkowski inequality states:

$$T\left(|f + g|^p\right)^{\frac{1}{p}} \leq \left(T\left(|f|^p\right)\right)^{\frac{1}{p}} + \left(T\left(|g|^p\right)\right)^{\frac{1}{p}}.$$

for all $f, g \in A$. In [3], K. Boulabiar proved the Hölder inequality for a restricted case (p, q are rational numbers), but the second author established the same inequality for the general setting, by using the arithmetic-geometric inequality (the Young inequality) $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q = 1$ and by the convexity of natural logarithm.

We take it for granted that the reader is familiar with the notions of vector lattices (or Riesz spaces) and operators between them. For terminology, notations and concepts that are not explained in this paper, we refer to the standard monographs [1, 4, 5].

2. PRELIMINARIES

In order to avoid unnecessary repetitions we will suppose that all vector lattices and ℓ-algebras under consideration are **Archimedean**.

Let A be a vector lattice, let $0 \leq v \in A$, the sequence $\{a_{n,}\ n = 1, 2, ...\}$ in A is called (v) *relatively uniformly convergent* to $a \in A$ if for every real number $\varepsilon > 0$, there exists a natural number n_ε such that $|a_n - a| \leq \varepsilon v$ for all $n \geq n_\varepsilon$. This will be denoted by $a_n \to a$ (v). If $a_n \to a$ (v) for some $0 \leq v \in A$, then the sequence $\{a_{n,}\ n = 1, 2, ...\}$ is called *(relatively) uniformly convergent* to a, which is denoted by $a_n \to a$*(r.u)*. The notion of (v) *relatively uniformly Cauchy* sequence is defined in the obvious way. A vector lattice is called *relatively uniformly complete* if every relatively uniformly Cauchy sequence in A has a unique limit. Relatively uniformly limits are unique if A is archimedean, see [4, Theorem 63.2].

A linear mapping *T* defined on a vector lattice *A* with values in a vector lattice *B* is called *positive* if $T(A^+) \subset B^+$ (notation $T \in \mathcal{L}^+(A,B)$ or $T \in \mathcal{L}^+(A)$ if $A = B$).

Let us recall some definitions and some basic facts about *f*-algebras. For more information about this field, we refer the reader to [1, 4, 5]. A (real) algebra *A* which is simultaneously a vector lattice such that the partial ordering and the multiplication in A are compatible, so $a, b \in A^+$ implies $ab \in A^+$ is called *lattice-ordered algebra*(briefly ℓ*-algebra*). An ℓ-algebra *A* is called an *f-algebra* if *A* verifies the property that $a \wedge b = 0$ and $c \geq 0$ imply $ac \wedge b = ca \wedge b = 0$. Any *f*-algebra is automatically commutative and has positive squares. An f-algebra with an identity element is called Φ-algebra. Let A be an f-algebra with unit element e, then for every $0 \leq f \in A$, the increasing sequence $0 \leq f_n = f \wedge ne$ converges (relatively) uniformly to f in A (for details, see e.g., [1, Theorem 8.22]).

3. THE MAIN RESULTS

The well-known Bernoulli inequality has many interesting consequences in analysis and integration theory. So it is natural to ask that if this inequality remains true in the vector lattice theory. This answer is affirmative.

Theorem 3.1. *(Bernoulli inequality) Let A be a uniformly complete Φ-algebras with e as identity element. If $1 < p < \infty$, then*

$$(e+f)^p \geq e + pf$$

for all $f \in A_+$.

Proof. We assume that $\frac{1}{p} = \frac{b}{a}$ be a rational such that $1 < p < \infty$ and let $f \in A_+$ such that $(e+pf) \leq me$. Let us define the following sequence

$$f_n = \inf_{\substack{\alpha = \frac{k}{n} \\ 1 \leq k \leq n}} \left\{ \alpha^{-\frac{1}{a}} \left(\frac{b}{a}(e+pf) + \alpha \left(\frac{a-b}{a} \right) e \right) \right\}. \tag{3.1}$$

This sequence will turn out to be the natural approximating Cauchy sequence for $f^{\frac{a}{b}}$. We claim that

$$0 \leq (f_n)^a - (e+pf)^b \leq \frac{C}{n} g$$

for some $0 \leq g \in A$, which means that $(f_n)^a \to (e+pf)^b \ (r.u)$, hence $f_n \to (e+pf)^{\frac{b}{a}} \ (r.u)$. Indeed,

$$\begin{aligned} (f_n)^a - (e+pf)^b &= \inf_{\substack{\alpha = \frac{k}{n} \\ 1 \leq k \leq n}} \left\{ \alpha^{-1} \left(\left(\frac{b}{a}(e+pf) + \alpha \left(\frac{a-b}{a} \right) e \right)^a - \alpha (e+pf)^b \right) \right\} \\ &= a^{-a} \inf_{\substack{\alpha = \frac{k}{n} \\ 1 \leq k \leq n}} \left\{ \alpha^{-1} \left((b(e+pf) + \alpha(a-b)e)^a - a^a \alpha (e+pf)^b \right) \right\}. \end{aligned}$$

Let $P((e+pf), \alpha e) = (b(e+pf) + \alpha(a-b)e)^a - a^a \alpha e (e+pf)^b$, which is a homogeneous polynomial in $(e+pf)$ and αe. Consider the corresponding inhomogeneous polynomial

$$P(X,1) = (bX + (a-b))^a - a^a X^b \in \mathbb{R}[X].$$

Since $P(1) = P'(1) = 0$, we have $P(X) = (X-1)^2 G(X)$ for some $G(X) \in \mathbb{R}[X]$. In other words

$$P(e+pf) = ((e+pf) - \alpha e)^2 G((e+pf), \alpha e).$$

Moreover

$$G((e+pf), \alpha e) = \beta_0 (e+pf) f^{a-2} + \beta_1^{a-2} (e+pf)(\alpha e) + \ldots + \beta_{a-2} (\alpha e)^{a-2}.$$

($\beta_i \in \mathbb{R},\ 0 \leq i \leq a-2$ and β_i does not depend on α).
Since $P(f(e+pf), \alpha e) = ((e+pf) - \alpha e)^2 G((e+pf), \alpha e)$,
then $|P((e+pf), \alpha e)| = ((e+pf) - \alpha e)^2 |G((e+pf), \alpha e)|$. Moreover,

$$G(f, \alpha e) \leq |\beta_0| (e+pf)^{a-2} + |\beta_1| (e+pf)^{a-2} e + \ldots + |\beta_{a-2}| e^{a-2}$$

and

$$-G(f, \alpha e) \leq |\beta_0| (e+pf)^{a-2} + |\beta_1| (e+pf)^{a-2} e + \ldots + |\beta_{a-2}| e^{a-2}.$$

Hence

$$|G(f, \alpha e)| \leq g$$

where $g = |\beta_0| (e + pf)^{a-2} + |\beta_1| (e + pf)^{a-2} e + ... + |\beta_{a-2}| e^{a-2}$.
We recall that by [3, Proposition 4.1]

$$\inf_{\substack{\alpha = \frac{k}{n} \\ 1 \leq k \leq n}} \alpha^{-1} ((e + pf) - \alpha e)^2 \leq n \frac{1}{n^2} e^2 = \frac{1}{n} e.$$

Therefore

$$\left| (f_n)^a - (e + pf)^b \right| = |P ((e + pf), \alpha e)| \leq a^{-a} \frac{1}{n} g.$$

Hence the sequence $\left\{ (f_n)^a - f^b \right\}_{n=1}^{\infty}$ is a uniformly Cauchy sequence in A, which converges to 0. Thus $(f_n)^a \rightarrow (e + pf)^b \ (r.u)$, hence $f_n \rightarrow (e + pf)^{\frac{b}{a}} \ (r.u)$. By formula (3.1), we deduce that

$$f_n \leq \left(\frac{b}{a} (e + pf) + \left(\frac{a - b}{a} \right) e \right).$$

and then, as $n \rightarrow \infty$

$$f^{\frac{b}{a}} \leq \left(\frac{b}{a} (e + pf) + \left(\frac{a - b}{a} \right) e \right). \tag{3.2}$$

Now let $0 \leq f \in A$. Since $f \wedge me \rightarrow f \ (r.u)$, see [1, Theorem 8.22], it follows that

$$(f \wedge me)^{\frac{b}{a}} \leq \left(\frac{b}{a} ((e + pf) \wedge me) + \left(\frac{a - b}{a} \right) e \right).$$

As $m \rightarrow \infty$, then

$$\begin{aligned} (e + pf)^{\frac{b}{a}} &\leq \frac{b}{a} (e + pf) + \left(\frac{a - b}{a} \right) e \\ &= e + f. \end{aligned}$$

Therefore

$$(e + pf)^{\frac{1}{p}} \leq e + f.$$

That is

$$e + pf \leq (e + f)^p . \tag{3.3}$$

Now let $1 < p < \infty$, then there exists a rational sequence p_n such that $p_n > 1$ and $p_n \rightarrow p$. Then by the inequality (3.3)

$$e + p_n f \leq (e + f)^{p_n} .$$

Since $(e + f)^{p_n} \rightarrow (e + f)^p \ (r.u)$ and $p_n f \rightarrow pf \ (r.u)$, then

$$e + pf \leq (e + f)^p$$

which completes the proof. □

Corollary 3.2. *Let A be a uniformly complete Φ-algebra with e as identity element. If $1 < p < \infty$, then*

$$(f + tg)^p \geq f^p + ptf^{p-1} g$$

for all $f, g \in A_+$, for all $t \in \mathbb{R}_+$.

Proof. We assume that f has an inverse in A. Then we apply the Bernoulli inequality to tgf^{-1}, then

$$(f+tg)^p \geq f^p + ptf^{p-1}g. \tag{3.4}$$

Now let $f \in A_+$, by the Birkhoff's inequality the sequence $\left(f \vee \frac{e}{n}\right)_{n\in\mathbb{N}}$ is uniformly convergent to g. Moreover $f \vee \frac{e}{n} \geq \frac{e}{n}$, then $f \vee \frac{e}{n}$ has an inverse in A. Hence by inequality (3.4),

$$\left(\left(f \vee \frac{e}{n}\right) + tg\right)^p \geq \left(f \vee \frac{e}{n}\right)^p + pt\left(f \vee \frac{e}{n}\right)^{p-1} g.$$

As $n \to \infty$, we have

$$(f+tg)^p \geq f^p + ptf^{p-1}g,$$

as required. □

From the Hölder inequality, the Minkowski inequality can be easily proved:
If $T\left(|f+g|^p\right) = 0$, then the inequality is trivial. We assume that $T\left(|f+g|^p\right) > 0$, then

$$\begin{aligned} T\left(|f+g|^p\right) &\leq T\left(|f+g|^{p-1}\left(|f|+|g|\right)\right) \\ &\leq T\left(|f+g|^{p-1}|f|\right) + T\left(|f+g|^{p-1}|g|\right). \end{aligned}$$

Applying the Hölder inequality to each component, and observing that $(p-1)\,q = p$, we obtain

$$\begin{aligned} T\left(|f+g|^p\right) &\leq \left(T\left(|f|^p\right)\right)^{\frac{1}{p}} T\left(|f+g|^{(p-1)q}\right)^{\frac{1}{q}} + \left(T\left(|g|^p\right)\right)^{\frac{1}{p}} T\left(|f+g|^{(p-1)q}\right)^{\frac{1}{q}} \\ &= \left(\left(T\left(|f|^p\right)\right)^{\frac{1}{p}} + \left(T\left(|g|^p\right)\right)^{\frac{1}{p}}\right) T\left(|f+g|^{(p-1)q}\right)^{\frac{1}{q}} \\ &= \left(\left(T\left(|f|^p\right)\right)^{\frac{1}{p}} + \left(T\left(|g|^p\right)\right)^{\frac{1}{p}}\right) T\left(|f+g|^p\right)^{\frac{1}{q}} \end{aligned}$$

and so

$$T\left(|f+g|^p\right)^{\frac{1}{p}} \leq \left(T\left(|f|^p\right)\right)^{\frac{1}{p}} + \left(T\left(|g|^p\right)\right)^{\frac{1}{p}}$$

which gives the desired result.

Next, we deduce Hölder inequality from Minkowski inequality.

Theorem 3.3. *(Hölder inequality) Let A be a uniformly complete Φ-algebra with e as unit element, let p,q be real numbers such that $\frac{1}{p}+\frac{1}{q}=1$ and let $T : A \to \mathbb{R}$ be a non trivial positive functional. Then, the Hölder inequality*

$$T\left(|fg|\right) \leq \left(T\left(|f|^p\right)\right)^{\frac{1}{p}} \left(T\left(|g|^q\right)\right)^{\frac{1}{q}}$$

holds for all $f, g \in A$.

Proof. Let $f, g \in A$, by the previous corollary,

$$pt\,|f|\,|g| \leq \left(|g|^{\frac{1}{p-1}} + t\,|f|\right)^p - |g|^{\frac{p}{p-1}}.$$

Then

$$T\left(pt\,|f|\,|g|\right) \leq T\left(\left(|g|^{\frac{1}{p-1}} + t\,|f|\right)^p\right) - T\left(|g|^{\frac{p}{p-1}}\right).$$

By the Minkowski inequality

$$ptT\left(|f|\,|g|\right) = ptT\left(|fg|\right) \leq \left[T\left(\left(|g|^{\frac{1}{p-1}}\right)^p\right)^{\frac{1}{p}} + \left(T\left(\left(t\,|f|\right)^p\right)\right)^{\frac{1}{p}}\right]^p - T\left(|g|^{\frac{p}{p-1}}\right).$$

Hence by the definition of the derivative

$$\begin{aligned} pT\left(|fg|\right) &\leq \liminf_{t\to 0^+} \frac{\left(\left[T\left(\left(|g|^{\frac{1}{p-1}}\right)^p\right)^{\frac{1}{p}} + \left(T\left(\left(t\,|f|\right)^p\right)\right)^{\frac{1}{p}}\right]^p - T\left(|g|^{\frac{p}{p-1}}\right)\right)}{t}. \\ &= pT\left(|g|^{\frac{1}{p-1}}\right)^{\frac{p-1}{p}} \left(T\left(\left(|f|\right)^p\right)\right)^{\frac{1}{p}} = p\left(T\left(|f|^p\right)\right)^{\frac{1}{p}} \left(T\left(|g|^q\right)\right)^{\frac{1}{q}} \end{aligned}$$

which gives the desired result. □

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