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ON SOME REMARKABLE PRODUCT OF THETA-FUNCTION

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ABSTRACT. On pages 338 and 339 in his first notebook, Ramanujan records eighteen values for a certain product of theta-function. All these have been proved by B. C. Berndt, H. H. Chan and L-C. Zhang [4]. Recently M. S. Mahadeva Naika and B. N. Dharmendra [7, 8] and Mahadeva Naika and M. C. Maheshkumar [9] have obtained general theorems to establish explicit evaluations of Ramanujan's remarkable product of theta-function. Following Ramanujan we define a new function $b_{M,\ N}$ as defined in (1.5). The main purpose of this paper is to establish some new general theorems for explicit evaluations of product of theta-function.

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1. INTRODUCTION

In Chapter 16 of his second notebooks [1, 2, 10], Ramanujan develops the theory of theta-function and his theta–function is defined by

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \tag{1.1}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.2}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty, \tag{1.3}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

On page 338 in his first notebook [10, p.338], Ramanujan defines

$$a_{M,\,N} = \frac{Ne^{\frac{-(N-1)\pi}{4}\sqrt{\frac{M}{N}}}\psi^2\left(e^{-\pi\sqrt{MN}}\right)\varphi^2\left(-e^{-2\pi\sqrt{MN}}\right)}{\psi^2\left(e^{-\pi\sqrt{\frac{M}{N}}}\right)\varphi^2\left(-e^{-2\pi\sqrt{\frac{M}{N}}}\right)}. \tag{1.4}$$

He then, on pages 338 and 339, offers a list of eighteen particular values. All these eighteen values have been established by Berndt, Chan and Zhang [4]. Following Ramanujan we define a new function by

$$b_{M,\,N} = \frac{Ne^{\frac{-(N-1)\pi}{4}\sqrt{\frac{M}{N}}}\psi^2\left(-e^{-\pi\sqrt{MN}}\right)\varphi^2\left(-e^{-2\pi\sqrt{MN}}\right)}{\psi^2\left(-e^{-\pi\sqrt{\frac{M}{N}}}\right)\varphi^2\left(-e^{-2\pi\sqrt{\frac{M}{N}}}\right)}. \tag{1.5}$$

Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with the moduli k, $k' := \sqrt{1-k^2}$, l and $l' := \sqrt{1-l^2}$ respectively, where $0 < k, l < 1$. For a fixed positive integer N, suppose that

$$N\frac{K'}{K} = \frac{L'}{L}. \tag{1.6}$$

Then a modular equation of degree N is a relation between k and l induced by (1.6). Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$. Then we say β is of degree N over α.
Define

$$g_n = 2^{-\frac{1}{4}} q^{-\frac{1}{24}} \chi(-q),$$

where

$$\chi(q) := (-q;q^2)_\infty.$$

Moreover, if $q = e^{-\pi\sqrt{\frac{M}{N}}}$ and β has degree N over α, then

$$g_{\frac{M}{N}} = \left(4\alpha(1-\alpha)^{-2}\right)^{-\frac{1}{24}} \ \textit{and}\ \ g_{MN} = \left(4\beta(1-\beta)^{-2}\right)^{-\frac{1}{24}}. \tag{1.7}$$

The main purpose of this paper is to obtain some new general theorems for the explicit evaluations of remarkable product of theta-function (1.5) and also several new explicit evaluations there from.

2. MAIN THEOREMS

In this section, we establish several new general formulas for explicit evaluations of $b_{M,\,N}$. In the following Theorem (2.1), the equivalent form of (1.5) is obtained.

Theorem 2.1. *We have*

$$b_{M,\,N}=\frac{Ne^{\frac{-(N-1)\pi}{4}\sqrt{\frac{M}{N}}}\psi^2\left(e^{-\pi\sqrt{MN}}\right)\varphi^2\left(-e^{-\pi\sqrt{MN}}\right)}{\psi^2\left(e^{-\pi\sqrt{\frac{M}{N}}}\right)\varphi^2\left(-e^{-\pi\sqrt{\frac{M}{N}}}\right)}, \tag{2.1}$$

where M is any positive rational and N is a positive integer.

Proof. The identity (1.5) can be rewritten as

$$b_{M,\,N}=\frac{Nq^{\frac{N-1}{4}}\psi^2\left(-q^N\right)\varphi^2\left(-q^{2N}\right)}{\psi^2\left(-q\right)\varphi^2\left(-q^2\right)},\quad q=e^{-\pi\sqrt{\frac{M}{N}}}. \tag{2.2}$$

If β is of degree N over α, then using Entry 10 (iii) and Entry 11 (ii) of Chapter 17 of Ramanujan's notebooks [2, pp.122–123] in (2.2), we find that

$$b_{M,\,N}=\frac{N}{m^2}\left(\frac{\beta}{\alpha}\left(\frac{1-\beta}{1-\alpha}\right)^2\right)^{\frac{1}{4}}. \tag{2.3}$$

Using Entry 10 (ii) of Chapter 17 of Ramanujan's notebooks [2, p.122], we have

$$\frac{\varphi^2(-q^N)}{\varphi^2(-q)}=\frac{1}{m}\left(\frac{1-\beta}{1-\alpha}\right)^{\frac{1}{2}}. \tag{2.4}$$

Using Entry 11 (i) of Chapter 17 of Ramanujan's notebooks [2, p.123], we have

$$\frac{q^{\frac{N-1}{4}}\psi^2(q^N)}{\psi^2(q)}=\frac{1}{m}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{4}}. \tag{2.5}$$

Using (2.4) and (2.5) in (2.3) with $q=e^{-\pi\sqrt{\frac{M}{N}}}$, we obtain (2.1). ∎

Theorem 2.2. *We have*

$$b_{2M,\,N}\ b_{\frac{2}{M},\,N}=1. \tag{2.6}$$

$$b_{2M,\,\frac{1}{N}}b_{2M,\,N}=1. \tag{2.7}$$

$$b_{M,\,2N}b_{\frac{1}{N},\,\frac{M}{2}}=1. \tag{2.8}$$

Proof of (2.6). Using equation (2.1), we find that

$$b_{2M,\,N}\ b_{\frac{2}{M},\,N}=N^2e^{\frac{-(N-1)\pi}{4}\left(\sqrt{\frac{2M}{N}}+\sqrt{\frac{2}{MN}}\right)} \tag{2.9}$$

$$\times\frac{\psi^2\left(e^{-\pi\sqrt{2MN}}\right)\varphi^2\left(-e^{-\pi\sqrt{2MN}}\right)\psi^2\left(e^{-\pi\sqrt{\frac{2N}{M}}}\right)\varphi^2\left(-e^{-\pi\sqrt{\frac{2N}{M}}}\right)}{\varphi^2\left(-e^{-\pi\sqrt{\frac{2}{MN}}}\right)\psi^2\left(e^{-\pi\sqrt{\frac{2}{MN}}}\right)\varphi^2\left(-e^{-\pi\sqrt{\frac{2M}{N}}}\right)\psi^2\left(e^{-\pi\sqrt{\frac{2M}{N}}}\right)}.$$

From Entry 27 (ii) of Chapter 16 of Ramanujan's notebooks [2, p.43], we have

$$e^{-\frac{\mu}{2}}\frac{\psi^2\left(e^{-2\mu}\right)}{\varphi^2\left(-e^{-\nu}\right)}=\frac{1}{4}\sqrt{\frac{\nu}{\mu}},\quad \mu\nu=\pi^2. \tag{2.10}$$

Putting $\mu = \pi\sqrt{\frac{M}{2N}}$ and $\nu = \pi\sqrt{\frac{2N}{M}}$ in (2.10), we find that

$$e^{-\frac{\pi}{2}\sqrt{\frac{M}{2N}}}\frac{\psi^2\left(e^{-\pi\sqrt{\frac{2M}{N}}}\right)}{\varphi^2\left(-e^{-\pi\sqrt{\frac{2N}{M}}}\right)} = \frac{1}{4}\sqrt{\frac{2N}{M}}. \tag{2.11}$$

Putting $\mu = \pi\sqrt{\frac{N}{2M}}$ and $\nu = \pi\sqrt{\frac{2M}{N}}$ in (2.10), we deduce that

$$e^{-\frac{\pi}{2}\sqrt{\frac{N}{2M}}}\frac{\psi^2\left(e^{-\pi\sqrt{\frac{2N}{M}}}\right)}{\varphi^2\left(-e^{-\pi\sqrt{\frac{2M}{N}}}\right)} = \frac{1}{4}\sqrt{\frac{2M}{N}}. \tag{2.12}$$

Putting $\mu = \pi\sqrt{\frac{MN}{2}}$ and $\nu = \pi\sqrt{\frac{2}{MN}}$ in (2.10), we find that

$$e^{-\frac{\pi}{2}\sqrt{\frac{MN}{2}}}\frac{\psi^2\left(e^{-\pi\sqrt{2MN}}\right)}{\varphi^2\left(-e^{-\pi\sqrt{\frac{2}{MN}}}\right)} = \frac{1}{4}\sqrt{\frac{2}{MN}}. \tag{2.13}$$

Putting $\mu = \pi\sqrt{\frac{1}{2MN}}$ and $\nu = \pi\sqrt{2MN}$ in (2.10), we deduce that

$$e^{-\frac{\pi}{2}\sqrt{\frac{1}{2MN}}}\frac{\psi^2\left(e^{-\pi\sqrt{\frac{2}{MN}}}\right)}{\varphi^2\left(-e^{-\pi\sqrt{2MN}}\right)} = \frac{1}{4}\sqrt{2MN}. \tag{2.14}$$

Using (2.11), (2.12), (2.13) and (2.14) in (2.9), we obtain the required result (2.6). ∎

Proofs of (2.7) and (2.8) are similar to the proof of (2.6). So we omit the proof.

Corollary 2.1. *We have*

$$b_{2,\,N} = 1. \tag{2.15}$$

Proof. Putting $M = 1$ in (2.6), we obtain the result (2.15). ∎

Theorem 2.3. *We have*

$$b_{2M,\,N} = b_{2N,\,M} = b_{\frac{2}{M},\,\frac{1}{N}} = b_{\frac{2}{N},\,\frac{1}{M}}. \tag{2.16}$$

Proof. Replacing M by 2M in (2.1), we deduce that

$$b_{2M,\,N} = Ne^{\frac{-(N-1)\pi}{4}\sqrt{\frac{2M}{N}}}\frac{\psi^2\left(e^{-\pi\sqrt{2MN}}\right)\varphi^2\left(-e^{-\pi\sqrt{2MN}}\right)}{\psi^2\left(e^{-\pi\sqrt{\frac{2M}{N}}}\right)\varphi^2\left(-e^{-\pi\sqrt{\frac{2M}{N}}}\right)}. \tag{2.17}$$

Putting $\mu = \pi\sqrt{\frac{M}{2N}}$ and $\nu = \pi\sqrt{\frac{2N}{M}}$ in (2.10), we find that

$$\psi^2\left(e^{-\pi\sqrt{\frac{2M}{N}}}\right) = \frac{1}{4}\sqrt{\frac{2N}{M}}e^{\frac{\pi}{2}\sqrt{\frac{M}{2N}}}\varphi^2\left(-e^{-\pi\sqrt{\frac{2N}{M}}}\right). \tag{2.18}$$

Putting $\mu = \pi\sqrt{\frac{N}{2M}}$ and $\nu = \pi\sqrt{\frac{2M}{N}}$ in (2.10), we deduce that

$$\varphi^2\left(-e^{-\pi\sqrt{\frac{2M}{N}}}\right) = 4\sqrt{\frac{N}{2M}}e^{-\frac{\pi}{2}\sqrt{\frac{N}{2M}}}\psi^2\left(e^{-\pi\sqrt{\frac{2N}{M}}}\right). \tag{2.19}$$

Using (2.18) and (2.19) in (2.17), we obtain the first equality of (2.16). ∎

The proofs of the other equalities are similar to the first equality. So we omit the details.

Theorem 2.4. *We have*

$$b_{M,\,2N}=b_{2N,\,M}b_{1,\,\frac{2N}{M}}. \tag{2.20}$$

Proof. Replacing N by 2N in (2.1), we obtain

$$b_{M,\,2N}=2Ne^{\frac{-(2N-1)\pi}{4}\sqrt{\frac{M}{2N}}}\frac{\psi^2\left(e^{-\pi\sqrt{2MN}}\right)\varphi^2\left(-e^{-\pi\sqrt{2MN}}\right)}{\psi^2\left(e^{-\pi\sqrt{\frac{M}{2N}}}\right)\varphi^2\left(-e^{-\pi\sqrt{\frac{M}{2N}}}\right)}. \tag{2.21}$$

Replacing M by 2N and N by M in (2.1), we deduce that

$$b_{2N,\,M}=Me^{\frac{-(M-1)\pi}{4}\sqrt{\frac{2N}{M}}}\frac{\psi^2\left(e^{-\pi\sqrt{2MN}}\right)\varphi^2\left(-e^{-\pi\sqrt{2MN}}\right)}{\psi^2\left(e^{-\pi\sqrt{\frac{2N}{M}}}\right)\varphi^2\left(-e^{-\pi\sqrt{\frac{2N}{M}}}\right)}. \tag{2.22}$$

Using (2.21), (2.22) and (2.1), we obtain the required result. ∎

Theorem 2.5. *We have*

$$\frac{1}{b_{M,\,3}}+b_{M,\,3}=\frac{1}{3}\left(\frac{g_{3M}^6}{g_{\frac{M}{3}}^6}+\frac{g_{\frac{M}{3}}^6}{g_{3M}^6}\right). \tag{2.23}$$

Proof. If β is of degree 3 over α, then using Entry 5 (vii) of Chapter 19 of Ramanujan's notebooks [2, p.230], we find that

$$m^2\left(\frac{\alpha(1-\alpha)^2}{\beta(1-\beta)^2}\right)^{\frac{1}{4}}+\frac{9}{m^2}\left(\frac{\alpha(1-\alpha)^2}{\beta(1-\beta)^2}\right)^{-\frac{1}{4}}=\left(\frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2}\right)^{\frac{1}{4}}+\left(\frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2}\right)^{-\frac{1}{4}}. \tag{2.24}$$

Using (1.7) and (2.3) with $N=3$ in the above identity (2.24), we obtain (2.23). ∎

Corollary 2.2. *We have*

$$b_{6,\,3}=\frac{1}{3}. \tag{2.25}$$

Proof. From the table in Chapter 34 of Ramanujan's notebooks [3, p.200], we have

$$g_{18}=\left(\sqrt{3}+\sqrt{2}\right)^{\frac{1}{3}} \tag{2.26}$$

and

$$g_{\frac{6}{3}}=g_2=1. \tag{2.27}$$

Using (2.26) and (2.27) in (2.23) with $M=6$, we obtain (2.25). ∎

Theorem 2.6. *We have*

$$\frac{1}{\sqrt{b_{M,\,5}}}+\sqrt{b_{M,\,5}}=\frac{1}{\sqrt{5}}\left(\frac{g_{5M}^3}{g_{\frac{M}{5}}^3}+\frac{g_{\frac{M}{5}}^3}{g_{5M}^3}\right). \tag{2.28}$$

Proof. If β is of degree 5 over α, then using Entry 13 (xii) of Chapter 19 of Ramanujan's notebooks [2, p.281], we find that

$$m\left(\frac{\alpha(1-\alpha)^2}{\beta(1-\beta)^2}\right)^{\frac{1}{8}}+\frac{5}{m}\left(\frac{\alpha(1-\alpha)^2}{\beta(1-\beta)^2}\right)^{-\frac{1}{8}}=\left(\frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2}\right)^{\frac{1}{8}}+\left(\frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2}\right)^{-\frac{1}{8}}. \tag{2.29}$$

Using (1.7) and (2.3) with $N=5$ in the above identity (2.29), we obtain (2.28). ∎

Theorem 2.7. *We have*

$$\frac{1}{\sqrt{b_{M,\,7}}}+\sqrt{b_{M,\,7}}=\frac{1}{7}\left(\frac{g_{7M}^6}{g_{\frac{M}{7}}^6}+\frac{g_{\frac{M}{7}}^6}{g_{7M}^6}-8\left(\frac{g_{7M}^2}{g_{\frac{M}{7}}^2}+\frac{g_{\frac{M}{7}}^2}{g_{7M}^2}\right)\right). \tag{2.30}$$

Proof. If β is of degree 7 over α, then using Entry 19 (v) of Chapter 19 of Ramanujan's notebooks [2, p.314], we find that

$$m^2\left(\frac{\alpha(1-\alpha)^2}{\beta(1-\beta)^2}\right)^{\frac{1}{4}}+\frac{49}{m^2}\left(\frac{\alpha(1-\alpha)^2}{\beta(1-\beta)^2}\right)^{-\frac{1}{4}}=\left(\frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2}\right)^{\frac{1}{4}} \tag{2.31}$$
$$+\left(\frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2}\right)^{-\frac{1}{4}}-8\left[\left(\frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2}\right)^{\frac{1}{12}}+\left(\frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2}\right)^{-\frac{1}{12}}\right].$$

Using (1.7) and (2.3) with $N=7$ in the above identity (2.31), we obtain (2.30). ∎

Theorem 2.8. *We have*

$$\frac{1}{\sqrt{b_{M,\,9}}}+\sqrt{b_{M,\,9}}=\frac{1}{3}\left(\frac{g_{9M}^3}{g_{\frac{M}{9}}^3}+\frac{g_{\frac{M}{9}}^3}{g_{9M}^3}-4\right). \tag{2.32}$$

Proof. If γ is of degree 9 over α, then using Entries 3 (x), (xi) of Chapter 20 of Ramanujan's notebooks [2, p.352], we find that

$$\sqrt{mm'}\left(\frac{\alpha(1-\alpha)^2}{\gamma(1-\gamma)^2}\right)^{\frac{1}{16}}+\frac{3}{\sqrt{mm'}}\left(\frac{\alpha(1-\alpha)^2}{\gamma(1-\gamma)^2}\right)^{-\frac{1}{16}} \tag{2.33}$$
$$=\left(\frac{\gamma(1-\alpha)^2}{\alpha(1-\gamma)^2}\right)^{\frac{1}{16}}+\left(\frac{\gamma(1-\alpha)^2}{\alpha(1-\gamma)^2}\right)^{-\frac{1}{16}}.$$

Using (1.7) and (2.3) with $N=9$ in the above identity (2.33), we obtain (2.32). ∎

Theorem 2.9. *We have*

$$\frac{1}{\sqrt{b_{M,\,13}}}+\sqrt{b_{M,\,13}}=\frac{1}{\sqrt{13}}\left(\frac{g_{13M}^3}{g_{\frac{M}{13}}^3}+\frac{g_{\frac{M}{13}}^3}{g_{13M}^3}-14\left(\frac{g_{13M}}{g_{\frac{M}{13}}}+\frac{g_{\frac{M}{13}}}{g_{13M}}\right)\right). \tag{2.34}$$

Proof. If β is of degree 13 over α, then using Entries 8 (iii), (iv) of Chapter 20 of Ramanujan's Notebooks [2, p.376], we find that

$$m\left(\frac{\alpha(1-\alpha)^2}{\beta(1-\beta)^2}\right)^{\frac{1}{8}}+\frac{13}{m}\left(\frac{\alpha(1-\alpha)^2}{\beta(1-\beta)^2}\right)^{-\frac{1}{8}}=\left(\frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2}\right)^{\frac{1}{8}}+\left(\frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2}\right)^{-\frac{1}{8}} \tag{2.35}$$
$$-4\left[\left(\frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2}\right)^{\frac{1}{24}}+\left(\frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2}\right)^{-\frac{1}{24}}\right].$$

Using (1.7) and (2.3) with $N=13$ in the above identity (2.35), we obtain (2.34). ∎

Theorem 2.10. *We have*

$$\frac{1}{\sqrt{b_{M,\,25}}}+\sqrt{b_{M,\,25}} \tag{2.36}$$
$$=\frac{1}{5}\left[\left(\frac{g_{25M}}{g_{\frac{M}{25}}}+\frac{g_{\frac{M}{25}}}{g_{25M}}\right)^3-4\left(\frac{g_{25M}}{g_{\frac{M}{25}}}+\frac{g_{\frac{M}{25}}}{g_{25M}}\right)^2-3\left(\frac{g_{25M}}{g_{\frac{M}{25}}}+\frac{g_{\frac{M}{25}}}{g_{25M}}\right)+8\right].$$

Proof. If γ is of degree 25 over α, then using Entries 15 (i), (ii) of Chapter 19 of Ramanujan's notebooks [2, p.291], we find that

$$\sqrt{mm'}\left(\frac{\alpha(1-\alpha)^2}{\gamma(1-\gamma)^2}\right)^{\frac{1}{16}}+\frac{5}{\sqrt{mm'}}\left(\frac{\alpha(1-\alpha)^2}{\gamma(1-\gamma)^2}\right)^{-\frac{1}{16}}=\left(\frac{\gamma(1-\alpha)^2}{\alpha(1-\gamma)^2}\right)^{\frac{1}{16}} \tag{2.37}$$
$$+\left(\frac{\gamma(1-\alpha)^2}{\alpha(1-\gamma)^2}\right)^{-\frac{1}{16}}-2\left[\left(\frac{\gamma(1-\alpha)^2}{\alpha(1-\gamma)^2}\right)^{\frac{1}{48}}+\left(\frac{\gamma(1-\alpha)^2}{\alpha(1-\gamma)^2}\right)^{-\frac{1}{48}}\right].$$

Using (1.7) and (2.3) with $N = 25$ in the above identity (2.37), we obtain (2.36). ∎

3. $b_{M,\,N}$ AND MIXED MODULAR EQUATIONS

We shall employ certain type of mixed modular equations to establish several values of $b_{M,\,N}$.

Theorem 3.1. *We have*

$$\frac{1}{\sqrt{A_M}}+\sqrt{A_M}=\frac{1}{3}\left[\left(V_M+V_M^{-1}\right)^3-7(V_M+V_M^{-1})\right], \tag{3.1}$$

where

$$A_M=b_{M,\,3}b_{25M,\,3}\ \ and\ \ V_M=\frac{g_{\frac{M}{3}}g_{\frac{25M}{3}}}{g_{3M}g_{75M}}.$$

Proof. If β, γ and δ are third, fifth and fifteenths degree over α respectively, then using Entries 11 (x) and (xi) of Chapter 20 of Ramanujan's notebooks [2, p.384], we find that

$$mm'\left(\frac{\alpha\gamma(1-\alpha)^2(1-\gamma)^2}{\beta\delta(1-\beta)^2(1-\delta)^2}\right)^{\frac{1}{8}}+\frac{9}{mm'}\left(\frac{\alpha\gamma(1-\alpha)^2(1-\gamma)^2}{\beta\delta(1-\beta)^2(1-\delta)^2}\right)^{-\frac{1}{8}} \tag{3.2}$$
$$=\left(\frac{\beta\delta(1-\alpha)^2(1-\gamma)^2}{\alpha\gamma(1-\beta)^2(1-\delta)^2}\right)^{\frac{1}{8}}+\left(\frac{\beta\delta(1-\alpha)^2(1-\gamma)^2}{\alpha\gamma(1-\beta)^2(1-\delta)^2}\right)^{-\frac{1}{8}}$$
$$-4\left(\frac{\beta\delta(1-\alpha)^2(1-\gamma)^2}{\alpha\gamma(1-\beta)^2(1-\delta)^2}\right)^{\frac{1}{24}}-4\left(\frac{\beta\delta(1-\alpha)^2(1-\gamma)^2}{\alpha\gamma(1-\beta)^2(1-\delta)^2}\right)^{-\frac{1}{24}}.$$

Using (1.7) and (2.3) with $N = 3$ in the above identity (3.2), we obtain (3.1). ∎

Theorem 3.2. *We have*

$$\frac{1}{\sqrt{A_M}}+\sqrt{A_M}=V_M^3+V_M^{-3}+4, \tag{3.3}$$

where

$$A_M=\frac{b_{25M,\,3}}{b_{M,\,3}}\ \ and\ \ V_M=\frac{g_{\frac{M}{3}}g_{75M}}{g_{3M}g_{\frac{25M}{3}}}.$$

Proof. If β, γ and δ are of third, fifth and fifteenths degree over α respectively, then by using Entries 11 (viii) and (ix) of Chapter 20 of Ramanujan's notebooks [2, p.384], we find that

$$\sqrt{\frac{m'}{m}}\left(\frac{\beta\gamma(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)^2(1-\delta)^2}\right)^{\frac{1}{16}}-\sqrt{\frac{m}{m'}}\left(\frac{\beta\gamma(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)^2(1-\delta)^2}\right)^{-\frac{1}{16}} \tag{3.4}$$
$$=\left(\frac{\alpha\delta(1-\beta)^2(1-\gamma)^2}{\beta\gamma(1-\alpha)^2(1-\delta)^2}\right)^{\frac{1}{16}}+\left(\frac{\alpha\delta(1-\beta)^2(1-\gamma)^2}{\beta\gamma(1-\alpha)^2(1-\delta)^2}\right)^{-\frac{1}{16}}.$$

Using (1.7) and (2.3) with $N = 3$ in the above identity (3.4), we obtain (3.3). ∎

Theorem 3.3. *We have*

$$\frac{1}{\sqrt{A_M}} + \sqrt{A_M} = V_M^3 + V_M^{-3} + 4, \tag{3.5}$$

where

$$A_M = \frac{b_{9M,\,5}}{b_{M,\,5}} \ \ and \ \ V_M = \frac{g_{\frac{M}{5}} g_{45M}}{g_{5M} g_{\frac{9M}{5}}}.$$

Proof. Using (1.7) and (2.3) with $N = 5$ in the above identity (3.4), we obtain (3.5). ∎

Theorem 3.4. *We have*

$$\frac{1}{\sqrt{A_M}} + \sqrt{A_M} = \left(V_M + V_M^{-1}\right)^3 + \left(V_M + V_M^{-1}\right), \tag{3.6}$$

where

$$A_M = \frac{b_{49M,\,3}}{b_{M,\,3}} \ \ and \ \ V_M = \frac{g_{\frac{M}{3}} g_{147M}}{g_{3M} g_{\frac{49M}{3}}}.$$

Proof. If β, γ and δ are of third, seventh and twenty-first degree over α respectively, then by using Entries 13 (i) and (ii) of Chapter 20 of Ramanujan's notebooks [2, p.401], we find that

$$\begin{aligned} &\frac{m'}{m}\left(\frac{\beta\gamma(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)^2(1-\delta)^2}\right)^{\frac{1}{8}} + \frac{m}{m'}\left(\frac{\beta\gamma(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)^2(1-\delta)^2}\right)^{-\frac{1}{8}} \\ &= \left(\frac{\alpha\delta(1-\beta)^2(1-\gamma)^2}{\beta\gamma(1-\alpha)^2(1-\delta)^2}\right)^{\frac{1}{8}} + \left(\frac{\alpha\delta(1-\beta)^2(1-\gamma)^2}{\beta\gamma(1-\alpha)^2(1-\delta)^2}\right)^{-\frac{1}{8}} \\ &+4\left(\frac{\alpha\delta(1-\beta)^2(1-\gamma)^2}{\beta\gamma(1-\alpha)^2(1-\delta)^2}\right)^{\frac{1}{24}} + 4\left(\frac{\alpha\delta(1-\beta)^2(1-\gamma)^2}{\beta\gamma(1-\alpha)^2(1-\delta)^2}\right)^{-\frac{1}{24}}. \end{aligned} \tag{3.7}$$

Using (1.7) and (2.3) with $N = 3$ in the above identity (3.7), we obtain (3.6). ∎

Theorem 3.5. *We have*

$$\frac{1}{\sqrt{A_M}} + \sqrt{A_M} = \left(V_M + V_M^{-1}\right)^3 + \left(V_M + V_M^{-1}\right), \tag{3.8}$$

where

$$A_M = \frac{b_{9M,\,7}}{b_{M,\,7}} \ \ and \ \ V_M = \frac{g_{\frac{M}{7}} g_{63M}}{g_{7M} g_{\frac{9M}{7}}}.$$

Proof. Using (1.7) and (2.3) with $N = 7$ in the above identity (3.7), we obtain (3.8). ∎

Theorem 3.6. *We have*

$$\frac{1}{\sqrt{A_M}} + \sqrt{A_M} = \left(V_M + V_M^{-1}\right)^3 + 4\left(V_M + V_M^{-1}\right)^2 + 5\left(V_M + V_M^{-1}\right), \tag{3.9}$$

where

$$A_M = \frac{b_{169M,\,3}}{b_{M,\,3}} \ \ and \ \ V_M = \frac{g_{\frac{M}{3}} g_{507M}}{g_{3M} g_{\frac{169M}{3}}}.$$

Proof. If β, γ and δ are of third, thirteenth and thirty-ninth degree over α respectively, then by using Entry 19 (iv) of Chapter 20 of Ramanujan's notebooks [2, p.426], we find that

$$\begin{aligned} &\sqrt{\frac{m'}{m}}\left(\frac{\beta\gamma(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)^2(1-\delta)^2}\right)^{\frac{1}{16}} + \sqrt{\frac{m}{m'}}\left(\frac{\beta\gamma(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)^2(1-\delta)^2}\right)^{-\frac{1}{16}} \\ &= \left(\frac{\alpha\delta(1-\beta)^2(1-\gamma)^2}{\beta\gamma(1-\alpha)^2(1-\delta)^2}\right)^{\frac{1}{16}} + \left(\frac{\alpha\delta(1-\beta)^2(1-\gamma)^2}{\beta\gamma(1-\alpha)^2(1-\delta)^2}\right)^{-\frac{1}{16}} \end{aligned} \tag{3.10}$$

$$+2\left(\frac{\alpha\delta(1-\beta)^2(1-\gamma)^2}{\beta\gamma(1-\alpha)^2(1-\delta)^2}\right)^{\frac{1}{48}}+2\left(\frac{\alpha\delta(1-\beta)^2(1-\gamma)^2}{\beta\gamma(1-\alpha)^2(1-\delta)^2}\right)^{-\frac{1}{48}}.$$

Using (1.7) and (2.3) with $N=3$ in the above identity (3.10), we obtain (3.9). ∎

Theorem 3.7. *We have*

$$\frac{1}{\sqrt{A_M}}+\sqrt{A_M}=\left(V_M+V_M^{-1}\right)^3+4\left(V_M+V_M^{-1}\right)^2+5\left(V_M+V_M^{-1}\right), \tag{3.11}$$

where

$$A_M=\frac{b_{9M,\,13}}{b_{M,\,13}}\ \ and\ \ V_M=\frac{g_{\frac{M}{13}}g_{117M}}{g_{13M}g_{\frac{9M}{13}}}.$$

Proof. Using (1.7) and (2.3) with $N=13$ in the above identity (3.10), we obtain (3.11). ∎

Theorem 3.8. *We have*

$$\frac{1}{\sqrt{A_M}}+\sqrt{A_M}=\left(V_M+V_M^{-1}\right)^3+4\left(V_M+V_M^{-1}\right)^2+5\left(V_M+V_M^{-1}\right)+4, \tag{3.12}$$

where

$$A_M=\frac{b_{49M,\,5}}{b_{M,\,5}}\ \ and\ \ V_M=\frac{g_{\frac{M}{5}}g_{245M}}{g_{5M}g_{\frac{49M}{5}}}.$$

Proof. If β, γ and δ are of fifth, seventh and thirty-fifth degree over α respectively, then by using Entries 18 (vi), (vii) of Chapter 20 of Ramanujan's notebooks [2, p.426], we find that

$$\begin{aligned}\sqrt{\frac{m'}{m}}\left(\frac{\beta\gamma(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)^2(1-\delta)^2}\right)^{\frac{1}{16}}-\sqrt{\frac{m}{m'}}\left(\frac{\beta\gamma(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)^2(1-\delta)^2}\right)^{-\frac{1}{16}}\\ =\left(\frac{\alpha\delta(1-\beta)^2(1-\gamma)^2}{\beta\gamma(1-\alpha)^2(1-\delta)^2}\right)^{\frac{1}{16}}+\left(\frac{\alpha\delta(1-\beta)^2(1-\gamma)^2}{\beta\gamma(1-\alpha)^2(1-\delta)^2}\right)^{-\frac{1}{16}}\\ +2\left(\frac{\alpha\delta(1-\beta)^2(1-\gamma)^2}{\beta\gamma(1-\alpha)^2(1-\delta)^2}\right)^{\frac{1}{48}}+2\left(\frac{\alpha\delta(1-\beta)^2(1-\gamma)^2}{\beta\gamma(1-\alpha)^2(1-\delta)^2}\right)^{-\frac{1}{48}}.\end{aligned} \tag{3.13}$$

Using (1.7) and (2.3) with $N=5$ in the above identity (3.13), we obtain (3.12). ∎

Theorem 3.9. *We have*

$$\frac{1}{\sqrt{A_M}}+\sqrt{A_M}=\left(V_M+V_M^{-1}\right)^3+4\left(V_M+V_M^{-1}\right)^2+5\left(V_M+V_M^{-1}\right)+4, \tag{3.14}$$

where

$$A_M=\frac{b_{25M,\,7}}{b_{M,\,7}}\ \ and\ \ V_M=\frac{g_{\frac{M}{7}}g_{175M}}{g_{7M}g_{\frac{25M}{7}}}.$$

Proof. Using (1.7) and (2.3) with $N=7$ in the above identity (3.13), we obtain (3.14). ∎

Theorem 3.10. *We have*

$$\frac{1}{\sqrt{A_M}}+\sqrt{A_M}=\frac{1}{3}\left[\left(V_M+V_M^{-1}\right)^3-4\left(V_M+V_M^{-1}\right)^2-3\left(V_M+V_M^{-1}\right)+12\right], \tag{3.15}$$

where

$$A_M=b_{M,\,3}\ b_{121M,\,3}\ \ and\ \ V_M=\frac{g_{\frac{M}{3}}g_{\frac{121M}{3}}}{g_{3M}g_{363M}}.$$

Proof. If β, γ and δ are third, eleventh and thirty-third degree over α respectively, then using Entries 14 (i) and (ii) of Chapter 20 of Ramanujan's notebooks [2, p.408], we find that

$$\sqrt{mm'}\left(\frac{\alpha\gamma(1-\alpha)^2(1-\gamma)^2}{\beta\delta(1-\beta)^2(1-\delta)^2}\right)^{\frac{1}{16}}+\frac{3}{\sqrt{mm'}}\left(\frac{\alpha\gamma(1-\alpha)^2(1-\gamma)^2}{\beta\delta(1-\beta)^2(1-\delta)^2}\right)^{-\frac{1}{16}} \tag{3.16}$$
$$=\left(\frac{\beta\delta(1-\alpha)^2(1-\gamma)^2}{\alpha\gamma(1-\beta)^2(1-\delta)^2}\right)^{\frac{1}{16}}+\left(\frac{\beta\delta(1-\alpha)^2(1-\gamma)^2}{\alpha\gamma(1-\beta)^2(1-\delta)^2}\right)^{-\frac{1}{16}}$$
$$-2\left(\frac{\beta\delta(1-\alpha)^2(1-\gamma)^2}{\alpha\gamma(1-\beta)^2(1-\delta)^2}\right)^{\frac{1}{48}}-2\left(\frac{\beta\delta(1-\alpha)^2(1-\gamma)^2}{\alpha\gamma(1-\beta)^2(1-\delta)^2}\right)^{-\frac{1}{48}}.$$

Using (1.7) and (2.3) with $N=3$ in the above identity (3.16), we obtain (3.15). ∎

Theorem 3.11. *We have*

$$\frac{1}{\sqrt{A_M}}+\sqrt{A_M}=V_M^3+V_M^{-3}, \tag{3.17}$$

where

$$A_M=\frac{b_{81M,\,3}}{b_{M,\,3}}\ \ and\ \ V_M=\frac{g_{\frac{M}{3}}g_{243M}}{g_{3M}g_{27M}}.$$

Proof. If β, γ and δ are of third, ninth and twenty-seventh degree over α respectively, then by using Entry 5 (i) of Chapter 20 of Ramanujan's notebooks [2, p.360] and its reciprocal equation, we find that

$$\sqrt{\frac{m''}{m}}\left(\frac{\beta\gamma(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)^2(1-\delta)^2}\right)^{\frac{1}{16}}-\sqrt{\frac{m}{m''}}\left(\frac{\beta\gamma(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)^2(1-\delta)^2}\right)^{-\frac{1}{16}} \tag{3.18}$$
$$=\left(\frac{\alpha\delta(1-\beta)^2(1-\gamma)^2}{\beta\gamma(1-\alpha)^2(1-\delta)^2}\right)^{\frac{1}{16}}-\left(\frac{\alpha\delta(1-\beta)^2(1-\gamma)^2}{\beta\gamma(1-\alpha)^2(1-\delta)^2}\right)^{-\frac{1}{16}}.$$

Using (1.7) and (2.3) with $N=3$ in the above identity (3.18), we obtain (3.17). ∎

Theorem 3.12. *We have*

$$\frac{1}{\sqrt{A_M}}+\sqrt{A_M}=V_M^3+V_M^{-3}, \tag{3.19}$$

where

$$A_M=\frac{b_{9M,\,3}}{b_{M,\,9}}\ \ and\ \ V_M=\frac{g_{\frac{M}{9}}g_{M}}{g_{9M}g_{81M}}.$$

Proof. Using (1.7) and (2.3) with $N=9$ in the above identity (3.18), we obtain (3.19). ∎

4. EXPLICIT EVALUATIONS OF $b_{M,\,N}$

In this section, we establish several explicit evaluations of $b_{M,\,N}$.

Theorem 4.1. *We have*

$$b_{5,\,3}=\frac{\sqrt{190-105\sqrt{3}}-\sqrt{186-105\sqrt{3}}}{2}, \tag{4.1}$$

$$b_{8,\,3}=\left(\sqrt{43+24\sqrt{3}}-\sqrt{42+24\sqrt{3}}\right)^{\frac{1}{2}}, \tag{4.2}$$

$$b_{20,\,3} = \left(2 - \frac{\sqrt{15}}{2}\right)^{\frac{1}{2}} \left(47 - 21\sqrt{5}\right)^{\frac{1}{2}}, \tag{4.3}$$

$$b_{22,\,3} = 6 + \sqrt{33} - 2\sqrt{17 + 3\sqrt{33}}, \tag{4.4}$$

$$b_{34,\,3} = 33 - 8\sqrt{17}, \tag{4.5}$$

$$b_{38,\,3} = 22 + 3\sqrt{57} - 2\sqrt{249 + 33\sqrt{57}}, \tag{4.6}$$

$$b_{42,\,3} = \frac{2+\sqrt{2}}{3} - \sqrt{\frac{1+2\sqrt{2}}{3}}, \tag{4.7}$$

$$b_{46,\,3} = \sqrt{3057 + 1248\sqrt{6}} - \sqrt{3056 + 1248\sqrt{6}}, \tag{4.8}$$

$$b_{66,\,3} = \frac{\sqrt{33}}{3} + \sqrt{42 + 10\sqrt{33}}\left(\frac{1}{24}\sqrt{66} - \frac{3}{8}\sqrt{2}\right), \tag{4.9}$$

$$b_{70,\,3} = \sqrt{54105 + 5280\sqrt{105}} - \sqrt{54104 + 5280\sqrt{105}}, \tag{4.10}$$

$$b_{110,\,3} = \sqrt{2537329 + 540960\sqrt{22}} - \sqrt{2537328 + 540960\sqrt{22}} \tag{4.11}$$

and

$$b_{174,\,3} = \frac{11\sqrt{6}}{3} + \left(8\sqrt{6} - 23\right)\sqrt{\frac{99 + 42\sqrt{6}}{29}}. \tag{4.12}$$

Proof of (4.1). :From the table in Chapter 34 of Ramanujan's notebooks [3, pp.190, 341], we find that

$$G_{15}G_{\frac{5}{3}} = \sqrt{2}. \tag{4.13}$$

Using Entries 12 (vi) and (vii) of Chapter 17 of Ramanujan's notebooks [2, p.124] in Entry 5(ii) of Chapter 19 of Ramanujan's notebooks [2, p.230], we find that

$$g_n^2 g_{9n}^2 = \frac{1}{2G_nG_{9n}}\left[G_n^3G_{9n}^3 + \sqrt{G_n^6G_{9n}^6 - 2}\right]. \tag{4.14}$$

Using (4.13) in (4.14) with $n = \frac{5}{3}$, we find that

$$g_{\frac{5}{3}}g_{15} = \frac{\sqrt{3}+1}{2}. \tag{4.15}$$

From Theorem 4.1(i) in [6], we have

$$2\sqrt{2}\left[g_n^3g_{9n}^3 + g_n^{-3}g_{9n}^{-3}\right] = \frac{g_{9n}^6}{g_n^6} - \frac{g_n^6}{g_{9n}^6}. \tag{4.16}$$

Using (4.15) in (4.16) with $n = \frac{5}{3}$, we deduce that

$$\frac{g_{15}^6}{g_{\frac{5}{3}}^6} = \sqrt{\frac{1710 - 945\sqrt{3}}{4}} + \sqrt{\frac{1706 - 945\sqrt{3}}{4}}. \tag{4.17}$$

Using (4.17) in (2.23) with $M = 5$, we obtain the required result (4.1). ∎

Proof of (4.2). From Theorem 4.5(i) in [6], we have

$$g_{\frac{8}{3}} g_{24} = \sqrt{\sqrt{3}+1}. \tag{4.18}$$

Using (4.18) in (4.16) with $n = \frac{8}{3}$, we find that

$$\frac{g_{24}^{12}}{g_{\frac{8}{3}}^{12}} = \left(44 + 27\sqrt{3}\right) + \left(33 + 18\sqrt{3}\right)\sqrt{2}. \tag{4.19}$$

Using (4.19) in (2.23) with $M = 8$, we obtain the required result (4.2). ∎

Proof of (4.4). From the table in Chapter 34 of Ramanujan's notebooks [3, p.201], we have

$$g_{66} = \left(\sqrt{3}+\sqrt{2}\right)^{\frac{1}{4}} \left(7\sqrt{2}+3\sqrt{11}\right)^{\frac{1}{12}} \left(\sqrt{\frac{7+\sqrt{33}}{8}} + \sqrt{\frac{\sqrt{33}-1}{8}}\right)^{\frac{1}{2}}. \tag{4.20}$$

Using (4.20) in (4.16) with $n = \frac{22}{3}$, we find that

$$g_{\frac{22}{3}} = \left(\sqrt{3}+\sqrt{2}\right)^{\frac{1}{4}} \left(7\sqrt{2}+3\sqrt{11}\right)^{\frac{1}{12}} \left(\sqrt{\frac{7+\sqrt{33}}{8}} - \sqrt{\frac{\sqrt{33}-1}{8}}\right)^{\frac{1}{2}}. \tag{4.21}$$

Using (4.20) and (4.21) in (2.23) with $M = 22$, we obtain the required result (4.4). ∎

Proof of the identity (4.3) is similar to the proof of the identity (4.1) and proofs of the identities (4.5)-(4.12) being similar to the proof of the identity (4.4). So we omit the details.

Theorem 4.2. *We have*

$$b_{6,\,5} = \left(\sqrt{2}-1\right)^2, \tag{4.22}$$

$$b_{38,\,5} = \left(17 - 12\sqrt{2}\right)^2 \tag{4.23}$$

and

$$b_{62,\,5} = \left(28 + 9\sqrt{10} - 3\sqrt{177+56\sqrt{10}}\right)^2. \tag{4.24}$$

Proof of (4.22). From the table in Chapter 34 of Ramanujan's notebooks [3, p.200], we have

$$g_{30} = \left(2+\sqrt{5}\right)^{\frac{1}{6}} \left(3+\sqrt{10}\right)^{\frac{1}{6}}. \tag{4.25}$$

From Theorem 4.1(ii) in [6], we have

$$2\left[g_n^2 g_{25n}^2 + g_n^{-2} g_{25n}^{-2}\right] = \frac{g_{25n}^3}{g_n^3} - \frac{g_n^3}{g_{25n}^3}. \tag{4.26}$$

Using (4.25) in (4.26) with $n = \frac{6}{5}$, we find that

$$g_{\frac{6}{5}}^6 = \left(2+\sqrt{5}\right)\left(-3+\sqrt{10}\right). \tag{4.27}$$

Using (4.25) and (4.27) in (2.28) with $M = 6$, we obtain the required result (4.22). ∎

As the proofs of the identities (4.23)-(4.24) being similar to the proof of the identity (4.22). So we omit the details.

Theorem 4.3. *We have*

$$b_{6,\,7} = 5 - 2\sqrt{6}, \tag{4.28}$$

$$b_{10,\,7} = \left(\sqrt{10} - 3\right)^2, \tag{4.29}$$

$$b_{14,\,7} = -\sqrt{\frac{2}{9}} + \sqrt{\frac{7 + 2\sqrt{14}}{49}} \tag{4.30}$$

and

$$b_{18,\,7} = \left(\sqrt{34 + 24\sqrt{2}} - \sqrt{33 + 24\sqrt{2}}\right)^2. \tag{4.31}$$

Proof of (4.28). From the table in Chapter 34 of Ramanujan's notebooks [3, p.201], we have

$$g_{42} = \left(2\sqrt{2} + \sqrt{7}\right)^{\frac{1}{6}} \left(\frac{\sqrt{3} + \sqrt{7}}{2}\right)^{\frac{1}{2}}. \tag{4.32}$$

From Theorem 4.1(iii) in [6], we have

$$\begin{aligned} &16\sqrt{2}\left[g_n^9 g_{49n}^9 + g_n^{-9} g_{49n}^{-9}\right] + 168\left[g_n^6 g_{49n}^6 + g_n^{-6} g_{49n}^{-6}\right] \\ &+336\sqrt{2}\left[g_n^3 g_{49n}^3 + g_n^{-3} g_{49n}^{-3}\right] + 658 = \frac{g_{49n}^{12}}{g_n^{12}} + \frac{g_n^{12}}{g_{49n}^{12}}. \end{aligned} \tag{4.33}$$

Using (4.32) in (4.33) with $n = \frac{6}{7}$, we find that

$$g_{\frac{6}{7}} = \left(2\sqrt{2} + \sqrt{7}\right)^{\frac{1}{6}} \left(\frac{-\sqrt{3} + \sqrt{7}}{2}\right)^{\frac{1}{2}}. \tag{4.34}$$

Using (4.32) and (4.34) in (2.30) with $M = 6$, we obtain the required result (4.28). ∎

As the proofs of the identities (4.29)-(4.31) being similar to the proof of the identity (4.28). So we omit the details.

Theorem 4.4. *We have*

$$b_{10,\,9} = \left(\sqrt{10 + 4\sqrt{6}} - \sqrt{9 + 4\sqrt{6}}\right)^2, \tag{4.35}$$

$$b_{22,\,9} = \left(\sqrt{253 + 44\sqrt{33}} - \sqrt{252 + 44\sqrt{33}}\right)^2 \tag{4.36}$$

and

$$b_{58,\,9} = \left(\sqrt{117370 + 47916\sqrt{6}} - \sqrt{117369 + 47916\sqrt{6}}\right)^2. \tag{4.37}$$

Proof of (4.35). From the table in Chapter 34 of Ramanujan's notebooks [3, p.202], we have

$$g_{90} = \left[(2 + \sqrt{5})(\sqrt{5} + \sqrt{6})\right]^{\frac{1}{6}} \left(\sqrt{\frac{3 + \sqrt{6}}{4}} + \sqrt{\frac{\sqrt{6} - 1}{4}}\right). \tag{4.38}$$

Using (4.38) in an identity from a page 145 of Chapter 4 in [5, eq(4.7.12),p.145] with changing q to $-q$, we obtain

$$g_{\frac{10}{9}} = \left[(2+\sqrt{5})(\sqrt{5}+\sqrt{6})\right]^{\frac{1}{6}} \left(\sqrt{\frac{3+\sqrt{6}}{4}} - \sqrt{\frac{\sqrt{6}-1}{4}}\right). \tag{4.39}$$

Using (4.38) and (4.39) in (2.32) with $M = 10$, we obtain the required result (4.35). ∎

As the proofs of the identities (4.36)-(4.37) being similar to the proof of the identity (4.35). So we omit the details.

Theorem 4.5. *We have*

$$b_{6,\,13} = \left(3-2\sqrt{2}\right)^2 \tag{4.40}$$

and

$$b_{10,\,13} = \left(\sqrt{65}-8\right)^2. \tag{4.41}$$

Proof of (4.40). From the table in Chapter 34 of Ramanujan's notebooks [3, p.202], we have

$$g_{78} = \left(\frac{3+\sqrt{13}}{2}\right)^{\frac{1}{2}} \left(5+\sqrt{26}\right)^{\frac{1}{6}}. \tag{4.42}$$

Using (4.42) in Entry 41 of Chapter 38 of Ramanujan's notebooks [3, p.378], we find that

$$g_{\frac{6}{13}} = \left(\frac{\sqrt{13}-3}{2}\right)^{\frac{1}{2}} \left(5+\sqrt{26}\right)^{\frac{1}{6}}. \tag{4.43}$$

Using (4.42) and (4.43) in (2.34) with $M = 6$, we obtain the required result (4.40). ∎

As the proof of the identity (4.41) being similar to the proof of the identity (4.40). So we omit the details.
Remark: $b_{M,\,N}$ are units in some quadratic field. We retain the details for our future paper.

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