The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 5, Issue 1, Article 12, pp. 1-17, 2008

ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THIRD ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Received 27 August, 2007; accepted 21 January, 2008; published 16 May, 2008.
Communicated by: S. Ntouyas

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ABSTRACT. This paper is concerned with the asymptotic behavior of solutions of nonlinear differential equations of the third order with quasiderivatives. Mainly, we present the necessary and sufficient conditions for the existence of nonoscillatory solutions with specified asymptotic behavior as t tends to infinity. These conditions are presented as integral criteria and involve only the coefficients of investigated differential equations. In order to prove some of the results, we use a topological approach based on the Schauder fixed point theorem.

Key words and phrases: Nonlinear differential equation, Third order, Nonoscillatory solution, Asymptotic behavior, Quasi-derivatives.

2000 *Mathematics Subject Classification.* 34K11, 34K25.

ISSN (electronic): 1449-5910

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The first author was supported by the grant VVGS 25/07–08 and the second one by the grant 1/3815/06 of the Grant Agency of Slovak Republic (VEGA)..

1. INTRODUCTION

Consider the third-order nonlinear differential equations with quasiderivatives of the form

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q(t)f(x(t)) = 0, \quad t \geq a. \tag{N}$$

Throughout the paper, we always assume that

$$r, p, q \in C([a,\infty), \mathbb{R}),\ r(t) > 0,\ p(t) > 0,\ q(t) > 0 \text{ on } [a,\infty), \tag{H1}$$

$$f \in C(\mathbb{R}, \mathbb{R}), \quad f(u)u > 0 \quad \text{for} \quad u \neq 0. \tag{H2}$$

For the sake of brevity, we introduce the following notation:

$$x^{[0]} = x, \quad x^{[1]} = \frac{1}{r}x', \quad x^{[2]} = \frac{1}{p}\left(\frac{1}{r}x'\right)' = \frac{1}{p}\left(x^{[1]}\right)', \quad x^{[3]} = \left(\frac{1}{p}\left(\frac{1}{r}x'\right)'\right)' = \left(x^{[2]}\right)'.$$

The functions $x^{[i]}$, i=0, 1, 2, 3, we call the *quasiderivatives* of x. In addition to (H1) and (H2), we will sometimes assume that

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} > 0. \tag{H3}$$

By a *solution* of an equation of the form (N), we mean a function $w : [a,\infty) \to \mathbb{R}$ such that quasiderivatives $w^{[i]}(t)$, $0 \leq i \leq 3$, exist and are continuous on the interval $[a,\infty)$ and it satisfies the equation (N) for all $t \geq a$. A solution w of equation (N) is said to be *proper* if it satisfies the condition

$$\sup\{|w(s)| : t \leq s < \infty\} > 0 \quad \text{for any } t \geq a.$$

A proper solution is said to be *oscillatory* if it has a sequence of zeros converging to ∞; otherwise it is said to be *nonoscillatory*. Furthermore, equation (N) is called *oscillatory* if it has at least one nontrivial oscillatory solution, and *nonoscillatory* if all its nontrivial solutions are nonoscillatory.

Let $\mathcal{N}(N)$ denote the set of all proper nonoscillatory solutions of equation (N). The set $\mathcal{N}(N)$ can be divided into the following four classes in the same way as in [1, 2, 4]:

$$\mathcal{N}_0 = \{x \in \mathcal{N}(N),\ \exists\, t_x : x(t)x^{[1]}(t) < 0,\ x(t)x^{[2]}(t) > 0 \text{ for } t \geq t_x\}$$
$$\mathcal{N}_1 = \{x \in \mathcal{N}(N),\ \exists\, t_x : x(t)x^{[1]}(t) > 0,\ x(t)x^{[2]}(t) < 0 \text{ for } t \geq t_x\}$$
$$\mathcal{N}_2 = \{x \in \mathcal{N}(N),\ \exists\, t_x : x(t)x^{[1]}(t) > 0,\ x(t)x^{[2]}(t) > 0 \text{ for } t \geq t_x\}$$
$$\mathcal{N}_3 = \{x \in \mathcal{N}(N),\ \exists\, t_x : x(t)x^{[1]}(t) < 0,\ x(t)x^{[2]}(t) < 0 \text{ for } t \geq t_x\}$$

Furthermore, with respect to asymptotic behavior of the solutions in the classes $\mathcal{N}_0 - \mathcal{N}_3$, we can divide the class $\mathcal{N}_0$ $[\mathcal{N}_3]$ into the following two disjoint subclasses

$$\mathcal{N}_0^B = \left\{x \in \mathcal{N}_0 : \lim_{t\to\infty} x(t) = l_x \neq 0\right\}, \quad \mathcal{N}_0^0 = \left\{x \in \mathcal{N}_0 : \lim_{t\to\infty} x(t) = 0\right\}$$
$$\left[\mathcal{N}_3^B = \left\{x \in \mathcal{N}_3 : \lim_{t\to\infty} x(t) = l_x \neq 0\right\}, \quad \mathcal{N}_3^0 = \left\{x \in \mathcal{N}_3 : \lim_{t\to\infty} x(t) = 0\right\}\right]$$

and also the class $\mathcal{N}_1$ $[\mathcal{N}_2]$ into the following two disjoint subclasses

$$\mathcal{N}_1^B = \left\{x \in \mathcal{N}_1 : \lim_{t\to\infty} |x(t)| = M_x < \infty\right\}, \quad \mathcal{N}_1^\infty = \left\{x \in \mathcal{N}_1 : \lim_{t\to\infty} |x(t)| = \infty\right\}$$
$$\left[\mathcal{N}_2^B = \left\{x \in \mathcal{N}_2 : \lim_{t\to\infty} |x(t)| = M_x < \infty\right\}, \quad \mathcal{N}_2^\infty = \left\{x \in \mathcal{N}_2 : \lim_{t\to\infty} |x(t)| = \infty\right\}\right].$$

If solution $x \in \mathcal{N}_0$, then its quasiderivatives satisfy the inequality $x^{[i]}(t)x^{[i+1]}(t) < 0$ for $i = 0, 1, 2$, for all sufficiently large t. Using the terminology as in [1, 2, 4, 12, 14], we call it a *Kneser solution*.

There are a lot of results (see, e.g., [1, 3, 4, 5, 15]) devoted to the oscillatory and asymptotic behavior of the linear case of equation (N), namely of the linear differential equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q(t)x(t) = 0, \quad t \geq a.$$

The nonlinear case, i.e. equation (N), has been largely studied in [1, 2, 4, 14]. In particular, many authors investigated the oscillatory and asymptotic properties of solutions of differential equations of the third order with deviating argument. Among the extensive literature on this field, we refer to [8, 9, 10, 12, 13, 16, 17, 18] and to the references contained therein.

The aim of this paper is to study the asymptotic behavior of nonoscillatory solutions of equation (N). For this purpose, we divide all proper nonoscillatory solutions of (N) into the above mentioned several classes with respect to their asymptotic behavior. Such a classification plays an important role in the study of the qualitative behavior of equation (N). Further, we use a topological approach based on the following fixed point theorem:

Theorem 1.1. *(Schauder fixed point theorem) Let Ω be a non-empty closed convex subset of a normed linear space E and let $T : \Omega \to \Omega$ be a continuous mapping such that $T(\Omega)$ is relatively compact in E. Then T has at least one fixed point in Ω.*

After the summarization of some known definitions and notation, in Section 2 we present the necessary and sufficient conditions for the existence of nonoscillatory solutions of equation (N) with a specified asymptotic behavior as t tends to infinity. These results are interesting by themselves by virtue of their necessary and sufficient character. Furthermore, our results are presented as integral criteria that involve only the functions p, r, q. Several examples illustrating the main theorems are also provided.

We point out that our assumption on the nonlinearity f is related with its behavior only in a neighbourhood of infinity. Moreover, not only monotonicity of the nonlinearity f is unnecessary but also no assumptions on the behavior of f in $\mathbb{R}$ are required. We also remark that the condition (H3) is needed only for some results concerning the class $\mathcal{N}_2$.

We close the introduction with the following notation:

$$I(u_i) = \int_a^\infty u_i(t)\,dt, \qquad I(u_i, u_j) = \int_a^\infty u_i(t) \int_a^t u_j(s)\,ds\,dt, \quad i, j = 1, 2$$

$$I(u_i, u_j, u_k) = \int_a^\infty u_i(t) \int_a^t u_j(s) \int_a^s u_k(z)\,dz\,ds\,dt, \quad i, j, k = 1, 2, 3,$$

where $u_i,\ i = 1, 2, 3$, are continuous positive functions on the interval $[a, \infty)$.
For simplicity, we will sometimes write $u(\infty)$ instead of $\lim_{t\to\infty} u(t)$.

2. MAIN RESULTS

We begin our consideration with several results concerning the asymptotic behavior of solutions of equation (N) in the class $\mathcal{N}_1$. The following result provides sufficient conditions for the existence of solutions in the class $\mathcal{N}_1^B$.

Theorem 2.1. *Let one of the following conditions be satisfied:*

(a) $I(p, r) < \infty$ *and* $I(q) < \infty$,

(b) $I(p,q) < \infty$ *and* $I(r) < \infty$.

Then equation (N) *has a bounded solution* x *in the class* $\mathcal{N}_1$*, i.e.* $\mathcal{N}_1^B \neq \emptyset$.

Proof. We prove the existence of a positive bounded solution of equation (N) in the class $\mathcal{N}_1$.

Suppose (a). Let $K = \max\{f(u) : u \in [c,d]\}$ where c, d are constants such that $0 < c < d$ and let $t_0 \geq a$ be such that

$$\int_{t_0}^{\infty} p(s) \int_{t_0}^{s} r(v)\, dv\, ds \leq 1 \quad \text{and} \quad \int_{t_0}^{\infty} q(s)\, ds \leq \frac{d-c}{K}. \tag{2.1}$$

Let us define the set

$$\Delta = \{u \in C([t_0,\infty),\mathbb{R}) : c \leq u(t) \leq d\},$$

where $C([t_0,\infty),\mathbb{R})$ will denote the Banach space of all continuous and bounded functions defined on the interval $[t_0,\infty)$ with the sup norm $\|u\| = \sup\{|u(t)|, t \geq t_0\}$. Clearly, Δ is a non-empty closed, convex and bounded subset of $C([t_0,\infty),\mathbb{R})$. For every $u \in \Delta$ we consider a mapping $T : \Delta \to C([t_0,\infty),\mathbb{R})$ given by

$$x_u(t) = (Tu)(t) = c + \int_{t_0}^{t} r(\tau) \int_{\tau}^{\infty} p(s) \int_{t_0}^{s} q(z) f(u(z))\, dz\, ds\, d\tau, \quad t \geq t_0.$$

In order to apply the Schauder fixed point theorem (Theorem 1.1) to the mapping T, it is sufficient to prove that T maps Δ into itself, T is a continuous mapping in Δ and $T(\Delta)$ is a relatively compact set in $C([t_0,\infty),\mathbb{R})$.

(i) T maps Δ into Δ. In fact, $x_u(t) \geq c$ and in view of (2.1), we obtain

$$\begin{aligned}
x_u(t) &= c + \int_{t_0}^{t} r(\tau) \int_{\tau}^{\infty} p(s) \int_{t_0}^{s} q(z) f(u(z))\, dz\, ds\, d\tau \\
&\leq c + K \int_{t_0}^{t} r(\tau) \int_{\tau}^{\infty} p(s) \int_{t_0}^{s} q(z)\, dz\, ds\, d\tau \\
&\leq c + K \left(\int_{t_0}^{\infty} q(z)\, dz \right) \left(\int_{t_0}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s)\, ds\, d\tau \right) \\
&= c + K \left(\int_{t_0}^{\infty} q(z)\, dz \right) \left(\int_{t_0}^{\infty} p(s) \int_{t_0}^{s} r(\tau)\, d\tau\, ds \right) \leq d.
\end{aligned}$$

(ii) T is continuous. Let $\{u_n\}$, $n \in N$ be a sequence of elements of Δ such that $\lim_{n\to\infty} \|u_n - u\| = 0$. Since Δ is closed, $u \in \Delta$. By the definition of T and in view of (2.1), we see that

$$|(Tu_n)(t) - (Tu)(t)| \leq \int_{t_0}^{\infty} G_n(z)\, dz, \quad t \geq t_0$$

where

$$G_n(z) = q(z)|f(u_n(z)) - f(u(z))|.$$

Thus

$$\|Tu_n - Tu\| \leq \int_{t_0}^{\infty} G_n(z)\, dz. \tag{2.2}$$

It is easy to see that $\lim_{n\to\infty} G_n(z) = 0$ which is a consequence of the convergence $u_n \to u$ in $C([t_0,\infty),\mathbb{R})$ and that the following inequality holds

$$\int_{t_0}^{\infty} G_n(z)\, dz \leq 2K \int_{t_0}^{\infty} q(z)\, dz.$$

Since $I(q) < \infty$, the Lebesgue's dominated convergence theorem yields

$$\lim_{n\to\infty} \int_{t_0}^{\infty} G_n(z)\, dz = 0.$$

Consequently, from (2.2) we have $\lim_{n\to\infty} \|Tu_n - Tu\| = 0,$ i.e. T is continuous.

(iii) $T(\Delta)$ is relatively compact. It suffices to show that the family of functions $T(\Delta)$ is uniformly bounded and equicontinuous on the interval $[t_0, \infty)$. The uniform boundedness of $T(\Delta)$ immediately follows from the facts that $T(\Delta) \subseteq \Delta$ and Δ is a bounded subset of $C([t_0, \infty), \mathbb{R})$. Now, we prove that $T(\Delta)$ is an equicontinuous family of functions on $[t_0, \infty)$. This will be accomplished if we show that for any given $\varepsilon > 0$, the interval $[t_0, \infty)$ can be decomposed into a finite number of subintervals in such a way that on each subinterval all functions of the family $T(\Delta)$ have oscillations less than ε (see, e.g. [11], p. 13).

Let $u \in \Delta$ and $t_2 > t_1 \geq t_0$. Then, taking into account (2.1), we have

$$\begin{aligned} (2.3)\quad |(Tu)(t_2) - (Tu)(t_1)| &\leq K \int_{t_1}^{t_2} r(\tau) \int_{\tau}^{\infty} p(s) \int_{t_0}^{s} q(z)\, dz\, ds\, d\tau \\ &\leq (d-c) \int_{t_1}^{\infty} p(s) \int_{t_1}^{s} r(\tau)\, d\tau\, ds \to 0 \qquad \text{as } t_1 \to \infty. \end{aligned}$$

We conclude from the above inequalities that for any given $\varepsilon > 0$ there exists $t^* > t_0$ such that for all $u \in \Delta$, we have

$$|(Tu)(t_2) - (Tu)(t_1)| < \varepsilon \qquad \text{if } t_2 > t_1 \geq t^*.$$

This shows that the oscillations of all functions of the family $T(\Delta)$ on $[t^*, \infty)$ are less than ε. Now, let $t_0 \leq t_1 < t_2 \leq t^*$. In view of (2.1), (2.3) and the fact that $I(p) < \infty$ (it follows from $I(p, r) < \infty$), we get

$$|(Tu)(t_2) - (Tu)(t_1)| \leq (d-c) \int_{t_1}^{t_2} r(\tau) \int_{\tau}^{\infty} p(s)\, ds\, d\tau \leq (d-c) M_1 |t_2 - t_1|$$

where $M_1 = \max \left\{ r(\tau) \int_{\tau}^{\infty} p(s)\, ds : \tau \in [t_0,\, t^*] \right\}$. Hence, for any given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u \in \Delta$

$$|(Tu)(t_2) - (Tu)(t_1)| < \varepsilon \quad \text{if } \ |t_2 - t_1| < \delta.$$

Consequently, the interval $[t_0, \infty)$ can be divided into a finite number of subintervals on which every function of the family $T(\Delta)$ has oscillation less than ε. Therefore $T(\Delta)$ is an equicontinuous family of functions on $[t_0, \infty)$. Hence $T(\Delta)$ is relatively compact.

Now, the Schauder theorem yields the existence of a fixed point $x \in \Delta$ for the mapping T such that

$$x(t) = c + \int_{t_0}^{t} r(\tau) \int_{\tau}^{\infty} p(s) \int_{t_0}^{s} q(z) f(x(z))\, dz\, ds\, d\tau, \quad t \geq t_0.$$

As

$$x^{[1]}(t) = \int_{t}^{\infty} p(s) \int_{t_0}^{s} q(z) f(x(z))\, dz\, ds > 0 \quad \text{and} \quad x^{[2]}(t) = -\int_{t_0}^{t} q(z) f(x(z))\, dz < 0,$$

it is clear that x is a positive bounded solution of equation (N) in the class $\mathcal{N}_1$, i.e. $x \in \mathcal{N}_1^B$.

Suppose (b). The proof is the same as in the case (a) except for some minor changes. Therefore, we omit it. This completes the proof. ∎

Remark 2.1. We observe that the existence of a negative bounded solution of equation (N) in the class $\mathcal{N}_1$ can be proved by using similar arguments. This fact about negative solution also holds for some next results.

The following example shows the meaning of Theorem 2.1.

Example 2.1. *We consider the differential equation*

$$\left(\left(t^2+1\right)\left(\left(t^2+1\right)x'(t)\right)'\right)' + \frac{8t}{\left(2t^2+1\right)^2}\, x^2(t)\operatorname{sgn} x(t) = 0\,, \quad t \geq 2. \tag{2.4}$$

This is the equation of the form (N) where $r(t) = p(t) = \dfrac{1}{t^2+1}$, $q(t) = \dfrac{8t}{\left(2t^2+1\right)^2}$ and $f(u) = u^2 \operatorname{sgn} u$. It is easy to verify that the assumptions of Theorem 2.1 are fulfilled and so equation (2.4) *has a solution in the class $\mathcal{N}_1^B$. One such solution is the function $x(t) = \dfrac{2t^2+1}{t^2+1}$.*

We also have the following result for the solutions of equation (N) in the class $\mathcal{N}_1^B$.

Theorem 2.2. *If $I(p,q) = \infty$, then $\mathcal{N}_1^B = \varnothing$.*

Proof. Assume that $x \in \mathcal{N}_1^B$. Without loss of generality, we suppose that there exists $T \geq a$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) < 0$ for all $t \geq T$. Let $x(\infty) = M_x < \infty$. As x is a positive increasing function and f is a continuous function on the interval $[x(T), M_x]$, there exists a positive constant m such that

$$m = \min\left\{f(u) : u \in [x(T), M_x]\right\}. \tag{2.5}$$

Integrating equation (N) twice in $[T,t]$, we obtain

$$x^{[1]}(t) = x^{[1]}(T) + x^{[2]}(T)\int_T^t p(s)\,ds - \int_T^t p(s)\int_T^s q(k)f(x(k))\,dk\,ds$$

and therefore

$$x^{[1]}(t) < x^{[1]}(T) - \int_T^t p(s)\int_T^s q(k)f(x(k))\,dk\,ds.$$

Using this inequality with (2.5), we have

$$x^{[1]}(t) < x^{[1]}(T) - m\int_T^t p(s)\int_T^s q(k)\,dk\,ds,$$

which gives a contradiction as $t \to \infty$, because function $x^{[1]}(t)$ is a positive for all $t \geq T$. The case $x(t) < 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) > 0$ for all $t \geq T^*$ (where $T^* \geq a$) can be treated similarly. ∎

From Theorem 2.1 and Theorem 2.2, one gets immediately the following result.

Corollary 2.3. *Let $I(r) < \infty$. Then a necessary and sufficient condition for equation (N) to have a solution x in the class $\mathcal{N}_1^B$ is that $I(p,q) < \infty$.*

For solutions in the class $\mathcal{N}_1^\infty$, the following theorem holds.

Theorem 2.4. *If $I(r) < \infty$, then $\mathcal{N}_1^\infty = \varnothing$.*

Proof. Let $x \in \mathcal{N}_1^\infty$. Without loss of generality, we suppose that there exists $T \geq a$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) < 0$ for all $t \geq T$. As $x^{[1]}$ is a positive decreasing function, we have that $x'(t) \leq x^{[1]}(T)r(t)$ for all $t \geq T$. Integrating this inequality in $[T,t]$, we obtain

$$x(t) \leq x(T) + x^{[1]}(T)\int_T^t r(s)\,ds,$$

which gives a contradiction as $t \to \infty$, because x is an unbounded solution. If $x(t) < 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) > 0$ for all $t \geq T^*$ (where $T^* \geq a$), similar arguments hold. ∎

Combining Corollary 2.3 and Theorem 2.4, we get the following.

Corollary 2.5. *Let $I(r) < \infty$. Then a necessary and sufficient condition for equation (N) to have a solution x in the class $\mathcal{N}_1$ is that $I(p, q) < \infty$.*

Now, we turn our attention to the solutions in the class $\mathcal{N}_2$. The following results concern the existence of solutions of equation (N) in the class $\mathcal{N}_2^B$.

Theorem 2.6. *Let one of the following conditions be satisfied:*

(a) $I(r, p) < \infty$ *and* $I(q) < \infty$,
(b) $I(q, p) < \infty$ *and* $I(r) < \infty$.

Then equation (N) has a bounded solution x in the class $\mathcal{N}_2$, i.e $\mathcal{N}_2^B \neq \varnothing$.

Proof. We prove the existence of a positive bounded solution of equation (N) in the class $\mathcal{N}_2$.

Suppose (a). Let $K = \max\{f(u) : u \in [c, d]\}$ where c, d are constants such that $0 < c < d$ and let $t_0 \geq a$ be such that

$$\int_{t_0}^{\infty} r(\tau) \int_{t_0}^{\tau} p(s)\, ds\, d\tau \leq 1 \quad \text{and} \quad \int_{t_0}^{\infty} q(s)\, ds \leq \frac{d-c}{K}. \tag{2.6}$$

Let us define the set Δ in the same way as in the proof of Theorem 2.1. For every $u \in \Delta$ we consider a mapping $T_1 : \Delta \to C([t_0, \infty), \mathbb{R})$ given by

$$x_u(t) = (T_1 u)(t) = c + \int_{t_0}^{t} r(\tau) \int_{t_0}^{\tau} p(s) \int_{s}^{\infty} q(z) f(u(z))\, dz\, ds\, d\tau, \quad t \geq t_0.$$

Taking into account (2.6) and using similar arguments as in the proof of Theorem 2.1, it is easy to verify that T_1 maps Δ into itself, T_1 is a continuous mapping in Δ and $T_1(\Delta)$ is a relatively compact set in $C([t_0, \infty), \mathbb{R})$. Consequently, by the Schauder fixed point theorem there exists a fixed point $x \in \Delta$ such that

$$x(t) = c + \int_{t_0}^{t} r(\tau) \int_{t_0}^{\tau} p(s) \int_{s}^{\infty} q(z) f(x(z))\, dz\, ds\, d\tau, \quad t \geq t_0.$$

As

$$x^{[1]}(t) = \int_{t_0}^{t} p(s) \int_{s}^{\infty} q(z) f(x(z))\, dz\, ds > 0 \quad \text{and} \quad x^{[2]}(t) = \int_{t}^{\infty} q(z) f(x(z))\, dz > 0,$$

it is clear that x is a positive bounded solution of equation (N) in the class $\mathcal{N}_2$, i.e. $x \in \mathcal{N}_2^B$.

Suppose (b). Using similar arguments as in the case (a), we are led to the conclusion that $\mathcal{N}_2^B \neq \varnothing$. Therefore, we omit it. The proof is now complete. ∎

Example 2.2. *Let us consider the differential equation*

$$\left(\frac{1}{t}\left(t^2 x'(t)\right)'\right)' + \frac{8t}{(1+t^2)^3 \operatorname{arctg}^3 t}\, x^3(t) = 0, \quad t \geq 1. \tag{2.7}$$

As $I(q, p) < \infty$ and $I(r) < \infty$, Theorem 2.6 yields that equation (2.7) *has a solution in the class $\mathcal{N}_2^B$. Really, one such solution is the function $x(t) = \operatorname{arctg} t$.*

Theorem 2.7. *If $I(q) = \infty$, then $\mathcal{N}_2^B = \varnothing$.*

Proof. Assume that $x \in \mathcal{N}_2^B$. Without loss of generality, we suppose that there exists $T \geq a$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$ for all $t \geq T$. Let $x(\infty) = M_x < \infty$. As x is a positive increasing function and f is a continuous function on the interval $[x(T), M_x]$, there exists a positive constant m such that

$$m = \min\left\{f(u) : u \in [x(T), M_x]\right\}. \tag{2.8}$$

By integrating equation (N) in $[T, t]$, we get

$$x^{[2]}(t) = x^{[2]}(T) - \int_T^t q(s)f(x(s))\,ds.$$

This equality with (2.8) yields that

$$x^{[2]}(t) < x^{[2]}(T) - m\int_T^t q(s)\,ds,$$

which gives a contradiction as $t \to \infty$, because function $x^{[2]}(t)$ is a positive for all $t \geq T$. The case $x(t) < 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) < 0$ for all $t \geq T^*$ (where $T^* \geq a$) can be treated in the similar way. ∎

From Theorem 2.6 and Theorem 2.7, one gets immediately the following result.

Corollary 2.8. *Let $I(r,p) < \infty$. Then a necessary and sufficient condition for equation (N) to have a solution x in the class $\mathcal{N}_2^B$ is that $I(q) < \infty$.*

The following result also holds.

Theorem 2.9. *Let (H3) hold. If $I(q) = \infty$, then $\mathcal{N}_2^\infty = \varnothing$.*

Proof. Let $x \in \mathcal{N}_2^\infty$. Without loss of generality, we assume that there exists $T \geq a$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$ for all $t \geq T$. Because $\left(x^{[2]}(t)\right)' = -q(t)f(x(t)) < 0$ for all $t \geq T$, $x^{[2]}(t)$ is a positive decreasing function and thus $0 \leq x^{[2]}(\infty) < \infty$. As $x(\infty) = \infty$, the assumption (H3) implies that there exists a positive number K and $T_1 \geq T$ such that

$$\frac{f(x(t))}{x(t)} \geq K \qquad \text{for all } t \geq T_1. \tag{2.9}$$

Integrating equation (N) in the interval $[T_1, t]$, we obtain

$$x^{[2]}(T_1) - x^{[2]}(t) = \int_{T_1}^t q(s)f(x(s))\,ds. \tag{2.10}$$

In view of (2.9) and the fact that x is an increasing function, the equality (2.10) gives

$$x^{[2]}(T_1) - x^{[2]}(t) \geq K\int_{T_1}^t q(s)x(s)\,ds \geq Kx(T_1)\int_{T_1}^t q(s)\,ds.$$

When $t \to \infty$, we get a contradiction with $I(q) = \infty$. The case $x(t) < 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) < 0$ for all $t \geq T^*$ (where $T^* \geq a$) can be treated similarly. ∎

Corollary 2.8 and Theorem 2.9 give the following result.

Corollary 2.10. *Let (H3) hold and $I(r,p) < \infty$. Then a necessary and sufficient condition for equation (N) to have a solution x in the class $\mathcal{N}_2$ is that $I(q) < \infty$.*

In the sequel, we present several results regarding the asymptotic behavior of solutions of equation (N) in the class $\mathcal{N}_3$. For the existence of solutions of (N) in the class $\mathcal{N}_3^0$, we state the following result.

Theorem 2.11. *If $I(r) < \infty$ and $I(p,q) < \infty$, then equation (N) has a solution x in the class $\mathcal{N}_3$ such that $\lim_{t\to\infty} x(t) = 0$, i.e. $\mathcal{N}_3^0 \neq \varnothing$.*

Proof. We prove the existence of a positive solution of equation (N) in the class $\mathcal{N}_3$ which approaches to zero as $t \to \infty$.

Let $t_0 \geq a$ be such that

$$K \int_{t_0}^{\infty} p(\tau) \int_{t_0}^{\tau} q(s)\, ds\, d\tau \leq 1, \tag{2.11}$$

where

$$K = \max\left\{ f(u) : u \in \left[0,\, 2\int_{t_0}^{\infty} r(s)\, ds\right]\right\}.$$

For convenience, we make use of the following notation:

$$H(t) = \int_t^{\infty} r(s)\, ds, \quad t \geq t_0.$$

Let us define the set

$$\Delta = \{u \in C([t_0,\infty),\mathbb{R}) \,:\, H(t) \leq u(t) \leq 2H(t)\},$$

where $C([t_0,\infty),\mathbb{R})$ denotes the Banach space of all continuous and bounded functions defined on the interval $[t_0,\infty)$ with the sup norm $\|u\| = \sup\{|u(t)|,\, t \geq t_0\}$. Clearly, Δ is a non-empty closed, convex and bounded subset of $C([t_0,\infty),\mathbb{R})$. For every $u \in \Delta$ we consider a mapping $T_2 : \Delta \to C([t_0,\infty),\mathbb{R})$ given by

$$x_u(t) = (T_2u)(t) = H(t) + \int_t^{\infty} r(\tau) \int_{t_0}^{\tau} p(s) \int_{t_0}^{s} q(z) f(u(z))\, dz\, ds\, d\tau, \quad t \geq t_0.$$

In order to apply to the mapping T_2 the Schauder fixed point theorem (Theorem 1.1), it is sufficient to prove that T_2 maps Δ into itself, T_2 is a continuous mapping in Δ and $T_2(\Delta)$ is a relatively compact set in $C([t_0,\infty),\mathbb{R})$.

(i) T_2 maps Δ into Δ. In fact, $x_u(t) \geq H(t)$ and in view of (2.11), we have

$$\begin{aligned}
x_u(t) &= H(t) + \int_t^{\infty} r(\tau) \int_{t_0}^{\tau} p(s) \int_{t_0}^{s} q(z) f(u(z))\, dz\, ds\, d\tau \\
&\leq H(t) + K \int_t^{\infty} r(\tau) \int_{t_0}^{\tau} p(s) \int_{t_0}^{s} q(z)\, dz\, ds\, d\tau \\
&\leq H(t) + K \left(\int_{t_0}^{\infty} p(s) \int_{t_0}^{s} q(z)\, dz\, ds\right)\left(\int_t^{\infty} r(\tau)\, d\tau\right) \leq H(t) + H(t) = 2H(t).
\end{aligned}$$

(ii) T_2 is continuous. Let $\{u_n\}$, $n \in N$ be a sequence of elements of Δ such that $\lim_{n\to\infty} \|u_n - u\| = 0$. Since Δ is closed, $u \in \Delta$. From the definition of T_2, we obtain

$$|(T_2u_n)(t) - (T_2u)(t)| \leq \int_{t_0}^{\infty} G_n(\tau)\, d\tau, \quad t \geq t_0$$

where

$$G_n(\tau) = r(\tau) \int_{t_0}^{\tau} p(s) \int_{t_0}^{s} q(z) |f(u_n(z)) - f(u(z))|\, dz\, ds.$$

Thus

$$\|T_2u_n - T_2u\| \leq \int_{t_0}^{\infty} G_n(\tau)\, d\tau. \tag{2.12}$$

It is easy to see that $\lim_{n\to\infty} G_n(\tau) = 0$, which is a consequence of the convergence $u_n \to u$ in $C([t_0,\infty),\mathbb{R})$ and that the following inequality holds

$$\int_{t_0}^{\infty} G_n(\tau)\,d\tau \le 2K \int_{t_0}^{\infty} r(\tau) \int_{t_0}^{\tau} p(s) \int_{t_0}^{s} q(z)\,dz\,ds\,d\tau.$$

Since $I(r,p,q) < \infty$, the Lebesgue's dominated convergence theorem yields

$$\lim_{n\to\infty} \int_{t_0}^{\infty} G_n(\tau)\,d\tau = 0.$$

Consequently, from (2.12), we have $\lim_{n\to\infty} \|T_2 u_n - T_2 u\| = 0$, i.e. T_2 is continuous.

(iii) $T_2(\Delta)$ is relatively compact. It suffices to show that the family of functions $T_2(\Delta)$ is uniformly bounded and equicontinuous on the interval $[t_0,\infty)$. The uniform boundedness of $T_2(\Delta)$ immediately follows from the facts that $T_2(\Delta) \subseteq \Delta$ and Δ is a bounded subset of $C([t_0,\infty),\mathbb{R})$. Now, we prove that $T_2(\Delta)$ is an equicontinuous family of functions on the interval $[t_0,\infty)$.

Let $u \in \Delta$ and $t_2 > t_1 \ge t_0$. From the definition of T_2, we have

$$(T_2u)(t_2) - (T_2u)(t_1) = -\int_{t_1}^{t_2} r(s)\,ds - \int_{t_1}^{t_2} r(\tau) \int_{t_0}^{\tau} p(s) \int_{t_0}^{s} q(z) f(u(z))\,dz\,ds\,d\tau \tag{2.13}$$

and so, taking into account (2.11), we obtain

$$\begin{aligned} |(T_2u)(t_2) - (T_2u)(t_1)| &\le H(t_1) + \int_{t_1}^{\infty} r(\tau) \int_{t_0}^{\tau} p(s) \int_{t_0}^{s} q(z) f(u(z))\,dz\,ds\,d\tau \\ &\le H(t_1) + K \left(\int_{t_0}^{\infty} p(s) \int_{t_0}^{s} q(z)\,dz\,ds \right) \left(\int_{t_1}^{\infty} r(\tau)\,d\tau \right) \le 2H(t_1). \end{aligned}$$

Since $H(t_1) \to 0$ as $t_1 \to \infty$, for any given $\varepsilon > 0$ there exists $T > t_0$ such that for all $u \in \Delta$, we have

$$|(T_2u)(t_2) - (T_2u)(t_1)| < \varepsilon \qquad \text{if } \ t_2 > t_1 \ge T.$$

This shows that the oscillations of all functions of the family $T_2(\Delta)$ on $[T,\infty)$ are less than ε. Now, let $t_0 \le t_1 < t_2 \le T$. Then the equality (2.13) yields

$$|(T_2u)(t_2) - (T_2u)(t_1)| \le M_1|t_2 - t_1| + KM_2|t_2 - t_1|$$

where

$$M_1 = \max\{r(s) : s \in [t_0,\,T]\}, \ \ M_2 = \max\left\{ r(\tau) \int_{t_0}^{\tau} p(s) \int_{t_0}^{s} q(z)\,dz\,ds : \tau \in [t_0,\,T] \right\}.$$

Hence, for any given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u \in \Delta$

$$|(T_2u)(t_2) - (T_2u)(t_1)| < \varepsilon \quad \text{if} \quad |t_2 - t_1| < \delta.$$

Consequently, we can divide the interval $[t_0,\infty)$ into a finite number of subintervals on which every function of the family $T_2(\Delta)$ has oscillation less than ε. Therefore $T_2(\Delta)$ is an equicontinuous family of functions on $[t_0,\infty)$ (see, e.g. [11], p. 13). Hence $T_2(\Delta)$ is relatively compact.

Now, according to the Schauder fixed point theorem there exists $x \in \Delta$ such that

$$x(t) = \int_t^{\infty} r(s)\,ds + \int_t^{\infty} r(\tau) \int_{t_0}^{\tau} p(s) \int_{t_0}^{s} q(z) f(x(z))\,dz\,ds\,d\tau, \quad t \ge t_0.$$

It is clear that x is a positive solution of the equation (N) in the class $\mathcal{N}_3$ which approaches to zero as $t \to \infty$, i.e. $x \in \mathcal{N}_3^0$. This completes the proof. ∎

We have the following result for solutions of equation (N) in the class $\mathcal{N}_3$.

Theorem 2.12. *If $I(r) = \infty$, then $\mathcal{N}_3 = \varnothing$.*

Proof. Let $x \in \mathcal{N}_3$. Without loss of generality, we suppose that there exists $T \geq a$ such that $x(t) > 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) < 0$ for all $t \geq T$. As $x^{[1]}$ is a negative decreasing function, we have that $x'(t) \leq x^{[1]}(T)r(t)$ for all $t \geq T$. Integrating this inequality in $[T, t]$, we obtain

$$x(t) \leq x(T) + x^{[1]}(T) \int_T^t r(s)\, ds.$$

When $t \to \infty$, we get a contradiction because the function $x(t)$ is a positive for all $t \geq T$. The case $x(t) < 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$ for all $t \geq T^*$ (where $T^* \geq a$) can be treated similarly. ∎

As a consequence of Theorems 2.11 and 2.12, we get the following result.

Corollary 2.13. *Assume that $I(p, q) < \infty$. Then a necessary and sufficient condition for equation (N) to have a solution x in the class $\mathcal{N}_3^0$ is that $I(r) < \infty$.*

The next results deal with the solutions of equation (N) in the class $\mathcal{N}_3^B$.

Theorem 2.14. *If $I(r, p, q) = \infty$, then $\mathcal{N}_3^B = \varnothing$.*

Proof. Let $x \in \mathcal{N}_3^B$. Without loss of generality, we suppose that there exists $T \geq a$ such that $x(t) > 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) < 0$ for all $t \geq T$. Let $x(\infty) = l_x > 0$. Integrating equation (N) three times in the interval $[T, t]$, we get

$$\begin{aligned} x(t) = x(T) + x^{[1]}(T) \int_T^t r(s)\, ds + x^{[2]}(T) \int_T^t r(s) \int_T^s p(z)\, dz\, ds \\ - \int_T^t r(s) \int_T^s p(z) \int_T^z q(\tau) f(x(\tau))\, d\tau\, dz\, ds. \end{aligned}$$

Thus

$$x(t) \leq x(T) - \int_T^t r(s) \int_T^s p(z) \int_T^z q(\tau) f(x(\tau))\, d\tau\, dz\, ds \qquad \text{for all } t \geq T. \tag{2.14}$$

The continuity of the function $f(u)$ on the interval $[l_x, x(T)]$ ensures the existence of a positive constant K such that

$$K = \min \{ f(u) : u \in [l_x, x(T)] \}. \tag{2.15}$$

The inequality (2.14) with (2.15) yields

$$x(t) \leq x(T) - K \int_T^t r(s) \int_T^s p(z) \int_T^z q(\tau)\, d\tau\, dz\, ds \qquad \text{for all } t \geq T.$$

When $t \to \infty$, we get a contradiction because the function $x(t)$ is a positive for all $t \geq T$. The case $x(t) < 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$ for all $t \geq T^*$ (where $T^* \geq a$) can be treated in the similar way. ∎

Theorem 2.15. *If $I(r, p, q) < \infty$, then equation (N) has a solution x in the class $\mathcal{N}_3$ such that $\lim_{t\to\infty} x(t) \neq 0$, i.e. $\mathcal{N}_3^B \neq \varnothing$.*

Proof. We prove the existence of a positive solution of equation (N) in the class $\mathcal{N}_3$ which approaches to nonzero constant as $t \to \infty$.

Let $K = \max \{ f(u) : u \in [c, d] \}$ where c, d are constants such that $0 < c < d$ and let $t_0 \geq a$ be such that

$$\int_{t_0}^{\infty} r(\tau) \int_{t_0}^{\tau} p(s) \int_{t_0}^{s} q(z)\, dz\, ds\, d\tau \leq \frac{d - c}{K}. \tag{2.16}$$

Let us define the set Δ in the same way as in the proof of Theorem 2.1. For every $u \in \Delta$ we consider a mapping $T_3 : \Delta \to C([t_0, \infty), \mathbb{R})$ given by

$$x_u(t) = (T_3u)(t) = c + \int_t^\infty r(\tau) \int_{t_0}^\tau p(s) \int_{t_0}^s q(z) f(u(z))\, dz\, ds\, d\tau, \quad t \geq t_0.$$

Taking into account (2.16) and using similar arguments as in the proof of Theorem 2.1, it is easy to verify that T_3 maps Δ into itself, T_3 is a continuous mapping in Δ and $T_3(\Delta)$ is a relatively compact set in $C([t_0, \infty), \mathbb{R})$. Consequently, the Schauder fixed point theorem ensures the existence of a fixed point $x \in \Delta$ such that

$$x(t) = c + \int_t^\infty r(\tau) \int_{t_0}^\tau p(s) \int_{t_0}^s q(z) f(x(z))\, dz\, ds\, d\tau, \quad t \geq t_0.$$

As

$$x^{[1]}(t) = -\int_{t_0}^t p(s) \int_{t_0}^s q(z) f(x(z))\, dz\, ds < 0 \quad \text{and} \quad x^{[2]}(t) = -\int_{t_0}^t q(z) f(x(z))\, dz < 0,$$

it is clear that x is a positive solution of the equation (N) in the class $\mathcal{N}_3$ which approaches to nonzero constant as $t \to \infty$, i.e. $x \in \mathcal{N}_3^B$. This completes the proof. ∎

Theorem 2.15 is illustrated by the following example.

Example 2.3. *Let us consider the differential equation*

$$\left(\frac{1}{t}\left(t^8 x'(t)\right)'\right)' + \frac{24t^4}{(3t+1)e^{\frac{3t+1}{t}}}\, x(t) e^{x(t)} = 0, \quad t \geq 1. \tag{2.17}$$

As $I(r,p,q) < \infty$, Theorem 2.15 secures that equation (2.17) *has a solution in the class $\mathcal{N}_3^B$. One such solution is the function $x(t) = \dfrac{3t+1}{t}$.*

Theorems 2.14 and 2.15 give the following corollary.

Corollary 2.16. *A necessary and sufficient condition for equation (N) to have a solution x in the class $\mathcal{N}_3^B$ is that $I(r,p,q) < \infty$.*

The following also holds.

Corollary 2.17. *Assume that $I(p,q) < \infty$. Then a necessary and sufficient condition for equation (N) to have a solution x in the class $\mathcal{N}_3$ is that $I(r) < \infty$.*

Finally, we consider the solutions of equation (N) in the class $\mathcal{N}_0$. We prove the following results for the existence of solutions of (N) in the class $\mathcal{N}_0^B$.

Theorem 2.18. *If $I(q,p,r) < \infty$, then equation (N) has a solution x in the class $\mathcal{N}_0$ such that $\lim_{t\to\infty} x(t) \neq 0$, i.e. $\mathcal{N}_0^B \neq \varnothing$.*

Proof. We prove the existence of a positive solution of equation (N) in the class $\mathcal{N}_0$ which approaches to nonzero constant as $t \to \infty$.

Let $K = \max\{f(u) : u \in [c,d]\}$ where c, d are constants such that $0 < c < d$ and let $t_0 \geq a$ be such that

$$\int_{t_0}^\infty q(z) \int_{t_0}^z p(s) \int_{t_0}^s r(\tau)\, d\tau\, ds\, dz \leq \frac{d-c}{K}. \tag{2.18}$$

Let us define the set Δ in the same way as in the proof of Theorem 2.1. For every $u \in \Delta$ we consider a mapping $T_4 : \Delta \to C([t_0, \infty), \mathbb{R})$ given by

$$x_u(t) = (T_4u)(t) = c + \int_t^\infty r(\tau) \int_\tau^\infty p(s) \int_s^\infty q(z) f(u(z))\, dz\, ds\, d\tau, \quad t \geq t_0.$$

(i) T_4 maps Δ into Δ. In fact, $x_u(t) \geq c$ and in view of (2.18), we have

$$\begin{aligned} x_u(t) &= c + \int_t^\infty r(\tau) \int_\tau^\infty p(s) \int_s^\infty q(z) f(u(z))\, dz\, ds\, d\tau \\ &\leq c + K \int_{t_0}^\infty r(\tau) \int_\tau^\infty p(s) \int_s^\infty q(z)\, dz\, ds\, d\tau \\ &= c + K \int_{t_0}^\infty q(z) \int_{t_0}^z p(s) \int_{t_0}^s r(\tau)\, d\tau\, ds\, dz \leq d. \end{aligned}$$

(ii) T_4 is continuous. Let $\{u_n\}$, $n \in N$ be a sequence of elements of Δ such that $\lim_{n\to\infty} \|u_n - u\| = 0$. Since Δ is closed, $u \in \Delta$. The definition of T_4 yields that

$$|(T_4 u_n)(t) - (T_4 u)(t)| \leq \int_{t_0}^\infty G_n(z)\, dz, \quad t \geq t_0$$

where

$$G_n(z) = q(z) |f(u_n(z)) - f(u(z))| \int_{t_0}^z p(s) \int_{t_0}^s r(\tau)\, d\tau\, ds.$$

Thus, we have the following

$$\|T_4 u_n - T_4 u\| \leq \int_{t_0}^\infty G_n(z)\, dz. \tag{2.19}$$

It is obvious that $\lim_{n\to\infty} G_n(z) = 0$ and

$$\int_{t_0}^\infty G_n(z)\, dz \leq 2K \int_{t_0}^\infty q(z) \int_{t_0}^z p(s) \int_{t_0}^s r(\tau)\, d\tau\, ds\, dz.$$

Since $I(q, p, r) < \infty$, applying the Lebesgue's dominated convergence theorem, we obtain from (2.19) that $\lim_{n\to\infty} \|T_4 u_n - T_4 u\| = 0$ which means that T_4 is continuous.

(iii) $T_4(\Delta)$ is relatively compact. It is easy to see that the family of functions $T_4(\Delta)$ is uniformly bounded. We need only to prove the equicontinuity of $T_4(\Delta)$ on the interval $[t_0, \infty)$.

Let $u \in \Delta$ and $t_2 > t_1 \geq t_0$. Then we have

$$(T_4 u)(t_2) - (T_4 u)(t_1) = -\int_{t_1}^{t_2} r(\tau) \int_\tau^\infty p(s) \int_s^\infty q(z) f(u(z))\, dz\, ds\, d\tau \tag{2.20}$$

and so

$$\begin{aligned} |(T_4 u)(t_2) - (T_4 u)(t_1)| &\leq K \int_{t_1}^\infty r(\tau) \int_\tau^\infty p(s) \int_s^\infty q(z)\, dz\, ds\, d\tau \\ &= K \int_{t_1}^\infty q(z) \int_{t_1}^z p(s) \int_{t_1}^s r(\tau)\, d\tau\, ds\, dz \to 0 \qquad \text{as } t_1 \to \infty. \end{aligned}$$

From the above facts, we conclude that for any given $\varepsilon > 0$ there exists $T > t_0$ such that for all $u \in \Delta$, we have

$$|(T_4 u)(t_2) - (T_4 u)(t_1)| < \varepsilon \qquad \text{if } t_2 > t_1 \geq T.$$

Now, let $t_0 \leq t_1 < t_2 \leq T$. The equality (2.20) and the fact that $I(q, p) < \infty$ (it follows from $I(q, p, r) < \infty$) give the following inequality

$$|(T_4 u)(t_2) - (T_4 u)(t_1)| \leq KM |t_2 - t_1|$$

where $M = \max\left\{r(\tau)\int_\tau^\infty p(s)\int_s^\infty q(z)\,dz\,ds : \tau \in [t_0,\,T]\right\}$. Hence, for any given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u \in \Delta$

$$|(T_4u)(t_2) - (T_4u)(t_1)| < \varepsilon \quad \text{if} \quad |t_2 - t_1| < \delta.$$

Consequently, we can divide the interval $[t_0, \infty)$ into a finite number of subintervals on which every function of the family $T_4(\Delta)$ has oscillation less than ε. Therefore $T_4(\Delta)$ is an equicontinuous family of functions on $[t_0, \infty)$ (see, e.g. [11], p. 13). Hence $T_4(\Delta)$ is relatively compact.

From the preceding considerations, we see that Schauder fixed point theorem (Theorem 1.1) can be applied to the mapping T_4. Hence, there exists a fixed point $x \in \Delta$ such that

$$x(t) = c + \int_t^\infty r(\tau)\int_\tau^\infty p(s)\int_s^\infty q(z)f(x(z))\,dz\,ds\,d\tau, \quad t \ge t_0.$$

It is clear that x is a positive solution of the equation (N) in the class $\mathcal{N}_0$ which approaches to nonzero constant as $t \to \infty$, i.e. $x \in \mathcal{N}_0^B$. This completes the proof. ∎

Theorem 2.19. *If $I(q,p,r) = \infty$, then $\mathcal{N}_0^B = \varnothing$.*

Proof. Let $x \in \mathcal{N}_0^B$. Without loss of generality, we suppose that there exists $T \ge a$ such that $x(t) > 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) > 0$ for all $t \ge T$. Let $x(\infty) = l_x > 0$. From equation (N), it follows that $\left(x^{[2]}(t)\right)' < 0$ for all $t \ge T$. Hence, $x^{[2]}(t)$ is a positive decreasing function. Integrating equation (N) three times in $[t,\infty)$ and taking into account the facts that $0 < x(\infty) < \infty$, $0 \le x^{[2]}(\infty) < \infty$ and $-\infty < x^{[1]}(\infty) \le 0$, we obtain

$$x(t) \ge \int_t^\infty r(\tau)\int_\tau^\infty p(s)\int_s^\infty q(z)f(x(z))\,dz\,ds\,d\tau. \tag{2.21}$$

The continuity of the function $f(u)$ on the interval $[l_x, x(T)]$ ensures the existence of a positive constant K such that

$$K = \min\{f(u) : u \in [l_x, x(T)]\}. \tag{2.22}$$

In view of (2.21) and (2.22), we have

$$x(t) \ge K\int_t^\infty r(\tau)\int_\tau^\infty p(s)\int_s^\infty q(z)\,dz\,ds\,d\tau \qquad \text{for all } t \ge T.$$

Hence, by interchanging the order of integration, we get that $I(q,p,r) < \infty$. For the case $x(t) < 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) < 0$ for all $t \ge T^*$ (where $T^* \ge a$), similar arguments hold. ∎

Theorems 2.18 and 2.19 give the following corollary.

Corollary 2.20. *A necessary and sufficient condition for equation (N) to have a solution x in the class $\mathcal{N}_0^B$ is that $I(q,p,r) < \infty$.*

Example 2.4. *The differential equation of the third order*

$$\left(\frac{1}{t^2}\left(\frac{1}{t}x'(t)\right)'\right)' + \frac{56}{t^8 \operatorname{arctg}\frac{t^2+1}{t^2}}\operatorname{arctg} x(t) = 0, \quad t \ge 1 \tag{2.23}$$

satisfies the condition of Theorem 2.18. Therefore, equation (2.23) *has a solution in the class $\mathcal{N}_0^B$. In fact, one such solution is the function $x(t) = \dfrac{t^2+1}{t^2}$.*

Now, we state sufficient condition for the existence of solutions of (N) in the class $\mathcal{N}_0^0$.

Theorem 2.21. *If $I(q,p) < \infty$ and $I(r) < \infty$, then equation (N) has a solution x in the class $\mathcal{N}_0$ such that $\lim_{t\to\infty} x(t) = 0$, i.e. $\mathcal{N}_0^0 \neq \varnothing$.*

Proof. We prove the existence of a positive solution of equation (N) in the class $\mathcal{N}_0$ which approaches to zero as $t \to \infty$.

Let $t_0 \geq a$ be such that

$$K \int_{t_0}^{\infty} q(s) \int_{t_0}^{s} p(\tau)\, d\tau\, ds \leq 1, \tag{2.24}$$

where

$$K = \max \left\{ f(u) : u \in \left[0,\, 2 \int_{t_0}^{\infty} r(s)\, ds \right] \right\}.$$

For convenience, we again make use of the following notation:

$$H(t) = \int_{t}^{\infty} r(s)\, ds, \qquad t \geq t_0.$$

Let us define the set Δ in the same way as in the proof of Theorem 2.11. For every $u \in \Delta$ we consider a mapping $T_5 : \Delta \to C([t_0, \infty), \mathbb{R})$ given by

$$x_u(t) = (T_5 u)(t) = H(t) + \int_{t}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s) \int_{s}^{\infty} q(z) f(u(z))\, dz\, ds\, d\tau, \quad t \geq t_0.$$

In order to apply the Schauder fixed point theorem to the mapping T_5, it is sufficient to prove that T_5 maps Δ into itself, T_5 is a continuous mapping in Δ and $T_5(\Delta)$ is a relatively compact set in $C([t_0, \infty), \mathbb{R})$.

(i) T_5 maps Δ into Δ. In fact, $x_u(t) \geq H(t)$ and in view of (2.24), we have

$$\begin{aligned} x_u(t) &= H(t) + \int_{t}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s) \int_{s}^{\infty} q(z) f(u(z))\, dz\, ds\, d\tau \\ &\leq H(t) + K \int_{t}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s) \int_{s}^{\infty} q(z)\, dz\, ds\, d\tau \\ &\leq H(t) + K \left(\int_{t_0}^{\infty} p(s) \int_{s}^{\infty} q(z)\, dz\, ds \right) \left(\int_{t}^{\infty} r(\tau)\, d\tau \right) \\ &= H(t) + K \left(\int_{t_0}^{\infty} q(z) \int_{t_0}^{z} p(s)\, ds\, dz \right) \left(\int_{t}^{\infty} r(\tau)\, d\tau \right) \leq H(t) + H(t) = 2H(t). \end{aligned}$$

(ii) T_5 is continuous. The proof is the same as the one of the continuity of mapping T_4 in Theorem 2.18. Therefore, we omit it.

(iii) $T_5(\Delta)$ is relatively compact. It suffices to show that the family of functions $T_5(\Delta)$ is uniformly bounded and equicontinuous on the interval $[t_0, \infty)$. The uniform boundedness of $T_5(\Delta)$ follows from the facts that $T_5(\Delta) \subseteq \Delta$ and Δ is a bounded subset of $C([t_0, \infty), \mathbb{R})$. Now, we prove that $T_5(\Delta)$ is an equicontinuous family of functions on the interval $[t_0, \infty)$.

Let $u \in \Delta$ and $t_2 > t_1 \geq t_0$. From the definition of T_5, we have

$$(T_5 u)(t_2) - (T_5 u)(t_1) = -\int_{t_1}^{t_2} r(\tau)\, d\tau - \int_{t_1}^{t_2} r(\tau) \int_{\tau}^{\infty} p(s) \int_{s}^{\infty} q(z) f(u(z))\, dz\, ds\, d\tau \tag{2.25}$$

and so, taking into account (2.24), we obtain

$$\begin{aligned} |(T_5 u)(t_2) - (T_5 u)(t_1)| &\leq H(t_1) + \int_{t_1}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s) \int_{s}^{\infty} q(z) f(u(z))\, dz\, ds\, d\tau \\ &\leq H(t_1) + K \left(\int_{t_1}^{\infty} p(s) \int_{s}^{\infty} q(z)\, dz\, ds \right) \left(\int_{t_1}^{\infty} r(\tau)\, d\tau \right) \leq 2H(t_1). \end{aligned}$$

Since $H(t_1) \to 0$ as $t_1 \to \infty$, for any given $\varepsilon > 0$ there exists $T > t_0$ such that for all $u \in \Delta$, we have

$$|(T_5 u)(t_2) - (T_5 u)(t_1)| < \varepsilon \qquad \text{if } \ t_2 > t_1 \geq T. \tag{2.26}$$

Now, let $t_0 \leq t_1 < t_2 \leq T$. The equality (2.25) and the fact that $I(q,p) < \infty$ yield

$$|(T_5 u)(t_2) - (T_5 u)(t_1)| \leq M_1 |t_2 - t_1| + K M_2 |t_2 - t_1|$$

where

$$M_1 = \max\{r(\tau) : \tau \in [t_0, T]\}, \quad M_2 = \max\left\{ r(\tau) \int_\tau^\infty p(s) \int_s^\infty q(z)\, dz\, ds : \tau \in [t_0, T] \right\}.$$

Hence, for any given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u \in \Delta$

$$|(T_5 u)(t_2) - (T_5 u)(t_1)| < \varepsilon \quad \text{if } \ |t_2 - t_1| < \delta. \tag{2.27}$$

In view of (2.26) and (2.27), we are able to decompose the interval $[t_0, \infty)$ into a finite number of subintervals on which every function of the family $T_5(\Delta)$ has oscillation less than ε. It follows that $T_5(\Delta)$ is relatively compact.

Now, according to the Schauder fixed point theorem there exists $x \in \Delta$ such that

$$x(t) = \int_t^\infty r(s)\, ds + \int_t^\infty r(\tau) \int_\tau^\infty p(s) \int_s^\infty q(z) f(x(z))\, dz\, ds\, d\tau, \quad t \geq t_0.$$

It is clear that x is a positive solution of the equation (N) in the class $\mathcal{N}_0$ which approaches to zero as $t \to \infty$, i.e. $x \in \mathcal{N}_0^0$. The proof is now complete. ∎

Remark 2.2. Similar investigation of the asymptotic behavior of solutions of the second order differential equations

$$(r(t)x'(t))' + q(t)x(t) = 0 \quad \text{and} \quad (r(t)x'(t))' + q(t)f(x(t)) = 0, \quad t \geq a,$$

where r, q, f satisfy (H1), (H2), has been given in [6] and [7], respectively. We also refer the reader to [11] for other results on this topic.

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