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CERTAIN INEQUALITIES FOR p_VALENT MEROMORPHIC FUNCTIONS WITH ALTERNATING COEFFICIENTS BASED ON INTEGRAL OPERATOR

A. EBADIAN , S. SHAMS AND SH. NAJAFZADEH

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, URMIA UNIVERSITY, URMIA, IRAN
a.ebadian@mail.urmia.ac.ir
sa40shams@yahoo.com

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, MARAGHEH UNIVERSITY, MARAGHEH, IRAN
Shnajafzadeh@yahoo.com

ABSTRACT. In this paper we introduce the class $\sigma_p^*(\beta)$ of functions $f(z) = Az^{-p} + \sum_{n=p}^{\infty}(-1)^{n-1}a_n z^n$ regular and multivalent in the $\Delta^* = \{z : 0 < |z| < 1\}$ and satisfying

$$Re\left\{\frac{z[\mathcal{J}(f(z))]'}{\mathcal{J}(f(z))}\right\} < -\beta$$

where $\mathcal{J}$ is a linear operator.

Coefficient inequalities, distortion bounds, weighted mean and arithmetic mean of functions for this class have been obtained.

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1. INTRODUCTION

Let Σ_p be the class of functions of the form

$$f(z) = Az^{-p} + \sum_{n=p}^{\infty} a_n z^n, \ A \geq 0 \tag{1.1}$$

that are regular in the punctured disk $\Delta^* = \{z : 0 < |z| < 1\}$ and σ_p be the subclass of Σ_p consisting of functions with alternating coefficients of the type

$$f(z) = Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1} a_n z^n, \ a_n \geq 0, \ A \geq 0. \tag{1.2}$$

Let

$$\Sigma_p^*(\beta) = \left\{ f \in \Sigma_p : Re\left(\frac{z[\mathcal{J}(f(z))]'}{\mathcal{J}(f(z))} \right) < -\beta, 0 \leq \beta < p \right\} \tag{1.3}$$

and let $\sigma_p^*(\beta) = \Sigma_p^*(\beta) \cap \sigma_p$ where

$$\mathcal{J}(f(z)) = (\gamma - p + 1) \int_0^1 (u^\gamma) f(uz) du, \ p < \gamma \tag{1.4}$$

is a linear operator.

With a simple calculation we obtain

$$\mathcal{J}(f(z)) = \begin{cases} Az^{-p} + \sum\limits_{n=p}^{\infty} (-1)^{n-1} \left(\frac{\gamma-p+1}{\gamma+n+1} \right) a_n z^n & f(z) \in \sigma_p, \\ Az^{-p} + \sum\limits_{n=p}^{\infty} \left(\frac{\gamma-p+1}{\gamma+n+1} \right) a_n z^n & f(z) \in \Sigma_p. \end{cases} \tag{1.5}$$

For more details about meromorphic p-valent functions, we can see the recent works of many authors in [1, 2, 3].

Also Uralegaddi and Ganigi [4] worked on meromorphic univalent functions with alternating coefficients.

2. COEFFICIENT ESTIMATES

Theorem 2.1. *Let* $f(z) = Az^{-p} + \sum\limits_{n=p}^{\infty} a_n z^n \in \Sigma_p$. *If*

$$\sum_{n=p}^{\infty} (n + \beta) \left(\frac{\gamma - p + 1}{\gamma + n + 1} \right) |a_n| \leq A(p - \beta) \tag{2.1}$$

then $f(z) \in \Sigma_p^*(\beta)$.

Proof. It is enough to show that

$$M = \left| \frac{\frac{z[\mathcal{J}f(z))]'}{\mathcal{J}f(z))} + p}{\frac{z[\mathcal{J}f(z))]'}{\mathcal{J}f(z))} - (p - 2\beta)} \right| < 1 \text{ for } |z| < 1.$$

But by 1.5

$$M = \left| \frac{-pAz^{-p} + \sum\limits_{n=p}^{\infty} n \left(\frac{\gamma-p+1}{\gamma+n+1} \right) a_n z^n + pAz^{-p} + \sum\limits_{n=p}^{\infty} p \left(\frac{\gamma-p+1}{\gamma+n+1} \right) a_n z^n}{-pAz^{-p} + \sum\limits_{n=p}^{\infty} n \left(\frac{\gamma-p+1}{\gamma+n+1} \right) a_n z^n - (p - 2\beta)Az^{-p} - \sum\limits_{n=p}^{\infty} (p - 2\beta) \left(\frac{\gamma-p+1}{\gamma+n+1} \right) a_n z^n} \right|$$

$$\leq \frac{\sum\limits_{n=p}^{\infty}\left[(n+p)\left(\frac{\gamma-p+1}{\gamma+n+1}\right)\right]|a_n|}{2A(p-\beta)-\sum\limits_{n=p}^{\infty}(n-p+2\beta)\left(\frac{\gamma-p+1}{\gamma+n+1}\right)|a_n|}.$$

The last expression is less than or equal to 1 provided

$$\sum_{n=p}^{\infty}\left[(n+p)\left(\frac{\gamma-p+1}{\gamma+n+1}\right)\right]|a_n| \leq 2A(p-\beta)-\sum_{n=p}^{\infty}(n-p+2\beta)\left(\frac{\gamma-p+1}{\gamma+n+1}\right)|a_n|$$

which is equivalent to

$$\sum_{n=p}^{\infty}(n+\beta)\left(\frac{\gamma-p+1}{\gamma+n+1}\right)|a_n| \leq A(p-\beta)$$

which is true by (2.1) so the proof is complete. ∎

The converse of the Theorem 2.1 is also true for functions in $\sigma_p^*(\beta)$, where p is an odd number.

Theorem 2.2. *A function $f(z)$ in σ_p is in $\sigma_p^*(\beta)$ if and only if*

$$\sum_{n=p}^{\infty}(n+\beta)\left(\frac{\gamma-p+1}{\gamma+n+1}\right)a_n \leq A(p-\beta). \tag{2.2}$$

Proof. According to Theorem 2.1 it is enough to prove the "only if" part. Suppose that

$$Re\left(\frac{z(\mathcal{J}f(z))'}{(\mathcal{J}f(z))}\right) = Re\left(\frac{-Apz^{-p}+\sum\limits_{n=p}^{\infty}n(-1)^{n-1}\left(\frac{\gamma-p+1}{\gamma+n+1}\right)a_nz^n}{Az^{-p}+\sum\limits_{n=p}^{\infty}(-1)^{n-1}\left(\frac{\gamma-p+1}{\gamma+n+1}\right)a_nz^n}\right) < -\beta. \tag{2.3}$$

By choosing values of z on the real axis so that $\frac{(z(\mathcal{J}f(z))'}{(\mathcal{J}f(z))}$ is real and clearing the denominator in (2.3) and letting $z \to -1$ through real values we obtain

$$Ap-\sum_{n=p}^{\infty}n\left(\frac{\gamma-p+1}{\gamma+n+1}\right)a_n \geq \beta\left(A+\sum_{n=p}^{\infty}\left(\frac{\gamma-p+1}{\gamma+n+1}\right)a_n\right)$$

which is equivalent to

$$\sum_{n=p}^{\infty}(n+\beta)\left(\frac{\gamma-p+1}{\gamma+n+1}\right)a_n \leq A(p-\beta).$$

∎

Corollary 2.3. *If $f(z) \in \sigma_p^*(\beta)$ then*

$$a_n \leq \frac{A(p-\beta)(\gamma+n+1)}{(n+\beta)(\gamma-p+1)} \textit{ for } n = p, p+1, \cdots. \tag{2.4}$$

The result is sharp for the functions of the type

$$f_n(z) = Az^{-p} + (-1)^{n-1}\frac{A(p-\beta)(\gamma+n+1)}{(n+\beta)(\gamma-p+1)}z^n. \tag{2.5}$$

3. DISTORTION BOUNDS AND IMPORTANT PROPERTIES OF $\sigma_p^*(\beta)$

In this section we obtain distortion bounds for functions in the class $\sigma_p^*(\beta)$ and prove some important properties of this class, where p is an odd number.

Theorem 3.1. *Let* $f(z) = Az^{-p} + \sum_{n=p}^{\infty}(-1)^{n-1}a_n z^n$, $a_n \geq 0$ *be in the class* $\sigma_p^*(\beta)$ *and* $\beta \geq \gamma+1$ *then*

$$Ar^{-p} - \frac{A(p-\beta)}{\gamma-p+1}r^p \leq |f(z)| \leq Ar^{-p} + \frac{A(p-\beta)}{\gamma-p+1}r^p. \tag{3.1}$$

Proof. Since $\beta \geq \gamma+1$ so $\frac{n+\beta}{\gamma+n+1} \geq 1$ then

$$(\gamma-p+1)\sum_{n=p}^{\infty} a_n \leq \sum_{n=p}^{\infty}\left(\frac{n+\beta}{\gamma+n+1}\right)(\gamma-p+1)a_n \leq A(p-\beta).$$

We have

$$\begin{aligned} |f(z)| &= |Az^{-p} + \sum_{n=p}^{\infty}(-1)^{n-1}a_n z^n| \\ &\leq \frac{A}{r^p} + r^p\sum_{n=p}^{\infty} a_n \leq \frac{A}{r^p} + r^p\frac{A(p-\beta)}{(\gamma-p+1)}. \end{aligned}$$

Similarly,

$$|f(z)| \geq \frac{A}{r^p} - \sum_{n=p}^{\infty} a_n r^n \geq \frac{A}{r^p} - r^p\sum_{n=p}^{\infty} a_n \geq \frac{A}{r^p} - \frac{A(p-\beta)}{\gamma-p+1}r^p.$$

∎

Theorem 3.2. *Let*

$$f(z) = Az^{-p} + \sum_{n=p}^{\infty} a_n z^n \text{ and } g(z) = Az^{-p} + \sum_{n=p}^{\infty}(-1)^{n-1}b_n z^n$$

be in the class $\sigma_p^*(\beta)$ *then the weighted mean of* f *and* g *defined by*

$$W_j(z) = \frac{1}{2}[(1-j)f(z) + (1+j)g(z)]$$

is also in the same class.

Proof. Since f and g belong to $\sigma_p^*(\beta)$ so by (2.2) we have

$$\begin{cases} \sum_{n=p}^{\infty}(n+\beta)\left(\frac{\gamma-p+1}{\gamma+n+1}\right)a_n \leq A(p-\beta), \\ \sum_{n=p}^{\infty}(n+\beta)\left(\frac{\gamma-p+1)}{\gamma+n+1}\right)b_n \leq A(p-\beta). \end{cases} \tag{3.2}$$

After a simple calculation we obtain

$$W_j(z) = Az^{-p} + \sum_{n=p}^{\infty}\left[\frac{1-j}{2}a_n + \frac{1+j}{2}b_n\right](-1)^{n-1}z^n.$$

But

$$\begin{aligned}
&\sum_{n=p}^{\infty}(n+\beta)\left(\frac{\gamma-p+1)}{\gamma+n+1}\right)\left[\frac{1-j}{2}a_n+\frac{1+j}{2}b_n\right] \\
&=\left(\frac{1-j}{2}\right)\sum_{n=p}^{\infty}(n+\beta)\left(\frac{\gamma-p+1)}{\gamma+n+1}\right)a_n+\left(\frac{1+j}{2}\right)\sum_{n=p}^{\infty}(n+\beta)\left(\frac{\gamma-p+1)}{\gamma+n+1}\right)b_n \\
&\overset{by(12)}{\leq}\left(\frac{1-j}{2}\right)A(p-\beta)+\left(\frac{1+j}{2}\right)A(p-\beta)=A(p-\beta).
\end{aligned}$$

Hence by Theorem 2.2 $W_j(z)\in\sigma_p^*(\beta)$. ∎

Theorem 3.3. *Let*

$$f_k(z)=Az^{-p}+\sum_{n=p}^{\infty}(-1)^{n-1}a_{n,k}z^n\in\sigma_p^*(\beta),k=1,2,\cdots,m$$

then the arithmetic mean of $f_k(z)$ defined by

$$F(z)=\frac{1}{m}\sum_{k=1}^{m}f_k(z) \tag{3.3}$$

is also in the same class.

Proof. Since $f_k(z)\in\sigma_p^*(\beta)$ so by (2.2) we have

$$\sum_{n=p}^{\infty}(n+\beta)\left(\frac{\gamma-p+1}{\gamma+n+1}\right)a_{n,k}\leq A(p-\beta)\ (k=1,2,\cdots,m). \tag{3.4}$$

After a simple calculation we obtain

$$\begin{aligned}
F(z)&=\frac{1}{m}\sum_{k=1}^{m}\left(Az^{-p}+\sum_{n=p}^{\infty}(-1)^{n-1}a_{n,k}z^n\right)\\
&=Az^{-p}+\sum_{n=p}^{\infty}(-1)^{n-1}\left(\frac{1}{m}\sum_{k=1}^{m}a_{n,k}\right)z^n.
\end{aligned}$$

But

$$\sum_{n=p}^{\infty}(n+\beta)\left(\frac{\gamma-p+1}{\gamma+n+1}\right)\left(\frac{1}{m}\sum_{k=1}^{m}a_{n,k}\right)\overset{by(14))}{\leq}\frac{1}{m}\sum_{k=1}^{m}A(p-\beta)=A(p-\beta)$$

which in view of Theorem 2.2 yields the proof of Theorem 3.3. ∎

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