The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

**Volume 5**, Issue 1, Article 1, pp. 1-14, 2008

# ON OSCILLATION OF SECOND-ORDER DELAY DYNAMIC EQUATIONS ON TIME SCALES

**S. H. SAKER**

*Received 18 May, 2006; accepted 2 May, 2007; published 19 February, 2008.*
*Communicated by: C. Tisdell*

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, MANSOURA UNIVERSITY, MANSOURA, 35516, EGYPT.
shsaker@mans.edu.eg

ABSTRACT. Some new oscillation criteria for second-order linear delay dynamic equation on a time scale $\mathbb{T}$ are established. Our results improve the recent results for delay dynamic equations and in the special case when $\mathbb{T} = \mathbb{R}$, the results include the oscillation results established by Hille [1948, Trans. Amer. Math. Soc. 64 (1948), 234-252] and Erbe [Canad. Math. Bull. 16 (1973), 49-56.] for differential equations. When $\mathbb{T} = \mathbb{Z}$ the results include and improve some oscillation criteria for difference equations. When $\mathbb{T} = h\mathbb{Z}$, $h > 0$, $\mathbb{T} = q^{\mathbb{N}}$ and $\mathbb{T} = \mathbb{N}^2$, i.e., for generalized second order delay difference equations our results are essentially new and can be applied on different types of time scales. An example is considered to illustrate the main results.

*Key words and phrases:* Oscillation, delay dynamic equations, time scales.

1991 *Mathematics Subject Classification.* 34K11, 39A10, 39A99.

ISSN (electronic): 1449-5910



The author thanks the referee for his/her comments.

## 1. INTRODUCTION

The study of dynamic equations on time scales, is an area of mathematics that has recently received a lot of attention. It has been created in order to unify the study of differential and difference equations, and it also extends these classical cases to cases " in between", e.g., to the so-called $q-$difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts.

Dynamic equations on a time scale have an enormous potential for applications such as in population dynamics. For example, it can model insect populations that are continuous while in season, die out in say winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population. For more details, we refer the reader to the books by Bohner and Peterson [4, 5] which summarize and organize much of time scale calculus.

In recent years there has been much research activity concerning the qualitative theory of dynamic equations on time scales. One of the main subject of the qualitative analysis of the dynamic equations is the oscillatory behavior. Recently, interesting results are established for oscillation, nonoscillation, stability and boundedness, we refer the reader to the papers [1, 2, 3, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 18, 22, 23, 24, 25, 28, 29]. Following this trend in this paper, we are concerned with oscillation of the second-order linear delay dynamic equation

$$x^{\Delta\Delta}(t) + p(t)x(\tau(t)) = 0, \tag{1.1}$$

on a time scale $\mathbb{T}$, here the function $p$ is a positive $rd-$continuous function defined on $\mathbb{T}$, $\tau(t) : \mathbb{T} \to \mathbb{T}$, $\tau(t) \le t$ and $\lim_{t\to\infty} \tau(t) = \infty$.

Let $T_0 = \min\{\tau(t) : t \ge 0\}$ and $\tau_{-1}(t) = \sup\{s \ge 0 : \tau(s) \le t\}$ for $t \ge T_0$. Clearly $\tau_{-1}(t) \ge t$ for $t \ge T_0$, $\tau_{-1}(t)$ is nondecreasing and coincides with the inverse of $\tau$ when the latter exists. A continuous function $x : \mathbb{T} \to \mathbb{R}$ is said to be a solution of (1.1) if it is $rd-$continuous on $[\tau_{-1}(t_0), \infty)$ along with its derivative and almost every where on $[\tau_{-1}(t_0), \infty)$.

Our attention is restricted to those solutions $x(t)$ of (1.1) which exist on some half line $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t > t_1\} > 0$ for any $t_1 \ge t_x$. A solution $x(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory. Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$.
On any time scale we define $f^{\sigma}(t) := (f \circ \sigma)(t) = f(\sigma(t))$. A function $f : \mathbb{T} \to \mathbb{R}$ is called $rd-$continuous function provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$ and the set of $rd-$continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T})$ =$C_{rd}(\mathbb{T}, \mathbb{R})$.

Erbe and Peterson [12] considered the second-order dynamic equation without delay

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)x^{\sigma} = 0, \quad t \in \mathbb{T}, \tag{1.2}$$

under the assumption that: There exists $t_0 \in \mathbb{T}$, such that $r(t)$ is bounded above on $[t_0, \infty)$, and $\inf \mu(t) > 0$. By using Riccati technique they proved that if

$$\int_{t_0}^{\infty} p(t)\Delta t = \infty,$$

then every solution of (1.2) is oscillatory on $[t_0, \infty)$.

Note that, the results given by Erbe and Peterson [12], can not be applied when $r$ is unbounded, $\mu(t) = 0$ and $p(t) = t^{-\alpha}$ when $\alpha > 1$.

Došlý and Hilger [9] considered also (1.2) and established some necessary and sufficient conditions for oscillation of all solutions on unbounded time scales. However, the oscillation criteria require additional assumptions on the unknown solutions, which may not be easy to check. The results in [9, 12] were improved recently by Saker [22], Bohner and Saker [6] and Erbe, Peterson and Saker [14].

Agarwal, Bohner and Saker [1] considered the delay dynamic equation (1.1) on a time scale and proved that if

$$\int_{t_0}^{\infty} \sigma(s)p(s)\Delta s = \infty, \tag{1.3}$$

then every bounded solution of (1.1) oscillates. Also they proved that if (1.3) holds and

$$\int_{t_0}^{\infty} \frac{\tau(s)}{\sigma(s)} p(s)\Delta s = \infty, \tag{1.4}$$

then every solution of (1.1) is oscillatory. Moreover they proved that if (1.3) holds and

$$\limsup_{t\to\infty} \left\{ t \int_t^{\infty} \frac{\tau(s)}{s} p(s)\Delta s \right\} = \infty, \tag{1.5}$$

then every solution of (1.1) oscillates. Also Agarwal, Bohner and Saker [1] proved that the oscillation of (1.1) is equivalent to the oscillation of a first order delay dynamic inequality and established some new oscillation criteria which depending on the oscillation results of first order delay dynamic equations established by Zhang and Deng [28]. They proved that every solution of (1.1) is oscillatory if

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\tau(s)\Delta s > \frac{1}{c} \text{ for some } c \in (0,1), \tag{1.6}$$

or

$$\limsup_{t\to\infty} \sup_{\lambda>0,\, -\lambda cp\tau\in\Re^+} f(\lambda) < 1, \tag{1.7}$$

where

$$f(\lambda) = \lambda e_{-\lambda cp\tau}(t,\ \tau(t)).$$

For definition of the exponential function $e_p(t, t_0)$ and its properties, we refer the reader to Chapter 1 in [4]. Note that the condition (1.6) is not sharp, since it is depending on the constant $c \in (0, 1)$.

Equation (1.1) in its general form involve some different types of differential and difference equations depending on the choice of the time scale $\mathbb{T}$. For example when $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, $\mu(t) = 0$,

$$f^{\Delta}(t) = f'(t), \quad \int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

and (1.1) becomes the second-order delay differential equation

$$x''(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1.8}$$

and (1.6) and (1.7) become

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\tau(s)ds > \frac{1}{c}, \tag{1.9}$$

and

$$f(\lambda) = \lambda e_{-\lambda cp\tau}(t, \tau(t)) = \lambda e^{-M\lambda}, \text{ where } M = \int_{\tau(t)}^{t} cp(s)\tau(s)ds,$$

for some $c \in (0,1)$. For $\lambda^* := 1/M$, we have $f'(\lambda^*) = 0$, and $f''(\lambda^*) < 0$. Thus

$$\sup_{\lambda>0,\, -\lambda cp\tau \in \Re^+} \lambda e_{-\lambda cp\tau}(t, \tau(t)) = f(\lambda^*) = \frac{1}{Me},$$

so that for some $c \in (0,1)$

$$\limsup_{t\to\infty} \sup_{\lambda>0,\, -\lambda cp\tau \in \Re^+} f(\lambda) < 1, \text{ if } \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\tau(s)ds > \frac{1}{ce}. \tag{1.10}$$

For oscillation of second-order differential equation (1.8) when $\tau(t) = t$, Hille [19] proved that if

$$\liminf_{t\to\infty} t \int_t^\infty p(s)ds > \frac{1}{4}, \tag{1.11}$$

then every solution of (1.8) oscillates.

Nehari [21] also considered (1.8), when $\tau(t) = t$, and proved that if

$$\liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} s^2 p(s)ds > \frac{1}{4}, \tag{1.12}$$

then every solution of (1.8) oscillates.

Wong [26] generalized the Hille-type condition for (1.8) and proved that if $\tau(t) \geq \alpha t$ with $0 < \alpha < 1$, and

$$\liminf_{t\to\infty} t \int_t^\infty p(s)ds > \frac{1}{4\alpha}, \tag{1.13}$$

then every solution of (1.8) is oscillatory.

Erbe [10] improved the condition (1.13), without any additional assumption on $\tau(t)$, and proved that if

$$\liminf_{t\to\infty} t \int_t^\infty p(s)\frac{\tau(s)}{s}ds > \frac{1}{4}, \tag{1.14}$$

then every solution of (1.8) is oscillatory.

When $\mathbb{T} = \mathbb{Z}$, we have $\sigma(t) = t + 1$, $\mu(t) = 1$, $f^\Delta(t) = \Delta f(t)$, $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$, and (1.1) becomes the delay difference equation

$$\Delta^2 x(t) + p(t)x(\tau(t)) = 0. \tag{1.15}$$

Li and Jiang [20] considered (1.15) when $\tau(t) = t$ and proved that if

$$\liminf_{t\to\infty} \frac{1}{t} \sum_{s=t_0}^{t-1} s^2 p(s) > \frac{1}{4},$$

then every solution of (1.15) oscillates. We note that the last condition of Li and Jiang [20] is the discrete analogy of the Nehari [21] condition (1.12).

When $\mathbb{T} = h\mathbb{Z}$, $h > 0$, we have $\sigma(t) = t + h$, $\mu(t) = h$, $x^\Delta(t) = \Delta_h x(t) = (x(t+h) - x(t))/h$, $\int_a^b f(t)\Delta t = \sum_{k=0}^{\frac{b-a-h}{h}} f(a + kh)h$, and (1.1) becomes the generalized second-order delay difference equation

$$\Delta_h^2 x(t) + p(t)x(\tau(t)) = 0. \tag{1.16}$$

When $\mathbb{T} = \{t : t = q^k, k \in \mathbb{N}_0, q > 1\}$, we have $\sigma(t) = qt$, $\mu(t) = (q-1)t$, $x^{\Delta}(t) = \Delta_q x(t) = (x(q\,t) - x(t))/(q-1)\,t$, $\int_{t_0}^{\infty} f(t)\Delta t = \sum_{k=0}^{\infty} f(q^k)\mu(q^k)$, and (1.1) becomes the second-order delay $q-$difference equation

$$\Delta_q^2 x(t) + p(t)x(\tau(t)) = 0. \tag{1.17}$$

When $\mathbb{T} = \mathbb{N}_0^2 = \{t^2 : t \in \mathbb{N}_0\}$, we have $\sigma(t) = (\sqrt{t}+1)^2$ and $\mu(t) = 1 + 2\sqrt{t}, x^{\Delta}(t) = \Delta_0 x(t) = (x((\sqrt{t}+1)^2) - x(t))/1 + 2\sqrt{t}$, and (1.1) becomes the second-order delay difference equation

$$\Delta_0^2 x(t) + p(t)x(\tau(t)) = 0. \tag{1.18}$$

When $\mathbb{T} = \mathbb{T}_n = \{t_n : n \in \mathbb{N}\}$ where $(t_n)$ is the harmonic numbers that are defined by $t_0 = 0$ and $t_n = \sum_{k=1}^{n} \frac{1}{k}$, $n \in \mathbb{N}_0$, we have $\sigma(t_n) = t_{n+1}$, $\mu(t_n) = \frac{1}{n+1}$, $x^{\Delta}(t) = \Delta_1 x(t_n) = (n+1)x(t_n)$ and (1.1) becomes the second-order delay difference equation

$$\Delta_1^2 x(t_n) + p(t)x(\tau(t_n))) = 0. \tag{1.19}$$

When $\mathbb{T}_2$=$\{\sqrt{n} : n \in \mathbb{N}_0\}$, we have $\sigma(t) = \sqrt{t^2+1}$,

$$\mu(t) = \sqrt{t^2+1} - t,\ x^{\Delta}(t) = \Delta_2 x(t) = (x(\sqrt{t^2+1}) - x(t))/\sqrt{t^2+1} - t,$$

and (1.1) becomes the second-order delay difference equation

$$\Delta_2^2 x(t) + p(t)x(\tau(t)) = 0. \tag{1.20}$$

When $\mathbb{T}_3$=$\{\sqrt[3]{n} : n \in \mathbb{N}_0\}$, we have $\sigma(t) = \sqrt[3]{t^3+1}$ and

$$\mu(t) = \sqrt[3]{t^3+1} - t,\ x^{\Delta}(t) = \Delta_3 x(t) = (x(\sqrt[3]{t^3+1}) - x(t))/\sqrt[3]{t^3+1} - t,$$

and (1.1) becomes the second-order delay difference equation

$$\Delta_3^2 x(t) + p(t)x(\tau(t)) = 0. \tag{1.21}$$

Note that the integration formula on a discrete time scale is defined by
$\int_a^b f(t)\Delta t =\sum_{t\in(a,b)} f(t)\mu(t)$.

The objective of this paper is to establish some new oscillation criteria for (1.1). First, we establish some new integral oscillation criteria for (1.1) which are formulated in terms of solutions of certain inequalities and enable us to obtain some new effective sufficient conditions for the oscillation of (1.1) which improve the conditions (1.4), (1.5), (1.6) and (1.7) established by Agarwal, Bohner and Saker in [1]. Second by employing the Riccati transformation technique and analyzing the Riccati dynamic inequality, we formulate another sufficient conditions for oscillation of (1.1) which can be applied on the delay case as well as in the nondelay case. Our results in the special case when $\mathbb{T} = \mathbb{R}$ include the oscillation conditions (1.11), (1.12) and (1.14) established by Hille [19], Nehari [21] and Erbe [10] and improve the condition (1.13) established by Wong [26] and also the results established by Yan [27]. In the case, when $\tau(t) = t$ and $\mathbb{T} = \mathbb{Z}$, i.e., for difference equations without delay, our results include the oscillation results established by Li and Jiang [20] for difference equations. The results are essentially new for the equations (1.16)-(1.21). An example is considered to illustrate the main results.

## 2. MAIN OSCILLATION RESULTS

In what follows and later, we assume that

$$\int_{t_0}^{\infty} \sigma(s)p(s)\Delta s = \infty. \tag{2.1}$$

We start with the following lemmas which we will use in the proof or our main results.

**Lemma 2.1.** [1]. *Let $x$ be a positive solution of (1.1) on $[t_0, \infty)$ and $T = \tau_{-1}(t_0)$. Then*

*(i)* $x^{\Delta}(t) \geq 0,\, x(t) \geq tx^{\Delta}(t)$ *for* $t \geq T$;
*(ii)* $x$ *is nondecreasing, while* $x(t)/t$ *is nonincreasing on* $[T, \infty)$.
Lemma 2.1 immediately implies the following lemma.

**Lemma 2.2.** *Let* $x(t)$ *be a positive solution of (1.1) on* $[t_0, \infty)$ *and* $T = \tau_{-1}(t_0)$. *Then the function* $y(t)$ *defined by* $y(t) = x^{\Delta}(t)$ *is a positive solution of the delay dynamic inequality*

$$y^{\Delta}(t) + \tau(t)p(t)y(\tau(t)) \leq 0. \tag{2.2}$$

From Lemma 2.2 we see that that the oscillation of (1.1) is equivalent to the oscillation of the first order dynamic inequality (2.2) and this is formulated in the following theorem.

**Theorem 2.3.** *Assume that the differential inequality (2.2) has no eventually positive solution. Then the equation (1.1) is oscillatory.*

Theorem 2.3 reduces the question of oscillation of (1.1) to that of the absence of eventually positive solutions of the dynamic inequality (2.2) and follows the argument in Agarwal, Bohner and Saker [1] by using Corollary 2 in [28], we obtain the following oscillation criteria which essentially improve the oscillation conditions (1.6) and (1.7), since the conditions (2.3) and (2.4) does not include any additional constants.

**Theorem 2.4.** *Each one of the conditions*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\tau(s)\Delta s > 1, \tag{2.3}$$

*and*

$$\limsup_{t\to\infty} \sup_{\lambda>0,\, -\lambda p\tau \in \Re^{+}} \lambda e_{-\lambda p\tau}(t, \tau(t)) < 1, \tag{2.4}$$

*guarantees the oscillation of (1.1).*

From Lemma 2.2, it is clear that the oscillation conditions of the second-order delay dynamic equation (1.1) depending on the estimation

$$x(\tau(t)) \geq \tau(t)x^{\Delta}(\tau(t)), \quad \text{for } t \geq T. \tag{2.5}$$

This inequality, however, can be improved and lead to new oscillation criteria which further also improve the conditions (2.3) and (2.4).

**Lemma 2.5.** *Let* $x$ *be a positive solution of (1.1) on* $[t_0, \infty)$ *and* $T = \tau_{-1}(t_0)$. *Then*

$$x(\tau(t)) \geq \delta(t)x^{\Delta}(\tau(t)) \ \textit{ for } t \geq \tau_{-1}(T), \tag{2.6}$$

*where*

$$\delta(t) = \tau(t) + \int_{T}^{\tau(t)} \sigma(s)\tau(s)p(s)\Delta s. \tag{2.7}$$

*Proof.* Integrate the identity $(x(t) - tx^{\Delta}(t))^{\Delta} = -\sigma(t)x^{\Delta\Delta}(t) = \sigma(t)p(t)x(\tau(t))$, from $T$ to $\tau(t)$, we have

$$x(\tau(t)) - \tau(t)x^{\Delta}(\tau(t)) - x(T) + x^{\Delta}(\tau(T)) = \int_{T}^{\tau(t)} \sigma(s)p(s)x(\tau(s))\Delta s.$$

This implies that

$$\begin{aligned} x(\tau(t)) &= \tau(t)x^{\Delta}(\tau(t)) + x(T) - x^{\Delta}(\tau(T)) + \int_{T}^{\tau(t)} \sigma(s)p(s)x(\tau(s))\Delta s \\ &\geq \tau(t)x^{\Delta}(\tau(t)) + \int_{T}^{\tau(t)} \sigma(s)p(s)x(\tau(s))\Delta s. \end{aligned}$$

Now, by using the estimation $x(\tau(t)) \geq \tau(t)x^{\Delta}(\tau(t))$, we obtain

$$x(\tau(t)) \geq \tau(t)x^{\Delta}(\tau(t)) + \int_T^{\tau(t)} \sigma(s)p(s)\tau(s)x^{\Delta}(\tau(s))\Delta s.$$

Using the fact that $x^{\Delta\Delta}(t) < 0$, we get

$$x(\tau(t)) \geq \tau(t)x^{\Delta}(\tau(t)) + x^{\Delta}(\tau(t)) \int_T^{\tau(t)} \sigma(s)p(s)\tau(s)\Delta s, \quad \text{for } t \geq \tau_{-1}(T).$$

The last inequality implies (2.6). The proof is complete. ∎

Lemma 2.5 immediately implies the following.

**Lemma 2.6.** *Let $x$ be a positive solution of (1.1) on $[t_0, \infty)$ and $T = \tau_{-1}(t_0)$. Then the function $y(t)$ defined by $y(t) = x^{\Delta}(t)$ is a positive solution of the delay dynamic inequality*

$$y^{\Delta}(t) + \delta(t)p(t)y(\tau(t)) \leq 0, \tag{2.8}$$

*where $\delta(t)$ is defined by (2.7).*

Lemma 2.6 immediately implies the following oscillation result.

**Theorem 2.7.** *Assume that the differential inequality (2.8) has no eventually positive solution. Then the equation (1.1) is oscillatory.*

Again, Theorem 2.7 reduces the question of oscillation of (1.1) to that of the absence of eventually positive solutions of the dynamic inequality (2.8) and, as in Theorem 2.8, we obtain the following results.

**Theorem 2.8.** . *Each one of the conditions*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s) \left( \tau(s) + \int_T^{\tau(s)} \sigma(u)\tau(u)p(u)\Delta u \right) \Delta s > 1, \tag{2.9}$$

*and*

$$\limsup_{t\to\infty} \sup_{\lambda>0,\, -\lambda p\delta\in\Re^+} f(\lambda) = \limsup \sup_{\lambda>0,\, -\lambda p\delta\in\Re^+} \lambda e_{-\lambda p\delta}(t, \tau(t)) < 1, \tag{2.10}$$

*guarantees the oscillation of (1.1).*

We note that, the above results can be applied only in the case when $\tau(t) < t$. In the following, we establish some new oscillation criteria which can be applied on the nondelay case as well as the delay case. In the following theorem, we extend Hille supremum condition for delay dynamic equation (1.1).

**Theorem 2.9.** *Assume that*

$$\limsup_{t\to\infty} t \int_t^{\infty} p(s)\frac{\tau(s)}{s}\Delta s > 1, \tag{2.11}$$

*then every solution of (1.1) is oscillatory*.

*Proof.* Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (1.1) such that $x(t) > 0$ for all $t \geq T_0$. Then there exists a $t_1$ such that $x(\tau(t)) > 0$ for every $t \geq T_0$. From (1.1), we have

$$-x^{\Delta\Delta}(t)) = p(t)x(\tau(t)). \tag{2.12}$$

Integrating (2.12) from $t$ to $T$, we obtain

$$\int_t^T p(s)x(\tau(s))\Delta s \leq x^{\Delta}(t),$$

and hence

$$\int_t^\infty p(s)x(\tau(s))\Delta s \leq x^{\Delta}(t). \tag{2.13}$$

Now by Lemma 2.1, since $x(t)/t$ is a nonincearsing function, we have

$$x(\tau(t)) \geq \frac{\tau(t)}{t}x(t), \, and \; x(t)/t \geq x^{\Delta}(t). \tag{2.14}$$

This and (2.13) imply that

$$x(t) \geq t\int_t^\infty p(s)\frac{\tau(s)}{s}x(s)\Delta s.$$

Since $x(t)$ is positive and increasing, it follows that

$$t\int_t^\infty p(s)\frac{\tau(s)}{s}\Delta s \leq 1,$$

which contradicts (2.11). The proof is complete. ∎

From Theorem 2.9, we can derive some sufficient conditions for oscillation of equations (1.8), (1.15)-(1.19). In the following we derive some sufficient conditions for oscillation of (1.8) and (1.15) and for the equations (1.16)-(1.19), the details are left to the reader.

**Corollary 2.10.** *Assume that* $\int_{t_0}^\infty p(s)\sigma(s)ds = \infty$. *If*

$$\limsup_{t\to\infty} t\int_t^\infty p(s)\frac{\tau(s)}{s}ds > 1, \tag{2.15}$$

*then every solution of (1.8) oscillates.*

**Corollary 2.11.** *Assume that* $\sum_{s=t_0}^\infty p(s)\sigma(s) = \infty$. *If*

$$\limsup_{t\to\infty} t\sum_{s=t}^\infty p(s)\frac{\tau(s)}{s} > 1,$$

*then every solution of (1.15) oscillates.*

Observe that (2.15) improves Corollary 2.6 of Yan [27], since our result does not require the additional constant $\varepsilon$, where $0 < \varepsilon < 1$.

In the following, we employ the Riccati transformation technique and establish some new oscillation criteria for (1.1). For convenience, we define

$$p_* := \liminf_{t\to\infty} \sigma(t)\int_{\sigma(t)}^\infty P(s)\Delta s \;\text{ and }\; q_* := \liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^t s^2 P(s)\Delta s, \tag{2.16}$$

where $P(s) = p(s)\frac{\tau(s)}{s}$. We assume that the step function $\mu(t)$ satisfies $\mu(t) \leq h_0$.

**Theorem 2.12.** *Let $x(t)$ be a nonoscillatory solution of (1.1) such that $x(\tau(t)) > 0$ for $t \geq t_1 > \tau_{-1}(t_0)$. Let $w(t) := \frac{x^\Delta(t)}{x(t)}$ and*

$$r := \liminf_{t\to\infty} \sigma(t)w^\sigma, \quad \textit{and} \quad R := \limsup_{t\to\infty} tw^\sigma, \tag{2.17}$$

*then*

$$p_* \leq r - r^2, \quad \textit{and} \quad q_* \leq R - R^2. \tag{2.18}$$

*Proof.* Since $x(t)$ be a nonoscillatory solution of (1.1) such that $x(\tau(t)) > 0$ for $t \geq t_1 > \tau_{-1}(t_0)$, we have by Lemma 2.1, that $x^\Delta(t) > 0$ for $t \geq t_1$ and this implies that $w(t) = \frac{x^\Delta(t)}{x(t)} > 0$ and satisfies

$$\begin{aligned} w^\Delta(t) &= \left(x^\Delta\right)^\sigma \left[\frac{1}{x(t)}\right]^\Delta + \frac{1}{x(t)}x^{\Delta\Delta}(t) = \frac{-x^\Delta(t)\left(x^\Delta\right)^\sigma}{x(t)x^\sigma)} + \frac{1}{x(t)}x^{\Delta\Delta}(t) \\ &= -\frac{\left(x^\Delta\right)^\sigma}{x^\sigma}\frac{x^\Delta(t)}{x(t)} + \frac{1}{x(t)}x^{\Delta\Delta}(t). \end{aligned}$$

From Lemma 2.1, since $x(t)/t$ is nonincreasing we have $\frac{x(\tau(t))}{x(t)} \geq \frac{\tau(t)}{t}$. This and (1.1), imply that $w(t)$ satisfies the Riccati dynamic inequality

$$w^\Delta(t) + P(t) + w(t)w^\sigma \leq 0, \quad \text{for } t \geq t_1. \tag{2.19}$$

Since $P(t) > 0$, and $w(t) > 0$ for $t \geq t_1$, we have

$$\frac{w^\Delta(t)}{w(t)w^\sigma} < -1, \quad for \;\; t \geq t_1,$$

i. e.,

$$\left(\frac{-1}{w(t)}\right)^\Delta < -1.$$

Integrating the last inequality from $t_1$ to $t$, we have

$$(t - t_1)w(t) < 1, \qquad t \geq t_1, \tag{2.20}$$

which implies that

$$\lim_{t\to\infty} w(t) = 0 \text{ and } \lim_{t\to\infty} \frac{1}{t}\int_{t_1}^t w(s)\Delta s = 0. \tag{2.21}$$

By (2.17), (2.20) and the fact that $w(t) \geq w^\sigma$, we see that

$$0 < r < 1, \quad \text{and} \quad 0 < R < 1. \tag{2.22}$$

Hence

$$r - r^2 > 0, \quad \text{and} \quad R - R^2 > 0. \tag{2.23}$$

Now, we prove that (2.18) holds. Integrating (2.19) from $\sigma(t)$ to $\infty$ ($\sigma(t) \geq t_1$) and using (2.21), we have

$$w^\sigma \geq \int_{\sigma(t)}^\infty P(s)\Delta s + \int_{\sigma(t)}^\infty w(s)w^\sigma\Delta s \;\text{ for } t \geq t_1. \tag{2.24}$$

From (2.19), we see that $w^\Delta(t) \leq 0$, and this implies that $w(t) \geq w^\sigma$. Using this in (2.19), we find that

$$w^\Delta(t) + P(t) + (w^\sigma)^2 \leq 0, \quad \text{for } t \geq t_1. \tag{2.25}$$

Multiplying (2.25) by $t^2$, and integrating from $t_1$ to $t$ ($t \geq t_1$) and using the integration by parts, we obtain

$$\begin{aligned}
\int_{t_1}^{t} s^2 P(s)\Delta s &\leq -\int_{t_1}^{t} s^2 w^{\Delta}(s)\Delta s - \int_{t_1}^{t} s^2 \left(w^{\sigma}\right)^2 \Delta s \\
&= \left[-t^2 w(t)\right]_{t_1}^{t} + \int_{t_1}^{t} (s^2)^{\Delta} w^{\sigma}\Delta s - \int_{t_1}^{t} s^2 \left(w^{\sigma}\right)^2 \Delta s \\
&= -t^2 w(t) + t_1^2 w(t_1) + \int_{t_1}^{t} (s+\sigma(s)) w^{\sigma}\Delta s - \int_{t_1}^{t} s^2 \left(w^{\sigma}\right)^2 \Delta s \\
&= -t^2 w(t) + t_1^2 w(t_1) + \int_{t_1}^{t} 2s w^{\sigma}\Delta s - \int_{t_1}^{t} s^2 \left(w^{\sigma}\right)^2 \Delta s \\
&\quad + \int_{t_1}^{t} \mu(s) w^{\sigma}\Delta s.
\end{aligned}$$

It follows that

$$tw(t) \leq \frac{t_1^2 w(t_1)}{t} + \frac{h_0}{t}\int_{t_1}^{t} w^{\sigma}\Delta s - \frac{1}{t}\int_{t_1}^{t} s^2 P(s)\Delta s + \frac{1}{t}\int_{t_1}^{t} \left[2sw^{\sigma} - s^2\left(w^{\sigma}\right)^2\right]\Delta s.$$

Then, since $w(t) \geq w^{\sigma}$, we have

$$\begin{aligned}
tw^{\sigma} &\leq \frac{t_1^2 w(t_1)}{t} + \frac{h_0}{t}\int_{t_1}^{t} w^{\sigma}\Delta s - \frac{1}{t}\int_{t_1}^{t} s^2 p(s)\Delta s \\
&\quad + \frac{1}{t}\int_{t_1}^{t} \left[2sw^{\sigma} - s^2\left(w^{\sigma}\right)^2\right]\Delta s. \qquad (2.26)
\end{aligned}$$

From (2.21), since $\lim_{t\to\infty} \frac{1}{t}\int_{t_1}^{t} w(s)\Delta s = 0$, and $w(t) \geq w^{\sigma}$, we get

$$\lim_{t\to\infty}\left[\frac{t_1^2 w(t_1)}{t} + \frac{h_0 \int_{t_1}^{t} w^{\sigma}\Delta s}{t}\right] = 0. \qquad (2.27)$$

Using the inequality $a^2 + b^2 \geq 2ab$, we have

$$\left[2sw^{\sigma} - s^2\left(w^{\sigma}\right)^2\right] \leq 1,$$

and this implies that

$$\lim_{t\to\infty}\sup \frac{1}{t}\int_{t_1}^{t} \left[2sw^{\sigma} - s^2\left(w^{\sigma}\right)^2\right]\Delta s \leq 1. \qquad (2.28)$$

From (2.24) and (2.26)-(2.28), we find that

$$\lim_{t\to\infty}\inf \sigma(t)w^{\sigma} = r \geq p_*, \text{ and } \lim_{t\to\infty}\sup tw^{\sigma} = R \leq 1 - q_*.$$

Then it follows that for any $0 < \epsilon < \min\{r, 1-R\}$ there exists $t_2 \geq t_1$ such that

$$r - \epsilon < \sigma(t)w^{\sigma} < r + \epsilon, \text{ and } R - \epsilon < tw^{\sigma} < R + \epsilon,$$

$$\sigma(t)\int_{\sigma(t)}^{\infty} P(s)\Delta s \geq p_* - \epsilon, \quad \textit{and} \quad \frac{1}{t}\int_{t_0}^{t} s^2 P(s)\Delta s > q_* - \epsilon, \ t \geq t_2.$$

From (2.24), we get

$$\begin{aligned} w^\sigma &\geq \int_{\sigma(t)}^{\infty} P(s)\Delta s + \int_{\sigma(t)}^{\infty} w(s)w^\sigma \Delta s \geq \int_{\sigma(t)}^{\infty} P(s)\Delta s + \int_{\sigma(t)}^{\infty} \frac{sw^\sigma sw^\sigma}{s^2}\Delta s \\ &\geq \int_{\sigma(t)}^{\infty} P(s)\Delta s + \int_{\sigma(t)}^{\infty} (r-\epsilon)^2 \frac{1}{s\sigma(s)}\Delta s \\ &\geq \int_{\sigma(t)}^{\infty} P(s)\Delta s + (r-\epsilon)^2 \int_{\sigma(t)}^{\infty} \left(\frac{-1}{s}\right)^{\Delta} \Delta s \\ &= \int_{\sigma(t)}^{\infty} P(s)\Delta s + \frac{1}{\sigma(t)} (r-\epsilon)^2 . \end{aligned}$$

This implies that,

$$\sigma(t)w^\sigma \geq \sigma(t) \int_{\sigma(t)}^{\infty} P(s)\Delta s + (r-\epsilon)^2 .$$

Then

$$r = \lim_{t\to\infty} \inf \sigma(t)w^\sigma \geq p_* - \epsilon + (r-\epsilon)^2 \text{ for } t \geq t_2. \tag{2.29}$$

Also, from (2.26), we have

$$\lim_{t\to\infty} \sup tw^\sigma \leq -q_* + \epsilon + (R+\epsilon)(2-R-\epsilon), \text{ for } t \geq t_2 . \tag{2.30}$$

Then, from (2.29) and (2.30) since $\epsilon$ is arbitrarily small, we have,

$$p_* \leq r - r^2, \text{ and } q_* \leq R - R^2.$$

Thus, (2.18) holds and this completes the proof. ∎

Now, from Theorem 2.12 we establish some new oscillation criteria for (1.1) which can be considered as the extensions of the oscillation conditions (1.12) and (1.14) established by Erbe [10] and Nehari [21] for differential equations.

**Theorem 2.13.** *If*

$$\lim_{t\to\infty} \inf \sigma(t) \int_{\sigma(t)}^{\infty} \left(p(s)\frac{\tau(s)}{s}\right) \Delta s > \frac{1}{4}, \tag{2.31}$$

*then every solution of (1.1) oscillates.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of (1.1) such that $x(\tau(t)) > 0$ for $t \geq t_1 > \tau_{-1}(t_0)$. Let $r = \lim_{t\to\infty} \inf \sigma(t)w^\sigma$. Then by Theorem 2.12, we have

$$p_* \leq r - r^2 \leq 1/4,$$

and this contradicts (2.31). The proof is complete. ∎

**Theorem 2.14.** *If*

$$\lim_{t\to\infty} \inf \frac{1}{t} \int_{t_0}^{t} s^2 \left(p(s)\frac{\tau(s)}{s}\right) \Delta s > \frac{1}{4}, \tag{2.32}$$

*then every solution of (1.1) oscillates.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of (1.1) such that $x(\tau(t)) > 0$ for $t \geq t_1 > \tau_{-1}(t_0)$. Let $R = \limsup_{t\to\infty} tw^\sigma$. Then by Theorem 2.12, we have

$$q_* \leq R - R^2 \leq 1/4,$$

and this contradicts (2.32). The proof is complete. ∎

When $\mathbb{T} = \mathbb{R}$, we see that the condition (2.31) in Theorem 2.13 becomes

$$\liminf_{t\to\infty} t \int_t^\infty \left( p(s) \frac{\tau(s)}{s} \right) ds > \frac{1}{4},$$

which is the oscillation condition (1.14) established by Erbe [10] and when $\tau(t) = t$, this condition becomes the condition (1.11) of Hille [19] . Also form Theorem 2.14 when $\mathbb{T} = \mathbb{R}$ the condition (2.32) becomes

$$\liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^t s^2 \left( p(s) \frac{\tau(s)}{s} \right) ds > \frac{1}{4},$$

and when and $\tau(t) = t$, this condition becomes

$$\liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^t s^2 p(s) ds > \frac{1}{4},$$

which is the oscillation condition (1.12) established by Nehari [21].

When $\mathbb{T} = \mathbb{N}$, we have from Theorems 2.13 and 2.14 the following oscillation results for (1.15).

**Corollary 2.15.** *Each one of the conditions*

$$\liminf_{t\to\infty} (t+1) \sum_{s=t+1}^{\infty} \left( p(s) \frac{\tau(s)}{s} \right) \Delta s > \frac{1}{4},$$

*and*

$$\liminf_{t\to\infty} \frac{1}{t} \sum_{s=t_0}^{t-1} s^2 \left( p(s) \frac{\tau(s)}{s} \right) > \frac{1}{4},$$

*guarantees the oscillation of (1.15).*

From Corollary 2.15 when $\tau(t) = t$, we have the following result that has been established by Li and Jiang [20].

**Corollary 2.16.** [20]. *If*

$$\liminf_{t\to\infty} \frac{1}{t} \sum_{s=t_0}^{t-1} s^2 p(s) > \frac{1}{4},$$

*then every solution of second-order difference equation*

$$\Delta^2 x(t) + p(t) x(t) = 0, \quad t \in [t_0, \infty),$$

*oscillates.*

The following example illustrates the main results.

**Example 2.1.** *Consider the second-order delay Euler dynamic equation*

$$x^{\Delta\Delta}(t) + \frac{\gamma}{t\tau(t)} x(\tau(t)) = 0, \quad t \in \mathbb{T}, \tag{2.33}$$

*for $\tau(t) \leq t$ and $\mathbb{T}$ =$[1, \infty)$. Here $p(t) = \frac{\gamma}{t\tau(t)}$. Note that (2.1) holds since*

$$\int_1^t \sigma(s) \frac{\gamma}{s\tau(s)} \Delta s \geq \int_1^t \frac{\gamma}{s} \Delta s = \infty.$$

*To apply Theorem 2.13, it remains to prove that condition (2.31) is satisfied. In our case (2.31) reads*

$$\begin{aligned}
\lim_{t\to\infty}\inf \sigma(t)\int_{\sigma(t)}^{\infty} p(s)\frac{\tau(s)}{s}\Delta s &= \gamma \lim_{t\to\infty}\inf \sigma(t)\int_{\sigma(t)}^{\infty} \frac{1}{s^2}\Delta s \\
&\geq \gamma \lim_{t\to\infty}\inf \sigma(t)\int_{\sigma(t)}^{\infty} \frac{1}{s\sigma(s)}\Delta s \\
&= \gamma \lim_{t\to\infty}\inf \sigma(t)\int_{\sigma(t)}^{\infty} \left(\frac{-1}{s}\right)^{\Delta}\Delta s = \gamma.
\end{aligned}$$

*So by Theorems 2.13, every solutions of (2.33) oscillates if* $\gamma > 1/4$. *Now, we apply Theorem 2.14. In our case (2.32) reads*

$$\begin{aligned}
\lim_{t\to\infty}\inf \frac{1}{t}\int_{t_0}^{t} s^2\left(p(s)\frac{\tau(s)}{s}\right)\Delta s &= \lim_{t\to\infty}\inf \frac{1}{t}\int_{1}^{t} s^2\frac{\gamma}{s\tau(s)}\frac{\tau(s)}{s}\Delta s \\
&= \gamma \lim_{t\to\infty}\inf \frac{1}{t}\int_{1}^{t}\Delta s = \gamma.
\end{aligned}$$

*So by Theorems 2.14, every solutions of (2.33) oscillates if* $\gamma > 1/4$.

**Remark 2.1.** Note that, the conditions (1.3), (1.4), (1.5), (1.6), (1.11), (1.13) cannot be applied for (2.33). So Theorems 2.13 and 2.14 improve the results established by Agarwal, Bohner and Saker [1] for second-order dynamic equation (2.33).

## REFERENCES


[1] R. P. AGARWAL, M. BOHNER and S. H. SAKER, Oscillation of second-order delay dynamic equations, *Canadian Appl. Math. Quart.*, **13** (2005), pp. 1-19.

[2] R. P. AGARWAL, D. O'REGAN and S. H. SAKER, Oscillation criteria for second-order nonlinear neutral delay dynamic equations, *J. Math. Anal. Appl.*, **300** (2004), pp. 203-217.

[3] E. A. BOHNER and J. HOFFACKER, Oscillation properties of an Emden-Fowler type equations on discrete time scales, *J. Diff. Eqns. Appl.*, **9** (2003), pp. 603-612.

[4] M. BOHNER and A. PETERSON, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[5] M. BOHNER, A. PETERSON, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[6] M. BOHNER and S. H. SAKER, Oscillation of second order nonlinear dynamic equations on time scales, *Rocky Mountain J. Math.*, **34** (2004), pp. 1239-1254.

[7] E. A. BOHNER, M. BOHNER and S. H. SAKER, Oscillation criteria for a certain class of second order Emden-Fowler dynamic equations, *Elect. Transc. Numerical. Anal.*, (to appear).

[8] M. BOHNER and C. C. TISDELL, Oscillation and nonoscillation of forced second order dynamic equations. *Pacifc J. Math.*, (in press)

[9] O. DOŠLY and S. HILGER, A necessary and sufficient condition for oscillation of the Sturm–Liouville dynamic equation on time scales, Special Issue on "*Dynamic Equations on Time Scales*", edited by R. P. AGARWAL, M. BOHNER, and D. O'REGAN, *J. Comp. Appl. Math.*, **141** (2002) pp. 147–158.

[10] L. ERBE, Oscillation criteria for second order nonlinear delay equations, *Canad. Math. Bull.*, **16** (1973), pp. 49-56.

[11] L. ERBE and A. PETERSON, Positive solutions for a nonlinear differential equation on a measure chain, *Mathl. Comp. Modelling, Boundary Value Problems and Related Topics,* **32** (2000), pp. 571–585.

[12] L. ERBE and A. PETERSON, Riccati equations on a measure chain, In G. S. LADDE, N. G. MEDHIN, and M. SAMBANDHAM, editors, *Proceedings of Dynamic Systems and Applications*, **3**, pp. 193–199, Atlanta, 2001. Dynamic publishers.

[13] L. ERBE and A. PETERSON, Boundedness and oscillation for nonlinear dynamic equations on a time scale, *Proc. Amer. Math. Soc*., **132** (2004), pp. 735–744.

[14] L. ERBE, A. PETERSON and S. H. SAKER, Kamenev-type oscillation criteria for second-order linear delay dynamic equations, *Dyn. Syst. Appl*., **15** (2006), pp. 65-78.

[15] L. ERBE, A. PETERSON and S. H. SAKER, Oscillation criteria for second-order nonlinear dynamic equations on time scales, *J. London Math. Soc*., **67** (2003), pp. 701-714.

[16] L. ERBE, A. PETERSON, and P. REHAK, Comparison theorems for linear dynamic equations on time scales, *J. Math. Anal. Appl*., **275**, (2002), pp. 418-438.

[17] L. ERBE, M. BOHNER and A. PETERSON, Oscillation for nonlinear second order dynamic equations on time scales, *J. Math. Anal. Appl*., **301** (2005), pp. 491-507.

[18] J. HOFFACKER and C. C. TISDELL, Stability and instability for dynamic equations on time scales, *Comput. Math. Appl.*, **49** (2005), pp. 1327-1334.

[19] E. HILLE, Non-oscillation theorems, *Trans. Amer. Math. Soc*., **64** (1948), pp. 234-252.

[20] X. LI and J. JIANG, Oscillation of second-order linear difference equations, *Mathl. Comp. Modelling*, **35** (2002), pp. 983-990.

[21] Z. NEHARI, Oscillation criteria for second-order linear differential equations, *Trans. Amer. Math. Soc*., **85** (1957), pp. 428-445.

[22] S. H. SAKER, Oscillation of nonlinear dynamic equations on time scales, *Appl. Math. Comp*., **148** (2004), pp. 81-91.

[23] S. H. SAKER, Oscillation criteria of second-order half-linear dynamic equations on time scales, *J. Comp. Appl. Math*., **177** (2005), pp. 375-387.

[24] S. H. SAKER, Boundedness of solutions of second-order forced nonlinear dynamic equations, *Rocky Mount. J. Math.*, **36** (2006), pp. 2027-2039.

[25] S. H. SAKER, Oscillation of second-order nonlinear neutral delay dynamic equations on time scales, *J. Comp. Appl. Math.*, **187** (2006), pp. 123-141.

[26] J. S. W. WONG, Second order oscillation with retarded arguments, in ”*Ordinary Differential Equations*”, pp. 581-596, Washington, 1971, Academic press, New York and London, 1972.

[27] J. YAN, Oscillatory properties for second order linear delay differential equations, *J. Math. Anal. Appl*., **122** (1987), pp. 380-384.

[28] B. G. ZHANG and X. DENG, Oscillation of delay differential equations on time scales, *Mathl. Comp. Modelling*, **36** (2002), pp. 1307-1318.

[29] B. G. ZHANGg and LIAN FU YUN, The distribution of generalized zeros of solutions of delay differential equations on time scales, *J. Diff. Eqns. Appl.*, **10** (2004), pp. 759-771.