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EXISTENCE OF SOLUTIONS FOR THIRD ORDER NONLINEAR BOUNDARY VALUE PROBLEMS

YUE HU AND ZUODONG YANG

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SCHOOL OF MATHEMATICS AND INFORMATION SCIENCE, HENAN POLYTECHNIC UNIVERSITY, HENAN JIAOZUO 454010, CHINA.
huu3y2@163.com

SCHOOL OF MATHEMATICS AND COMPUTER SCIENCE, NANJING NORMAL UNIVERSITY, JIANGSU NANJING 210097, CHINA;

COLLEGE OF ZHONGBEI, NANJING NORMAL UNIVERSITY, JIANGSU NANJING 210046, CHINA.
zdyang_jin@263.net
yangzuodong@njnu.edu.cn

ABSTRACT. In this paper, the existence of solution for a class of third order quasilinear ordinary differential equations with nonlinear boundary value problems

$$(\Phi_p(u''))' = f(t, u, u', u''),\ u(0) = A,\ u'(0) = B,\ R(u'(1), u''(1)) = 0$$

is established. The results are obtained by using upper and lower solution methods.

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1. INTRODUCTION

We consider the nonlinear equation

$$(\Phi_p(u''))' = f(t, u, u', u''), \quad t \in I = [0, 1] \tag{1.1}$$

satisfying the conditions

$$u(0) = A, \quad u'(0) = B, \quad R(u'(1), u''(1)) = 0, \tag{1.2}$$

where A and B are constants, and $\Phi_p(s) = |s|^{p-2}s, p > 1$. Equations of the above form are mathematical models occuring in studies of the p-Laplace equation, generalized reaction-diffusion theory [1], non-Newtonian fluid theory, and the turbulent flow of a gas in porous medium [2]. In the non-Newtonian fluid theory, the quantity p is characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids.

For the equation

$$(\Phi_p(u'))' = f(t, u, u'), \quad t \in I = [0, 1] \tag{1.3}$$

with different boundary conditions has been studied by many authors, see, for example, [3]-[11] and references therein. Our results were motivated by the paper [3], [4], [14]-[17], [21] which studied periodic and Neumann with nonlinear boundary conditions for Eq. (1.3). On the contrary, it seems that little is known about the result for problem (1.1)-(1.2). When $p = 2$, the related some results have been obtained by [18]-[21] for problem (1.1)-(1.2). In this paper, we obtain the existence of solutions to the problem (1.1)-(1.2), extended to results and complement to the results by [18]-[21].

2. PRELIMINARIES

In this section, we present results for second order Volterra type intergro-differential equation, which help to prove our main results.

Let us consider the following boundary value problem

$$(\Phi_p(u'))' = f(t, u, Tu, u') \tag{2.1}$$

$$u(0) = D, \quad R(u(1), u'(1)) = 0, \tag{2.2}$$

where $Tu(t) = \phi(t) + \int_0^t K(t,s)u(s)ds$, function $K(t,s) \in C([0,1] \times [0,1]), \phi(t) \in C[0,1]$, $K(t,s) \geq 0$ on $[0,1] \times [0,1]$, and D is a constant.

As in [12], we give the following definition:

Definition 2.1. We say that a function $\alpha(t) \in C^1[0,1]$ is a lower solution of Eq. (2.1) if $\Phi_p(\alpha') \in C^1(0,1)$ and satisfies

$$(\Phi_p(\alpha'))' \geq f(t, \alpha, T\alpha, \alpha'), \text{ for } t \in I = [0,1].$$

Analogously, we say that $\beta \in C^1[0,1]$ is a upper solution of Eq. (2.1) if $\Phi_p(\beta') \in C^1(0,1)$ and satisfies

$$(\Phi_p(\beta'))' \leq f(t, \beta, T\beta, \beta'), \text{ for } t \in I = [0,1].$$

Assume that $f(t,u,v,w)$ satisfies the following conditions:

(H_1) $f(t,u,v,w)$ is nonincreasing in v.

(H_2) $f(t,u,v,w) \in C([0,1] \times \mathbf{R^3}, \mathbf{R})$, for any positive constants $r_1, r_2 > 0$, there exist positive function $h(x) \in C[0,\infty)$ satisfying

$$\int_0^\infty \Phi_p^{-1}(u)/h(\Phi_p^{-1}(u))du = \infty.$$

and while $0 \le t \le 1, |u| \le r_1, |v| \le r_2, w \in \mathbf{R}, |f(t,u,v,w)| \le h(|w|)$.

From [12], we give the following Lemma

Lemma 2.1. *Let $\alpha(t)$ and $\beta(t)$ be a lower and an upper solution of Eq. (2.1), respectively, with $\alpha \le \beta$ in I. Assume that hypotheses $(H_1)-(H_2)$ are satisfied. Then boundary value problem*

$$(\Phi_p(u'))' = f(t,u,Tu,u'), \quad u(0) = A, \quad u(1) = B$$

has at least one solution $\alpha(t) \le u(t) \le \beta(t)$ for all $\alpha(0) \le A \le \beta(0), \ \alpha(1) \le B \le \beta(1)$.

Lemma 2.2. *Let $\alpha(t)$ and $\beta(t)$ be a lower and an upper solution of Eq. (2.1), respectively, with $\alpha \le \beta$ in I. Assume that $(H_1)-(H_2)$ are satisfied, and $R(u,v)$ is nondecreasing in v with continuous on $\mathbf{R^2}$, and*

$$R(\alpha(1), \alpha'(1)) \le 0 \le R(\beta(1), \beta'(1)).$$

Then boundary value problem

$$(\Phi_p(u'))' = f(t,u,Tu,u'), \quad u(0) = D, \quad R(u(1), u'(1)) = 0 \tag{2.3}$$

has at least one solution $\alpha(t) \le u(t) \le \beta(t)$ for all $\alpha(0) \le D \le \beta(0)$.

Proof. First, we assume that $\alpha(1) = \beta(1), \alpha(0) = \beta(0)$. Then, by $\alpha(t) \le \beta(t)$, it follows that $\alpha'(1) \ge \beta'(1)$. On the other hand, it is clear that $\alpha'(1) \le \beta'(1)$ from $R(\alpha(1), \alpha'(1)) \le R(\beta(1), \beta'(1))$, which means $\alpha'(1) = \beta'(1)$. Then the problem

$$(\Phi_p(u'))' = f(t,u,Tu,u'), \quad u(0) = D, \quad u(1) = \alpha(1)$$

has at least one solution $\alpha(t) \le u(t) \le \beta(t)$, which is a solution of problem (2.3).

Next we consider that $\alpha(1) < \beta(1)$ and $\alpha(0) < \beta(0)$ (if $\alpha(1) = \beta(1), \alpha(0) < \beta(0)$ or $\alpha(1) < \beta(1), \alpha(0) = \beta(0)$ similar be proved). Applying Lemma 2.1, we know that the problem

$$(\Phi_p(u'))' = f(t,u,Tu,u'), \quad u(0) = D, \quad u(1) = \alpha(1)$$

has at least one solution $\alpha_0(t)$, and $\alpha(t) \le \alpha_0(t) \le \beta(t)$, it follows that $\alpha_0'(1) \le \alpha'(1)$. From the assumptions in R, we see that

$$R(\alpha_0(1), \alpha_0'(1)) \le R(\alpha(1), \alpha'(1)) \le 0. \tag{2.4}$$

If "$=$" in (2.4) is true, then $\alpha_0(t)$ is a solution of problem (2.3). Thus the proof is complete. Otherwise, we consider the following boundary value problem:

$$(\Phi_p(u'))' = f(t,u,Tu,u'), \quad u(0) = D, \quad u(1) = \beta(1).$$

Clearly, the same reasoning gets to a solution $\beta_0(t)$ and such that

$$\alpha_0(t) \le \beta_0(t) \le \beta(t), \quad 0 \le t \le 1,$$

$$R(\beta_0(1), \beta_0'(1)) \ge R(\beta(1), \beta'(1)) \ge 0. \tag{2.5}$$

Consequently, if "=" in (2.5) is true, then the proof is completed. Otherwise we choose $d_1 = (\beta_0(1) + \alpha_0(1))/2$, and we consider the problem

$$(\Phi_p(u'))' = f(t,u,Tu,u'), \quad u(0) = D, \quad u(1) = d_1.$$

Applying Lemma 2.1, we obtain a solution $u_1(t)$ from the above problem, and $\alpha_0 \le u_1(t) \le \beta_0$. If $R(u(1), u'(1)) = 0$, then the proof is completed. If $R(u(1), u'(1)) > 0$, then let $\alpha_1(t) = \alpha_0(t), \beta_1(t) = u_1(t)$; if $R(u(1), u'(1)) < 0$, then let $\alpha_1(t) = u_1(t), \beta_1(t) = \beta_0(t)$. Hence $\beta_1(1) - \alpha_1(1) = \frac{1}{2}[\beta_0(1) - \alpha_0(1)]$. By induction method, that we have obtained $\alpha_n(t), \beta_n(t) (n = 1, 2, \cdots, m)$, which satisfy

$$\alpha_{n-1}(t) \le \alpha_n(t) \le \beta_n(t) \le \beta_{n-1}(t), \quad 0 \le t \le 1, \tag{2.6}$$

$$\beta_n(1) - \alpha_n(1) = \frac{1}{2}[\beta_{n-1}(1) - \alpha_{n-1}(1)]. \tag{2.7}$$

Then, choosing $d_{m+1} = \frac{1}{2}[\beta_m(1) + \alpha_m(1)]$, and we consider the following problem

$$(\Phi_p(u'))' = f(t, u, Tu, u'), \quad u(0) = D, \quad u(1) = d_{m+1}.$$

Consequently, by the same method used to obtain $\alpha_1(t)$ and $\beta_1(t)$, we have $\alpha_{m+1}(t)$ and $\beta_{m+1}(t)$, which satisfy

$$\alpha_m(t) \le \alpha_{m+1}(t) \le \beta_{m+1}(t) \le \beta_m(t), \quad 0 \le t \le 1,$$

$$\beta_{m+1}(1) - \alpha_{m+1}(1) = \frac{1}{2}[\beta_m(1) - \alpha_m(1)].$$

Hence, we prove the relations (2.6) and (2.7) for every n.

In view of the fact choosing $\alpha_n(t)$ and $\beta_n(t)$, it easy to see that

$$R(\alpha_n(1), \alpha'_n(1)) < 0, \quad R(\beta_n(1), \beta'_n(1)) > 0. \tag{2.8}$$

From (2.8), we imply that

$$\beta_n(1) - \alpha_n(1) = \frac{1}{2^n}[\beta_0(1) - \alpha_0(1)]. \tag{2.9}$$

In addition, Nagumo condition shows that $\{\alpha_n(t)\}, \{\beta_n(t)\}, \{\alpha'_n(t)\}, \{\beta'_n(t)\}$ are equicontinuous and uniformly bounded in $0 \le t \le 1$. Therefore, applying the Arzela-Ascoli theorem to the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$, there exist two subsequences $\{\beta_{n_j}(t)\}$ and $\{\alpha_{n_i}(t)\}$ such that as $j \to \infty, \beta_{n_j}(t) \to u_0(t), \beta'_{n_j}(t) \to u'_0(t)$, uniformly on $[0, 1]$, and as $i \to \infty$, $\alpha_{n_i}(t) \to \overline{u}_0(t), \alpha'_{n_i}(t) \to \overline{u}'_0(t)$, uniformly on $[0, 1]$. Therefore $u_0(t)$ and $\overline{u}_0(t)$ satisfies (2.1), and we have from (2.8)

$$R(u_0(1), u'_0(1)) \ge 0, \quad R(\overline{u}_0(1), \overline{u}'_0(1)) \le 0. \tag{2.10}$$

From (2.6), it is obvious that

$$u_0(t) \ge \overline{u}_0(t), \quad 0 \le t \le 1. \tag{2.11}$$

On the other hand, by (2.9), we can show that $\overline{u}_0(1) = u_0(1)$. Thus, we have from (2.11)

$$\overline{u}'_0(1) \ge u'_0(1). \tag{2.12}$$

From (2.10) and (2.12), it follows that

$$0 \le R(u_0(1), u'_0(1)) \le R(\overline{u}_0(1), \overline{u}'_0(1)) \le 0. \tag{2.13}$$

From (2.13), it is easy to show the following relations

$$0 = R(u_0(1), u'_0(1)) = R(\overline{u}_0(1), \overline{u}'_0(1)).$$

Hence, we complete the proof. ∎

3. MAIN RESULTS

In this section, we discuss the existence of solutions for boundary value problem (1.1)-(1.2).

Definition 3.1. We say that a function $\alpha(t) \in C^2[0, 1]$ is a lower solution of Eq.(1.1) if $\Phi_p(\alpha'') \in C^1(0, 1)$ and satisfies

$$(\Phi_p(\alpha''))' \ge f(t, \alpha, \alpha', \alpha''), \text{ for } t \in I.$$

Analogously, we say that $\beta \in C^2[0, 1]$ is a upper solution of Eq. (1.1) if $\Phi_p(\beta'') \in C^1(0, 1)$ and satisfies

$$(\Phi_p(\beta''))' \le f(t, \beta, \beta', \beta''), \text{ for } t \in I.$$

We obtain the following main theorem

Theorem 3.1. *Assume that*

(i) $f(t, u, u', u'')$ *is nonincreasing in* u *and continuous on* $[0, 1] \times \mathbf{R}^3$*;*

(ii) Nagumo Condition, for (t, u, u') *on* $[0, 1] \times \mathbf{R}^2$*,*

$$f(t, u, u', u'') = O(|u''|^2), \quad \textit{as } |u''| \to \infty;$$

(iii) $R(u, v)$ *is nondecreasing in* v *and continuous on* $\mathbf{R}^2$*;*

(iv) there exists an upper solution $\beta(t)$ *and a lower solution* $\alpha(t)$ *of Eq.(1.1) on* $I = [0, 1]$ *such that*

$$\alpha(t) \le \beta(t), \quad \alpha'(t) \le \beta'(t), 0 \le t \le 1,$$
$$\alpha(0) \le A \le \beta(0), \quad \alpha'(0) \le B \le \beta'(0),$$
$$R(\alpha'(1), \alpha''(1)) \le 0 \le R(\beta'(1), \beta''(1)),$$

then the boundary value problem (1.1)-(1.2) has a solution $u(t)$ *such that* $\alpha(t) \le u(t) \le \beta(t)$*.*

Proof. Set $u' = z$, then we have $u(t) = A + \int_0^t z(s)ds$. Thus, the boundary value problem (1.1)-(1.2) can be written as a boundary value problem for the second order integro-differential equation of Volterra type as below

$$(\Phi_p(z'))' = f(t, A + \int_0^t z(s)ds, z, z'), \tag{3.1}$$

$$z(0) = B, \quad R(z(1), z'(1)) = 0. \tag{3.2}$$

In order to employ Lemma 2.2, we construct the lower and upper solutions for the boundary value problem (3.1)-(3.2) by using $\alpha(t), \beta(t)$ and hypotheses (i)-(iv). We set

$$\overline{\alpha}(t) = \alpha(t) + \delta_1, \quad \overline{\beta}(t) = \beta(t) + \delta_2, \tag{3.3}$$

where $\delta_1 = A - \alpha(0), \delta_2 = \beta(0) - A$. Then, it is clear that $\overline{\alpha}(0) = A = \overline{\beta}(0)$. Moreover, if we write

$$\overline{\alpha}'(t) = \alpha_+(t), \quad \overline{\beta}'(t) = \beta_+(t), \tag{3.4}$$

it is easy show that

$$\alpha_+(t) \le \beta_+(t) \tag{3.5}$$

because of (3.4) and (iv).

Note that $\overline{\alpha}(t) = A + \int_0^t \alpha_+(s)ds, \overline{\beta}(t) = A + \int_0^t \beta_+(s)ds$. Now, using (3.4)-(3.5), (iv), and the monotonicity of f from (i), we obtain

$$(\Phi_p(\alpha_+'))' \ge f(t, A + \int_0^t \alpha_+(s)ds, \alpha_+(t), \alpha_+'(t)),$$

$$(\Phi_p(\beta_+'))' \le f(t, A + \int_0^t \beta_+(s)ds, \beta_+(t), \beta_+'(t)),$$

$$\alpha_+(0) \le B \le \beta_+(0), \quad R(\alpha_+(1), \alpha_+'(1)) \le 0 \le R(\beta_+(1), \beta_+'(1)).$$

Thus, we see that the functions $\alpha_+(t)$ and $\beta_+(t)$ are the lower and the upper solutions respectively for the boundary value problem (3.1)-(3.2). Hence, by Lemma 2.2, we have

$$\alpha_+(t) \le z(t) \le \beta_+(t), \quad 0 \le t \le 1,$$

where $z(t)$ is a solution of the boundary value problem (3.1)-(3.2). Finally, from the relation $z(t) = u'(t)$, we can recover $u(t) = A + \int_0^t z(s)ds$. ∎

Example 3.1. *We consider the following third-order boundary value problem:*

$$(\Phi_p(u''))' + (t-u)^2 - t(4+t^2)u' - (u')^2 \sin(u'') = 0, \quad 0 < t < 1, \tag{3.6}$$

$$u(0) = 0, \quad u'(0) = B, \quad (u'(1))^3 + (u''(1))^2 = 0, \tag{3.7}$$

where $\Phi_p(u) = |u|^{p-2}u, p > 1, \ -1 \le B \le 1$. *Let*

$$f(t,u,v,w) = (t-u)^2 - t(4+t^2)v - v^2 \sin w, \quad R(v,w) = v^3 + w^2.$$

It is easily to prove that $\alpha(t) = -t, \beta(t) = t$ *are lower and upper solutions of BVP (3.6)-(3.7), respectively.* f *is continuous on* $[0,1] \times \mathbf{R^3}$ *and nonincreasing in* u *when* $\alpha(t) \le u(t) \le \beta(t), t \in [0,1]$. R *are continuous on* $\mathbf{R^2}$, $R(v,w)$ *is increasing in* w. *Furthermore, we obtain* f *satisfies Nagumo condition in*

$$D = \{(t,u,v,w) \in [0,1] \times \mathbf{R^3} : -t \le u(t) \le t, -1 \le u'(t) \le 1\}.$$

Therefore, by Theorem 3.1, there exists at least one solution $u(t)$ *for BVP (3.6)-(3.7) such that*

$$-t \le u(t) \le t, \quad -1 \le u'(t) \le 1, \quad t \in [0,1].$$

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