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ON INTERACTION OF DISCONTINUOUS WAVES IN A GAS WITH DUST PARTICLES

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ABSTRACT. In this paper, the interaction of the strong shock with the weak discontinuity has been investigated for the system of partial differential equations describing one dimensional unsteady plane flow of an inviscid gas with large number of dust particles. The amplitudes of the reflected and transmitted waves after interaction of the weak discontinuity through a strong shock are evaluated by exploiting the results of general theory of wave interaction.

Key words and phrases: Shock wave, Weak Discontinuity, Interaction of shock waves.

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1. Introduction

The problem of the interaction of an acoustic wave with a shock has been studied by Swan & Fowles [1] and Van Moorhen & George [2]. The evolution of a weak discontinuity for a hyperbolic quasi-linear system of equations satisfying the Bernoulli's law has been studied quite extensively in the literatures (see for instance Varley & Cumberbatch [3], Jeffrey [4] and Boillat & Ruggeri [5]). The fact that a shock undergoes an acceleration jump as a consequence of an interaction with a weak wave [6] has been accounted for in the works of Brun [7] and Boillat & Ruggeri [8]. Radha, Sharma & Jeffrey [9] have shown that the general theory of wave interaction problem which originated from the work of Jeffrey [10] leads to the results obtained by Brun [7] and Boillat & Ruggeri [8]. The theory has been successfully applied to study the interaction of a weak discontinuity wave with a bore and to evaluate the amplitudes of reflected and transmitted waves after interaction in shallow water [11].

We consider a system of partial differential equations describing the one dimensional unsteady plane flow of an inviscid dusty gas. It is assumed that the dusty gas consists of a perfect gas and a large number of small particles. The present work is concerned with the cases when the mass concentration of the particles is comparable with that of the gas. The volume occupied by the particles is negligible because the density of the solid particles is much larger than that of the gas. The amplitudes of reflected and transmitted waves after interaction of the weak discontinuity through a strong shock are evaluated by exploiting the results of general theory of wave interaction [9].

2. Basic Equations

The governing equations can be written in the matrix form as [12] & [13]

$$U_t + AU_x = f, \tag{2.1}$$

where $U = (\rho, u, p, \sigma, v, \theta)^{tr}$, and $f = (0, -\sigma D/(m\rho), -(\gamma-1)\sigma\{(u+v)D+Q\}, 0, D/m, Q/(mC_m))^{tr}$, and

$$A = \begin{pmatrix} u & \rho & 0 & 0 & 0 & 0 \\ 0 & u & 1/\rho & 0 & 0 & 0 \\ 0 & \gamma p & u & 0 & 0 & 0 \\ 0 & 0 & 0 & v & \sigma & 0 \\ 0 & 0 & 0 & 0 & v & 0 \\ 0 & 0 & 0 & 0 & 0 & v \end{pmatrix}.$$

Here, x is the distance, t the time, ρ , u, p are the density, velocity, and pressure of the gas and σ, v, θ are the mass concentration, velocity and temperature of the particles, respectively; $\gamma = C_p/C_v$ is the specific heat ratio of the perfect gas, m is the mass of a particle and C_p, C_v & C_m are specific heat of gas at constant pressure, specific heat of the gas at constant volume and the specific heat of the particles, respectively. The equation of state for the thermally perfect gas is given by

$$p = \rho RT,$$

where T is the temperature of the gas and R is the gas constant.

The gas and the particles interact through the drag force D and the heat transfer rate Q experienced by the particles not in equilibrium with the gas. In [12] the forms of D and Q are taken in the following form

$$D = \frac{1}{8}\pi d^2 \rho (u-v)|u-v|C_D, \qquad Q = \pi d \mu C_p Pr^{-1}(T-\theta)Nu,$$

where d, the diameter of the particle, $Re = \rho|u - v|d/\mu$, the Reynolds number based upone d and the relative velocity of the particle to the gas, $C_D = (0.48 + 28Re^{-0.85})$, the coefficient of drag, $Nu = (2.0 + 0.6Pr^{1/3}Re^{1/2})$, the Nusselt number, $Pr = \mu C_p/k$, the prandtl number, μ, the viscosity and k, the thermal conductivity of the gas.

We consider the motion of a shock front, $x = \chi(t)$, propagating into an inhomogeneous medium specified by

$$u_0 = 0,\ \ p_0 = \text{constant},\ \ \rho_0 = \rho_0(x),\ \ \sigma_0 = \sigma_0(x),\ \ v_0 = v_0(x),\ \ \theta_0 = \theta_0(x). \tag{2.2}$$

Equation (2.1) can be written in the conservation form

$$G_t(x, t, U) + F_x(x, t, U) = H(x, t, U), \tag{2.3}$$

where U is the solution vector behind the shock and G, F and H are given by

$$\begin{aligned} G &= \left(\rho, \rho u, \rho(C_V T + u^2/2), \sigma, \sigma v, \sigma(C_m\theta + v^2/2)\right)^{tr}, \\ F &= \left(\rho u, \rho u^2 + p, \rho u(C_p T + u^2/2), \sigma v, \sigma v^2, +\sigma v(C_m\theta + v^2/2)\right)^{tr}, \\ H &= (0, -\sigma D/m, -\sigma(vD + Q)/m, 0, \sigma D/m, \sigma(vD + Q)/m)^{tr}. \end{aligned}$$

Let V be the shock velocity; then the Rankine-Hugoniot jump conditions across the shock front may be derived from (2.3) as

$$\begin{aligned} &\rho(\chi(t), t) = \frac{(\gamma+1){\rho_0}^2(\chi(t))V^2}{(\gamma-1)\rho_0(\chi(t))V^2 + 2\gamma p_0}, \quad u(\chi(t), t) = \frac{2}{(\gamma+1)}\frac{\rho_0(\chi(t))V^2 - \gamma p_0}{\rho_0(\chi(t))V}, \\ &p(\chi(t), t) = \frac{2\rho_0(\chi(t))V^2 - (\gamma-1)p_0}{(\gamma+1)}, \\ &\sigma(\chi(t), t) = \sigma_0(\chi(t)), \qquad v(\chi(t), t) = v_0(\chi(t)), \qquad \theta(\chi(t), t) = \theta_0(\chi(t)). \end{aligned} \tag{2.4}$$

where a subscript 0 refers to the medium ahead of the shock.

3. EVOLUTION OF THE WEAK DISCONTINUITY

The matrix A in (2.1) has eigenvalues

$$\lambda^{(1)} = (u + a), \qquad \lambda^{(2)} = u, \qquad \lambda^{(3)} = (u - a), \qquad \lambda^{(4)} = v \quad \text{(a triple root)} \tag{3.1}$$

with the corresponding left and right eigen vectors

$$\begin{aligned} &L^{(1)} = (0, \rho a, 1, 0, 0, 0), && R^{(1)} = (1/(2a^2), 1/(2\rho a), 1/2, 0, 0, 0)^{tr}, \\ &L^{(2)} = (-a^2, 0, 0, 0, 0, 0), && R^{(2,1)} = (-a^{-2}, 0, 0, 0, 0, 0)^{tr}, \\ &L^{(3)} = (0, -\rho a, 1, 0, 0, 0), && R^{(3)} = (-1/(2a^2), -1/(2\rho a), 1/2, 0, 0, 0)^{tr}, \\ &L^{(4,1)} = (0, 0, 0, 0, 1, 1), && R^{(4,1)} = (0, 0, 0, 1, 0, 1)^{tr}, \\ &L^{(4,2)} = (0, 0, 0, 0, 1, 1/2), && R^{(4,2)} = (0, 0, 0, 1, 0, 2)^{tr}, \\ &L^{(4,3)} = (0, 0, 0, 0, 1, 1/3), && R^{(4,1)} = (0, 0, 0, 1, 0, 3)^{tr}, \end{aligned} \tag{3.2}$$

where $a = (\gamma p/\rho)^{1/2}$ is the frozen speed of sound.

The transport equation for the weak discontinuities across the i^{th} characteristics of a hyperbolic system of n equations of the type (2.1) is given by (see Radha, Sharma & Jeffrey [9])

$$\begin{aligned} &L_b^{(i,k)}\frac{d\Lambda_i}{dt} + L_b^{(i,k)}(U_{bx} + \Lambda_i)(\nabla\lambda^{(i)})_b\Lambda_i + \{(\nabla L^{(i,k)})_b\Lambda_i\}^{tr}\frac{dU_b}{dt} \\ &+(L_b^{(i,k)}\Lambda_i)((\nabla\lambda^{(i)})_b U_{bx} + {\lambda_{bx}}^{(i)}) - (\nabla(L^{(i,k)}f))_b\Lambda_i = 0, \end{aligned} \tag{3.3}$$

where the matrix A possesses p distinct real eigenvalues $\lambda^{(i)}$, $i = 1, 2, 3..., p$, assumed to be ordered so that $\lambda^{(p)} < \lambda^{(p)} < \lambda^{(p-1)} < .... < \lambda^{(1)}$ with multiplicities m_i such that $\sum_{i=1}^{p} m_i = n$, together with n linearly independent left (respectively, right) eigenvectors $L^{(i,k)}$ (respectively, $R^{(i,k)}$), $k = 1, 2, ..., m_i$ corresponding to the eigenvalues $\lambda^{(i)}$. Here,

$$\Lambda_i = \sum_{k=1}^{m_i} \alpha_k^{(i)}(t) R_b^{(i,k)}, \tag{3.4}$$

is the jump in U_x across the $C^{(1)}$ discontinuity propagating along the curve determined by $dx/dt = \lambda^{(1)}$ originating from the point (x_0, t_0) in the region behind the shock, the subscript b refers to the state just behind the shock and ahead of the i^{th} characteristic curve, $\alpha_k^{(i)}$ is the amplitude of the $C^{(1)}$ wave, ∇ denotes the gradient operator with respect to U.

Substituting the values of U, f from equation (2.1), $\lambda^{(1)}$, $L^{(1)}$ and Λ_1 from (3.1), (3.2) and (3.4) respectively, in the equation (3.3) we obtain the following Bernoulli type equation

$$\frac{d\alpha^{(1)}}{dt} + \frac{(\gamma+1)}{4\rho a}\alpha^{(1)2} + J^*\alpha^{(1)} = 0, \tag{3.5}$$

where

$$\begin{aligned} J^* = &\frac{(\gamma+1)}{4\rho a}(\rho a u_x + p_x) + \left(2u_x - \frac{a\rho_x}{\rho} + \frac{\gamma}{\rho a}p_x\right) \\ &+ \frac{1}{2a^2}\frac{\sigma}{m}\left[a_\rho D + \{a + (\gamma-1)(u+v)\}D_\rho + (\gamma-1)Q_\rho\right] \\ &+ \frac{1}{2\rho a}\frac{\sigma}{m}\left[(\gamma-1)D + \{a + (\gamma-1)(u+v)\}D_u + (\gamma-1)Q_u\right] \\ &+ \frac{1}{2}\frac{\sigma}{m}[a_p D + (\gamma-1)Q_p]. \end{aligned}$$

Once, the values of the parameters behind the shock ρ, u, p, σ, v, θ and the forms of D and Q are known, the above equation can be solved to obtain the value of $\alpha^{(1)}$.

4. Collision of the Weak Discontinuity with the Strong Shock Wave

In order to study the amplitudes of the reflected and transmitted weak discontinuities, we write the conservation equation (2.3) in the regions behind and ahead of the shock (i.e. to the left and to the right of the discontinuity curve, $dx/dt = \Pi$ which propagates with the speed Π)

$$\begin{aligned} &G_t(x,t,U) + F_x(x,t,U) = H(x,t,U), \\ &G_{*t}(x,t,U_*) + F_{*x}(x,t,U_*) = H_*(x,t,U_*), \end{aligned} \tag{4.1}$$

where U and U_* are the solution vectors behind and ahead of the shock and G, F and H are as in (2.3).

Let (x_p, t_p) be the point at which the fastest $C^{(1)}$ discontinuity of $(4.1)_1$, moving along the characteristic $\phi_1(x,t) = 0$ and originating from the point (x_0, t_0) intersects the discontinuity line. As in [9], the amplitudes of the incident, reflected and transmitted waves on the discontinuity line are given by the relations

$$\begin{aligned} &\Lambda_1(P) = \sum_{k=1}^{m_1} \alpha_k^{(1)}(t_p) R_s^{(1,k)}, \qquad \Lambda_i^{(R)}(P) = \sum_{k=1}^{m_i} \alpha_k^{(i)}(t_p) R_s^{(i,k)}, \\ &\Lambda_i^{*(T)}(P) = \sum_{k=1}^{m_i^*} \beta_k^{(i)}(t_p) R_s^{*(i,k)}, \end{aligned} \tag{4.2}$$

where a subscript s refers to the values evaluated at the shock. The evolutionary equations to determine the jump in the shock acceleration $[\dot{\Pi}]$, the coefficients of the amplitudes of reflected waves $\alpha_k^{(i)}$ and transmitted waves $\beta_k^{(i)}$ after interaction are given by the matrix equation

$$\begin{aligned}[t] &[\dot{\Pi}](G-G_*)_s + (\nabla G)_s \sum_{i=p-q+1}^{p} \left(\sum_{k=1}^{m_i} \alpha_k^{(i)} \left(\Pi - \lambda^{(i)}\right)^2 R_s^{(i,k)} \right) \\ &\quad - (\nabla^* G_*)_s \sum_{j=1}^{q} \left(\sum_{k=1}^{m_j} \beta_k^{(j)} \left(\Pi - \lambda_*^{(j)}\right)^2 R_s^{*(j,k)} \right) \\ &\quad = -(\nabla G)_s \sum_{k=1}^{m_i} \alpha_k^{(1)} \left(\Pi - \lambda^{(1)}\right)^2 R_s^{(1,k)}, \end{aligned} \tag{4.3}$$

which is a system of n inhomogeneous algebraic equations.

We have now at $t = t_p$

$$\lambda^{(1)} = \frac{2+\Gamma}{(\gamma+1)}\Pi, \quad \lambda^{(2)} = \frac{2}{(\gamma+1)}\Pi, \quad \lambda^{(3)} = \frac{2-\Gamma}{(\gamma+1)}\Pi, \quad \lambda^{(4)} = v_{0c}\Pi, \tag{4.4}$$

and

$$\lambda_*^{(1)} = (\gamma p_0/\rho_0)^{1/2}, \quad \lambda_*^{(2)} = 0, \quad \lambda_*^{(3)} = -(\gamma p_0/\rho_0)^{1/2}, \quad \lambda_*^{(4)} = v_{0c}, \tag{4.5}$$

where $\Gamma^2 = 2\gamma(\gamma-1)$. Since, the pressure ahead of the shock wave (where the variables are designated by asterisk) is very small when compared with the pressure behind, it follows that the Lax evolutionary conditions for a physical shock [14] for an integer l in the interval $1 \le l \le p$

$$\begin{aligned} &\lambda^{(p)} < \lambda^{(p-1)} < \ldots < \lambda^{(l+1)} < \Pi < \lambda^{(l)} < \ldots < \lambda^{(1)} \\ &\lambda_*^{(p)} < \lambda_*^{(p-1)} < \ldots < \lambda_*^{(l)} < \Pi < \lambda_*^{(l-1)} < \ldots < \lambda_*^{(1)}, \end{aligned} \tag{4.6}$$

are satisfied, i.e.

$$\lambda^{(3)} < \lambda^{(2)} < \Pi < \lambda^{(4)} < \lambda^{(1)}, \quad \text{and} \quad \lambda_*^{(3)} < \lambda_*^{(2)} < \lambda_*^{(4)} < \lambda_*^{(1)} < \Pi. \tag{4.7}$$

In effect, this asserts that when the incident wave with velocity $\lambda^{(1)}$ at $t = t_p$ encounters the shock, it gives rise two reflected waves with velocities $\lambda^{(2)}$ and $\lambda^{(3)}$ and three transmitted waves with velocities $\lambda^{(4)}$ along the characteristics issuing from the collision point. The reflection and transmission coefficients $\alpha^{(2)}$, $\alpha^{(3)}$, $\beta_1^{(4)}$, $\beta_2^{(4)}$ and $\beta_3^{(4)}$ and the jump in shock acceleration $[\dot{\Pi}] = \dot{\Pi}_{t_p^+} - \dot{\Pi}_{t_p^-}$ at the collision time $t = t_p$ can be determined from the algebraic system of equations

$$\begin{aligned} &(G-G_*)_s[\dot{\Pi}] + (\nabla G)_s R_s^{(2)} (\Pi - \lambda_s^{(2)})^2 \alpha^{(2)} + (\nabla G)_s R_s^{(3)} (\Pi - \lambda_s^{(3)})^2 \alpha^{(3)} \\ &-(\nabla^* G_*)_s \sum_{i=1}^{3} R_s^{*(4,i)} (\Pi - \lambda_{*s}^{(4)})^2 \beta_i^{(4)} = -(\nabla G)_s R_s^{(1)} (\Pi - \lambda_s^{(1)})^2 \alpha^{(1)}. \end{aligned} \tag{4.8}$$

In view of relations (2.2), (2.4) and (4.8) the balance equation at the time t_p can be written as the following system of algebraic equations in the unknowns $[\dot{\Pi}]$, $\alpha^{(2)}$, $\alpha^{(3)}$, $\beta_1^{(4)}$, $\beta_2^{(4)}$, $\beta_3^{(4)}$

$$\begin{aligned}
&\frac{2\rho_0(\chi(t))}{(\gamma-1)}[\dot{\Pi}] - \frac{(\gamma-1)^2}{\Gamma^2}\alpha^{(2)} - \frac{(\gamma+\Gamma-1)^2}{2\Gamma^2}\alpha^{(3)} = -\frac{(\gamma-\Gamma-1)^2}{2\Gamma^2}\alpha^{(1)},\\
&\frac{2\rho_0(\chi(t))}{(\gamma-1)}[\dot{\Pi}] - \frac{2(\gamma-1)^2}{\Gamma^2(\gamma+1)}\alpha^{(2)} - \frac{(\gamma+\Gamma-1)^2(\Gamma+2)}{2\Gamma^2(\gamma+1)}\alpha^{(3)}\\
&\qquad\qquad = -\frac{(\gamma-\Gamma-1)^2(\Gamma+2)}{2\Gamma^2(\gamma+1)}\alpha^{(1)},\\
&\frac{4\rho_0(\chi(t))}{(\gamma-1)}[\dot{\Pi}] - \frac{2\gamma-1)^2}{\Gamma^2(\gamma+1)}\alpha^{(2)} + \frac{(\gamma+\Gamma-1)^2(\gamma-\Gamma-1)}{\Gamma^2(\gamma+1)}\alpha^{(3)}\\
&\qquad\qquad = -\frac{(\gamma-\Gamma-1)^2(\gamma+\Gamma+1)}{\Gamma^2(\gamma+1)}\alpha^{(1)},\\
&\beta_1^{(4)} + v_{0c}\Pi\beta_2^{(4)} + \{C_m(\theta_{0c}+v_{0c}/2)\Pi^2 + C_m\sigma_{0c}\rho_0\}\beta_3^{(4)} = 0,\\
&\beta_1^{(4)} + v_{0c}\Pi\beta_2^{(4)} + \{C_m(\theta_{0c}+v_{0c}/2)\Pi^2 + 2C_m\sigma_{0c}\rho_0\}\beta_3^{(4)} = 0,\\
&\beta_1^{(4)} + v_{0c}\Pi\beta_2^{(4)} + \{C_m(\theta_{0c}+v_{0c}/2)\Pi^2 + 3C_m\sigma_{0c}\rho_0\}\beta_3^{(4)} = 0,
\end{aligned} \tag{4.9}$$

The first three equations of the above algebraic system (4.9) yields on solving

$$\begin{aligned}
[\dot{\Pi}] &= \frac{(\gamma-\Gamma-1)^2}{8\Gamma^2}\frac{(\gamma-1)}{\rho_0(\chi(t))}\frac{2\gamma+(\Gamma+2)(\Gamma+1)}{\gamma(\gamma+1)+(\Gamma+2)}\alpha^{(1)}\\
\alpha^{(2)} &= \frac{(\gamma-\Gamma-1)^2}{4(\gamma-1)^2}\frac{\gamma(\gamma-\Gamma-1)}{\gamma(\gamma+1)+(\Gamma+2)}\alpha^{(1)}\\
\alpha^{(3)} &= \frac{(\gamma-\Gamma-1)^2}{(\gamma+\Gamma-1)^2}\frac{\Gamma-\gamma(\gamma-1)}{\gamma(\gamma+1)+(\Gamma+2)}\alpha^{(1)}.
\end{aligned} \tag{4.10}$$

where $\alpha^{(1)}$ can be determined from (3.5). From the last three equations of (4.9) it is evident that although the coefficients $\beta_1^{(4)}$, $\beta_2^{(4)}$, $\beta_3^{(4)}$ can have some nonzero values other than the trivial solution, they do not depend upon the coefficient $\alpha^{(1)}$ i.e. the incident $C^{(1)}$ wave. Hence, there are two reflected waves in the characteristics with the velocities $\lambda^{(2)}$ and $\lambda^{(3)}$, respectively, and the initial discontinuity of the incident wave has no effect on transmitted waves propagating with the velocity $\lambda^{(4)}$.

Equations (4.10) demonstrate that the amplitudes of the reflected waves are proportional to the incident wave and as would be expected, in the absence of the incident wave (i.e. $\alpha^{(1)} = 0$), the jump in the shock acceleration vanishes and there are no reflected waves. Moreover, an increase in the magnitude of the initial discontinuity of the incident wave causes the reflection coefficients and the jump in shock acceleration to increase in magnitude.

5. RESULTS AND CONCLUSION

In this paper, the amplitudes of reflected and transmitted waves and the jump in the shock acceleration after interaction of the weak discontinuity through a strong shock are evaluated for a gas with large number of dust particles. It has been observed that there are two reflected waves along the characteristic lines with velocities $\lambda^{(2)}$ and $\lambda^{(3)}$ and the transmission is independent of the incident discontinuity. In the absence of the incident wave the jump in the shock acceleration vanishes and there are no reflected or transmitted waves. An increase in the magnitude of the initial discontinuity of the incident wave causes the reflection coefficients and the jump in shock acceleration to increase in magnitude.

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