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THE ITERATED VARIATIONAL METHOD FOR THE EIGENELEMENTS OF A CLASS OF TWO-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. The iterated variational method is considered in the approximation of eigenvalues and eigenfunctions for a class of two point boundary value problems. The implementation of this method is easy and competes well with other methods. Numerical examples are presented to show the efficiency of the method proposed. Comparison with the work of others is also illustrated.

Key words and phrases: Iterated variational method, Eigenvalues, Two-point boundary value problems, Singular problems.

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1. INTRODUCTION

We consider the following class of eigenvalue problems of the form

$$[p(x)y'(x)]' + \alpha q(x)y(x) = g(x), \tag{1.1}$$

on $(0,1)$ with some specified boundary conditions, where $q(x), p(x) > 0$ with $q(x)$ and $p(x)$ are smooth functions. This problem is regular with the analogous singular problem having the form given in (1.1) subject to boundary conditions of the form

$$ay(0) + by'(0) = 0, \quad cy(1) + d\,y'(1) = 0, \tag{1.2}$$

where again $q(x), p(x) \geq 0, \quad \frac{1}{p} \in L_{loc}(0,1]$, $\frac{1}{p} \notin L^1(0,1)$, and a, b, c, d are constants. Under these conditions, (1.1) is termed a singular eigenvalue problem. Such linear and nonlinear boundary value problems have been the subject of numerical investigation of many recent papers. General existence and uniqueness results for solutions of the singular problem were given in Fink *et al.* [5] and Baxley [3] while for the eigenvalue problem was considered by Nassif [6], Elder [4] and Bailey *et al.* [2].

Now since the application of these problems to applied sciences and engineering is so important, enough work was done to investigate them. A well-known example of those problems is the analysis of the behavior of magneto-electro-elastic material which has the ability of converting the magnetic, electric, and mechanical energies from one to another. It is worth noting that this engineering problem is a promising candidate in designing and manufacturing intelligent and smart systems and structures. The mathematical modeling of characteristics of such material, see Wang and Zong [7] , is a very active area of research and investigations. Therefore, the solution of such model represents a challenge since it involves solving a particular singular boundary value problem.

The numerical treatment of such regular and singular boundary value problems has always been far from trivial. For that reason several authors have been extensively involved in the solution of such class of problems and numerous innovative methods and approach have enriched the scientific literature; each with its particular merits and advantages. Attili [1] used the homotopy perturpation method and Adomian decomposition method to solve this type of problem. We will treat the regular and the singular problem given by (1.1) using the iterated variation method. The method is proved to be efficient and the details will be given in the coming sections. It is worth mentioning that it is a promising method for solving (1.1). Not much was done using this method in terms of numerical approximation to eigenvalue problems, making it the subject of interest of further investigation.

The rest of the paper is organized as follows. We will present the iterated variational method in the next section while the numerical results and comparisons will be done in the last section.

2. METHODOLOGY

Rewrite the Sturm-Liouville problem (1.1) of the form or

$$y''(x) + \frac{p'(x)}{p(x)}y'(x) + \alpha\frac{q(x)}{p(x)}y(x) = \frac{g(x)}{p(x)}, \qquad x \in (0,1). \tag{2.1}$$

The present analytical technique based on dividing the problem (2.1) into two parts as follows

$$D_2[y] + D_1[y] = \frac{g(x)}{p(x)}, \qquad x \in [0,1]. \tag{2.2}$$

where D_1 and D_2 are linear operators defined as

$$D_2[y] = y''(x), \quad \text{and} \quad D_1[y] = \frac{p'(x)}{p(x)}y'(x) + \alpha\frac{q(x)}{p(x)}y(x), \qquad x \in [0,1].$$

By defining a correction functional through the following iteration form as follows

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)\left(D_2[y_n(\xi)] + D_1[\tilde{y}_n(\xi)] - \frac{g(\xi)}{p(\xi)}\right)d\xi, \tag{2.3}$$

where λ is the Lagrange multiplier which should be obtained by a variational theory and $\tilde{y_n}$ is a restricted variation with the property $\delta\tilde{y_n} = 0$. It should be noted that the initial choice of y_0 can be chosen as the solution of equation $D_2[y] = 0$. In order to determine the Lagrange multiplier we could first rewrite equation (2.3) in the form

$$y_{n+1}(x) = \{1 - \lambda'(x)\}\, y_n(x) + \lambda(x)y_n'(x) + \int_0^x \{\lambda''(\xi)y_n(\xi) + D_1[\tilde{y}_n(\xi)]\}\, d\xi. \tag{2.4}$$

Making the above functional given in (2.4) stationary, we obtain

$$\delta y_{n+1}(x) = \{1 - \lambda'(x)\}\, \delta y_n(x) + \lambda(x)\delta y_n'(x) + \int_0^x \{\lambda''(\xi)\delta y_n(\xi) + D_1[\delta\tilde{y}_n(\xi)]\}\, d\xi, \tag{2.5}$$

and consequently, we obtain the following stationary conditions

$$\begin{aligned} \lambda''(\xi) &= 0 \\ \lambda(\xi)_{|_{\xi=x}} &= 0 \\ 1 - \lambda'(\xi)_{|_{\xi=x}} &= 0. \end{aligned}$$

It can be easily shown that the Lagrange multiplier $\lambda = \xi - x$ satisfies all the above stationary conditions. Furthermore, the first initial condition (1.2) will be used to find the first constant in y_0 whereas the second condition will be used in the calculations of the eigenvalues α in each iteration step.

3. NUMERICAL RESULTS

Herein, we apply the variational iteration technique presented in the previous section to solve some eigenvalue problems. Numerical results will be presented in this section.

Example 3.1. *Consider the eigenvalue problem*

$$y''(x) + \alpha y(x) = 0, \qquad x \in (0,1) \tag{3.1}$$

subject to

$$y'(0) = 0, \quad y(1) = 0. \tag{3.2}$$

It should be noted that the analytical eigenvalues and the corresponding eigenfunctions are known for this example and given by

$$\alpha_n = \left(\frac{2k+1}{2}\pi\right)^2, \qquad y_n(x) = \cos\left(\frac{2k+1}{2}\pi x\right) \tag{3.3}$$

where $k = 0, 1, \cdots$.

According to the analysis presented in the previous section, we can easily see that

$$y_0 = B,$$

where the first condition of (3.2) has been applied. Here B is a nonzero constant. Consequently, the functional iteration (2.3) is ready to construct a convergent sequence to the solution $y(x)$. Table (3.1) lists the approximations to the first eigenvalue $\alpha_1 = 2.467401101$ using the functional iteration (2.3). The results illustrate the rapid convergence of this method. The approximated

n	α_1^*	$\lvert\alpha_1^* - \alpha_1\rvert$
1	2.000000	0.467401
2	2.535898	0.068497
3	2.464604	0.002797
4	2.467479	0.000078

Table 3.1: The approximations to the first eigenvalue

solution after thirteen iterates is given by

$$\begin{aligned} y_{13}(x) &= B(1 - 0.5\,\alpha\,x^2 + 0.041667\,\alpha^2 x^4 - 0.0013889\,\alpha^3 x^6 + 0.000024802\,\alpha^4 x^8 \\ &- 0.00000027557\,\alpha^5 x^{10} + 0.0000000020877\,\alpha^6 x^{12} - 1.1471 \times 10^{-11}\,\alpha^7 x^{14} \\ &+ 4.7795 \times 10^{-14}\,\alpha^8 x^{16} - 1.5619 \times 10^{-16}\,\alpha^9 x^{18} + 4.1103 \times 10^{-19}\,\alpha^{10} x^{20} \\ &- 8.8968 \times 10^{-22}\,\alpha^{11} x^{22} + 1.6117 \times 10^{-24}\,\alpha^{12} x^{24} - 2.4796 \times 10^{-27}\,\alpha^{13} x^{26}). \end{aligned}$$

Satisfying the second boundary condition in the approximated solution, y_{13}, and then solving the resulted equation for α we obtain those eigenvalues

$$\alpha_1^* = 2.4674011, \alpha_2^* = 22.2066099, \alpha_3^* = 61.6844723,$$

which are in excellent agreement with the exact results. At the end of the discussion of this example, it is worthy mentioning here that the results of this example are in a good agreement with the results of Attili [1] who uses the homotopy perturpation method and Adomain Decomposition method.

Example 3.2. *Consider the eigenvalue problem*

$$y''(x) + (3 + \alpha)y(x) = 0, \qquad x \in [0, \pi] \tag{3.4}$$

subject to

$$y'(0) = 0, \quad y'(\pi) = 0. \tag{3.5}$$

It is well-known that the analytical eigenvalues and the corresponding eigenfunctions of this equation are given by

$$\alpha_k = k^2 - 3, \qquad y_k(x) = \cos kx. \tag{3.6}$$

where $k = 0, 1, \cdots$. Those values will be used for comparison purposes.

Similar to the solution steps discussed in the previous example, we apply the first condition on the solution of $y''(x) = 0$ to obtain an initial solution, $y_0 = B$, which is required for starting the functional iteration (2.3). The first eigenvalue, $\alpha_1 = -3$, is found exactly by the first iteration, $n = 1$. The results of the first five iterates, summarized in table (3.2), show excellent approximations for $\alpha_2 = -2$ when $n \geq 4$.

n	α_2^*	$\lvert\alpha_2^* - \alpha_2\rvert$
1	—	—
2	—	—
3	—	—
4	-2.039673859	0.039673859
5	-1.995476483	0.004523517
6	-2.000282816	0.000282816
7	-1.999986473	0.000013527
8	-2.000000443	0.000000443
9	-1.999999918	0.000000082
10	-1.999999932	0.000000068

Table 3.2: The Approximations for α_2

The approximated solution after thirteen iterates is given by

$$\begin{aligned} y_{13}(x) &= B(1 - 0.5\,(3+\alpha)\,x^2 - 0.08333\,(3+\alpha)\,(-1.5-0.5\,\alpha)\,x^4 \\ &+ 0.002778\,(3+\alpha)^2\,(-1.5-0.5\,\alpha)\,x^6 - 0.00004960\,(3+\alpha)^3 \\ &\times (-1.5-0.5\,\alpha)\,x^8 + 0.0000005511\,(3+\alpha)^4\,(-1.5-0.5\,\alpha)\,x^{10} \\ &- 0.000000004175\,(3+\alpha)^5\,(-1.5-0.5\,\alpha)\,x^{12} + 2.294\times 10^{-11} \\ &\times (3+\alpha)^6\,(-1.5-0.5\,\alpha)\,x^{14} - 9.559\times 10^{-14}\,(3+\alpha)^7\,(-1.5-0.5\,\alpha)\,x^{16} \\ &+ 3.124\times 10^{-16}\,(3+\alpha)^8\,(-1.5-0.5\,\alpha)\,x^{18} - 8.221\times 10^{-19}\,(3+\alpha)^9 \\ &\times (-1.5-0.5\,\alpha)\,x^{20} + 1.779\times 10^{-21}\,(3+\alpha)^{10}\,(-1.5-0.5\,\alpha)\,x^{22} \\ &- 3.223\times 10^{-24}\,(3+\alpha)^{11}\,(-1.5-0.5\,\alpha)\,x^{24} + 4.959\times 10^{-27}\,(3+\alpha)^{12} \\ &\times (-1.5-0.5\,\alpha)\,x^{26}). \end{aligned}$$

Satisfying the second boundary condition in the approximated solution, y_{13}, and then solving the resulted equation for α we obtain the third and fourth approximated eigenvalues, $\alpha_3^* = .9999995921$ and $\alpha_4^* = 6.033557406$, which are in excellent agreement with the corresponding exact results $\alpha_3 = 1$ and $\alpha_4 = 6$, respectively.

Example 3.3. *Consider the eigenvalue problem*

$$y''(x) + y'(x) + \alpha y(x) = 0, \qquad x \in [0,1] \tag{3.7}$$

subject to

$$y(0) = 0, \quad y(1) = 0. \tag{3.8}$$

The exact eigenvalues and the corresponding eigenfunctions of this equation are found theoretically and given by

$$\alpha_k = (k\pi)^2 + \frac{1}{4}, \qquad y_k(x) = e^{-x/2}\sin(k\pi x). \tag{3.9}$$

where $k = 0, 1, \cdots$.

The initial solution, y_0, is found to be

$$y_0(x) = Ax,$$

where A is a nonzero constant. The implementation of the variational iteration (2.3) gives the following approximated values for the first eigenvalue, $\alpha_1 = 10.11960440$, as reported in table

(3.3). It is clearly seen that the speed of convergence is slower than that in Examples 3.1 and

n	α_1^*	$\lvert\alpha_1^* - \alpha_1\rvert$
1	—	—
2	—	—
3	—	—
4	—	—
5	9.712272286	0.407332114
6	10.20571352	0.08610912
7	10.10723513	0.01236927
8	10.12121726	0.00161286
9	10.11941996	0.00018444
10	10.11962329	0.00001889

Table 3.3: The approximated values of the first eigenvalue

3.2. The thirteens iterates, $y_{13}(x)$, gives a good estimation, $\alpha_2^* = 39.72882695$, for the second eigenvalue, $\alpha_2 = 39.72841762$.

Example 3.4. *Consider the eigenvalue problem*

$$y''(x) + (\frac{1}{x} + \alpha)y(x) = 0, \qquad x \in (0, 1) \tag{3.10}$$

subject to

$$y(0) = 0, \quad y(1) = 0. \tag{3.11}$$

The available results obtained by different numerical schemes are summarized in table (3.4). It can be seen that present method is in an excellent comparison with ADM until the fifth eigenvalue, α_5^*. However, the discrepancies between both methods seems to be increasing after α_5^*. Extra comparisons are done with the ADM by tracking some corresponding eigenfunctions as shown in Figures (1)-(7). It is found that the ADM results show that the corresponding eigenfunction for α_5^* breaks down and not satisfying the boundary condition $y(1) = 0$ (see Attili [1]) whereas in the present study they are smooth and more accurate.

α_n^*	Baily et al. [7]	Attili [1]	Present
α_0^*	7.37399	7.37398501517	7.373985015
α_1^*	36.33602	36.3360195952	36.33601960
α_2^*	85.29258	85.2925820941	85.29258209
α_3^*	154.09862	154.098623739	154.0986237
α_4^*	242.70555	242.705559362	242.7055594
α_5^*		351.091215481	351.0911671
α_6^*		478.629381125	479.2434134
α_7^*			627.1550441

Table 3.4: The first few eigenvalues

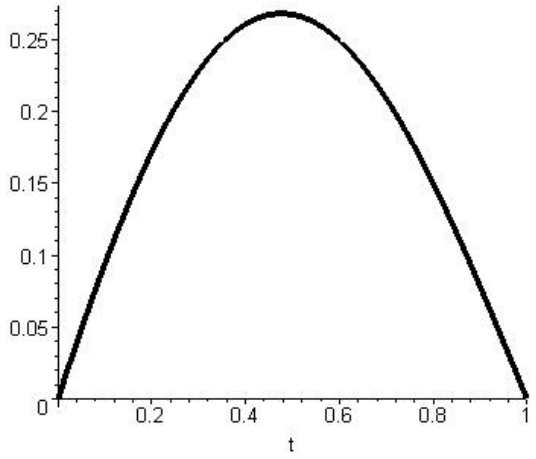

Figure 1: The first eigenfunction

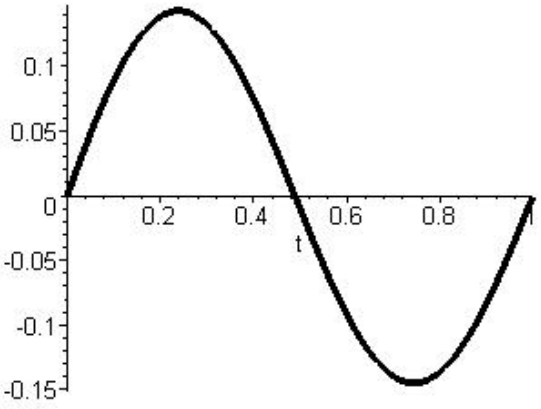

Figure 2: The second eigenfunction

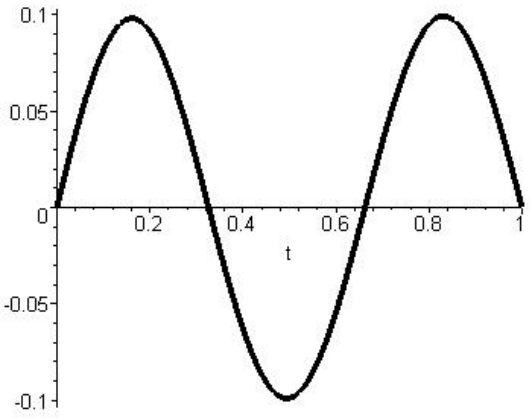

Figure 3: The third eigenfunction

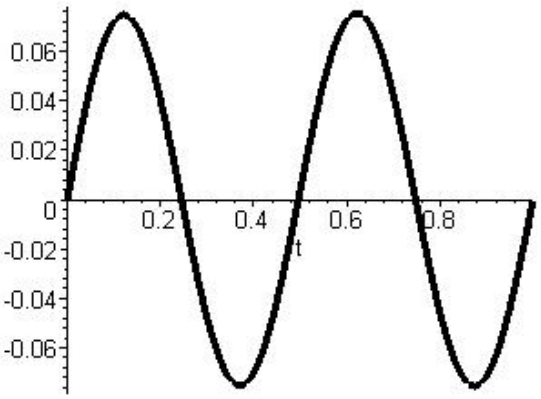

Figure 4: The fourth eigenfunction

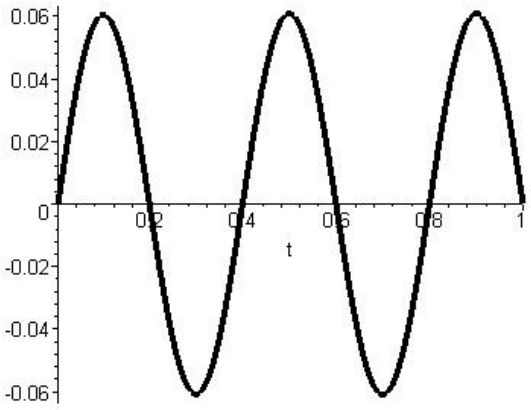

Figure 5: The fifth eigenfunction

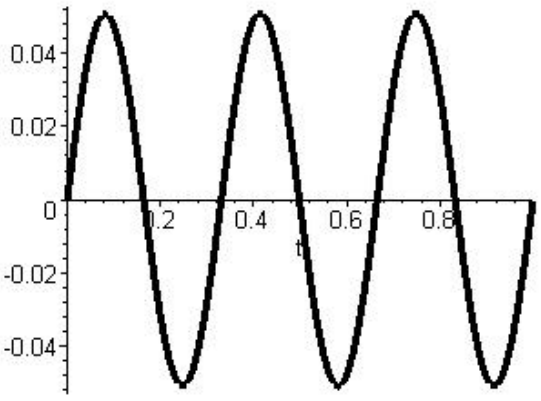

Figure 6: The sixth eigenfunction

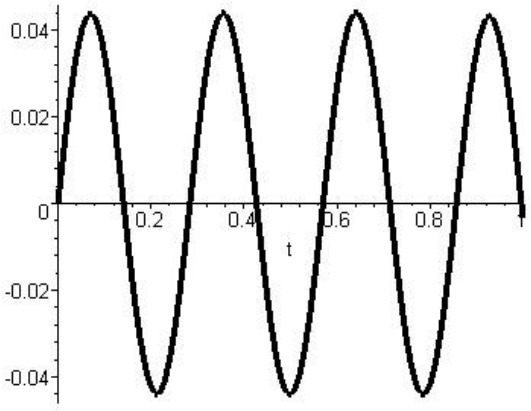

Figure 7: The seventh eigenfunction

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