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EXISTENCE OF LARGE SOLUTIONS TO NON-MONOTONE SEMILINEAR ELLIPTIC EQUATIONS

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ABSTRACT. We study the existence of large solutions of the semilinear elliptic equation $\Delta u = p(x)f(u)$ where f is not monotonic. We prove existence, on bounded and unbounded domains, under the assumption that f is Lipschitz continuous, $f(0) = 0$, $f(s) > 0$ for $s > 0$ and there exists a nonnegative, nondecreasing Hölder continuous function g and a constant M such that $g(s) \leq f(s) \leq Mg(s)$ for large s. The nonnegative function p is allowed to be zero on much of the domain.

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1. INTRODUCTION

We consider the semilinear elliptic equation

$$\Delta u = p(x)f(u),\ x \in \Omega \subseteq \mathbb{R}^n,\ n \geq 3, \tag{1.1}$$

where Ω is open and connected, the nonnegative function p can be zero on much of the domain, and f is a Lipschitz continuous function on $[0, \infty)$ that satisfies $f(0) = 0,\ f(s) > 0$ for $s > 0$. In addition, we assume that there exists a nonnegative, nondecreasing Hölder continuous function g and positive constants M and s_0 such that

$$g(s) \leq f(s) \leq Mg(s) \text{ for all } s \geq s_0. \tag{1.2}$$

We are interested in the existence of large solutions of (1.1) on Ω; i.e., solutions for which $u(x) \to \infty$ as $x \to \partial\Omega$ if Ω is bounded, and if Ω is unbounded, we also require that $u(x) \to \infty$ for $|x| \to \infty$ within Ω.

Unlike almost all previous work (See, for example, [1, 2, 6, 7, 8, 10, 12, 13], and their references.), we do not require f to be nondecreasing. The usual requirement that f be monotonic is necessary, in part, because the proofs depend on the maximum principle. However, where f is not monotonic, the maximum principle cannot be applied directly to equation (1.1).

The only existence result for non-monotonic f we are aware of is given by Goncalves and Roncalli [5]. They proved existence under the conditions $\liminf_{s\to\infty} f(s)/s^b > 0$ and $0 < \sup_{s>0} f(s)/s^a < \infty$, $1 < b \leq a < \infty$. These conditions reduce to the existence of position constants c_0 and c_1 such that $c_0 s^a \leq f(s) \leq c_1 s^a$ for s large, and hence is a special case of our results.

For nondecreasing f, we know that (1.1) has a large solution on a bounded domain if and only if f satisfies (see [7])

$$\int_1^\infty \left[\int_0^s f(t)\,dt\right]^{-1/2} ds < \infty. \tag{1.3}$$

We prove here that this remains true for nonmonotone f (Theorem 2.2). For unbounded domains, we prove results analogous to those for increasing f. In particular, we show that if p decays rapidly as $|x| \to \infty$, then, as in the bounded domain case, (1.1) has a large solution if and only if f satisfies (1.3) (see Corollary 3.4). Our proofs, although comparable to those in [7], require substantial innovations to compensate for the lack of monotonicity.

We note that similar results for systems comparable to (1.1) such as

$$\begin{aligned} \Delta u &= p(x)f(v) \\ \Delta v &= q(x)h(u) \end{aligned}$$

remain an open problem. Indeed, existence results for large solutions of such systems are known only under the rather restrictive conditions that $\Omega = \mathbb{R}^n$, p and q are spherically symmetric and both f and h are nondecreasing (see [3], [11]).

2. EXISTENCE OF SOLUTIONS ON BOUNDED DOMAINS

We first make some preliminary definitions and observations before establishing our existence theorems. In particular, we define the functions G and H as follows:

$$G(s) = \begin{cases} A\min\{f(t) : s \leq t \leq s_0,\}\ , \ 0 \leq s \leq s_0 \\ Af(s_0)g(s)/g(s_0)\ ,\ s \geq s_0. \end{cases} \tag{2.1}$$

$$H(s) = \begin{cases} K\max\{f(t) : 0 \leq t \leq s\}\ ,\ 0 \leq s \leq s_0, \\ KF_0 g(s)/g(s_0)\ ,\ s \geq s_0, \end{cases} \tag{2.2}$$

where $0 < A \leq min\{1, \frac{g(s_0)}{f(s_0)}\}$, $F_0 = \max\{f(t) : 0 \leq t \leq s_0\}$, K is a constant chosen so that $K \geq \max\{1, Mg(s_0)/F_0\}$, and M comes from (1.2). We note without proof that G and H are nondecreasing $C^\alpha_{loc}([0,\infty))$ functions which are positive when their argument is positive, and satisfy

$$G(s) \leq f(s) \leq H(s) \text{ for } s \geq 0\,. \tag{2.3}$$

We say that the nonnegative function p is *c-positive* if for any $x_0 \in \Omega$ satisfying $p(x_0) = 0$, there exists a domain Ω_0 such that $x_0 \in \Omega_0,\ \overline{\Omega}_0 \subset \Omega$, and $p(x) > 0$ for all $x \in \partial\Omega_0$. Thus p can be zero on much of the domain.

Lemma 2.1. *Suppose Ω is a bounded domain in $\mathbb{R}^n$ with a $C^{2,\gamma}$ boundary, and p is a nonnegative $C^\alpha(\overline{\Omega})$ function that is c-positive on Ω. Suppose f is Lipschitz continuous on $[0,\infty)$, $f(0) = 0$, $f(s) > 0$ for $s > 0$ and satisfies (1.2). Then for any nonnegative constant c, the boundary value problem*

$$\begin{aligned} &\Delta v = p(x)f(v),\ x \in \Omega, \\ &v(x) = c,\ x \in \partial\Omega \end{aligned} \tag{2.4}$$

has a nonnegative classical solution v on Ω.

Proof. From [4] (See Theorem 14.10) we have that for any nonnegative constant c there exist nonnegative classical solutions v_1 and v_2 to the following boundary value problems

$$\begin{aligned} \Delta v_1 &= p(x)G(v_1),\ x \in \Omega, \\ v_1(x) &= c,\ x \in \partial\Omega, \\ \Delta v_2 &= p(x)H(v_2),\ x \in \Omega, \\ v_2(x) &= c,\ x \in \partial\Omega. \end{aligned}$$

We claim that $v_1 \geq v_2$ on $\overline{\Omega}$. Indeed, suppose $v_1 < v_2$ at some point in $\overline{\Omega}$. Let $\varepsilon > 0$ be small enough such that $\max_{\overline{\Omega}}[v_2(x) - v_1(x) - \varepsilon h(r)] > 0$, where $h(r) = (1 + r^2)^{-1/2},\quad r = |x|$. Then $0 < v_2(x_0) - v_1(x_0) - \varepsilon h(r) \equiv \max_{\overline{\Omega}}[v_2(x) - v_1(x) - \varepsilon h(r)]$ and hence at x_0 we have $0 \geq \Delta(v_2 - v_1 - \varepsilon h(r)) = p(x_0)[H(v_2(x_0)) - G(v_1(x_0))] - \varepsilon\Delta h(r) \geq -\varepsilon\Delta h(r) > 0$, a contradiction. The last inequality holds because $n \geq 3$. Thus, $v_2 \leq v_1$ in $\overline{\Omega}$.

Now, letting $\overline{v} = v_1$ and $\underline{v} = v_2$ we have that $\underline{v} \leq \overline{v}$ in Ω and

$$\begin{aligned} &\Delta\overline{v} = p(x)G(\overline{v}) \leq p(x)f(\overline{v}),\ x \in \Omega, \\ &\Delta\underline{v} = p(x)H(\underline{v}) \geq p(x)f(\underline{v}),\ x \in \Omega. \end{aligned}$$

Thus, $\overline{v}$ and $\underline{v}$ are upper and lower solutions, respectively, of $\Delta v = p(x)f(v)$ on Ω, and hence the monotone iteration scheme (see [14]) gives the existence of a classical solution, v, to equation (2.4) on Ω with $\underline{v} \leq v \leq \overline{v}$. ∎

The following is our main result for this section.

Theorem 2.2. *Suppose Ω, p and f satisfy the hypothesis of the lemma above. Then equation (1.1) has a nonnegative large solution in Ω if and only if f satisfies (1.3).*

Proof. Suppose f satisfies (1.3). Let v_k and w_k be the nonnegative solutions of (see [7])

$$\begin{aligned} \Delta v_k &= p(x)G(v_k),\ x \in \Omega, \\ v_k(x) &= k,\ x \in \partial\Omega, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \Delta w_k &= p(x)H(w_k),\ x \in \Omega, \\ w_k(x) &= k,\ x \in \partial\Omega. \end{aligned} \tag{2.6}$$

Then v_k and w_k are monotonically increasing. We shall construct a monotone sequence of functions $\{u_k\}$ which satisfies, for each k,

$$\Delta u_k = p(x)f(u_k),\ x \in \Omega,$$
$$u_k(x) = k,\ x \in \partial\Omega.$$

We start with $k = 1$. Letting $\overline{u_1} = v_1$ and $\underline{u_1} = w_1$ we have that there exists a nonnegative classical solution u_1 of

$$\begin{aligned} \Delta u_1 &= p(x)f(u_1),\ x \in \Omega, \\ u_1(x) &= 1,\ x \in \partial\Omega, \end{aligned}$$

with $w_1 = \underline{u_1} \leq u_1 \leq \overline{u_1} = v_1$. We then consider the following system of equations

$$\begin{aligned} \Delta v_2 &= p(x)G(v_2),\ x \in \Omega, \\ v_2(x) &= 2,\ x \in \partial\Omega, \end{aligned}$$

$$\begin{aligned} \Delta u_1 &= p(x)f(u_1),\ x \in \Omega, \\ u_1(x) &= 1,\ x \in \partial\Omega. \end{aligned}$$

Letting $\overline{u_2} = v_2$ and $\underline{u_2} = u_1$ we have that there exists a nonnegative classical solution u_2 of

$$\begin{aligned} \Delta u_2 &= p(x)f(u_2),\ x \in \Omega, \\ u_2(x) &= 2,\ x \in \partial\Omega, \end{aligned}$$

with $w_1 \leq u_1 \leq u_2 \leq \overline{u_2} = v_2$. Continuing this line of reasoning we have that there exists a nonnegative classical solution u_k to

$$\begin{aligned} \Delta u_k &= p(x)f(u_k),\ x \in \Omega, \\ u_k(x) &= k,\ x \in \partial\Omega, \end{aligned}$$

with $w_1 \leq u_{k-1} \leq u_k \leq v_k$, $k \geq 2$. Clearly the sequence $\{u_k\}$ is monotone. We note that since f satisfies (1.3), G does as well. Hence it can be shown (see Theorem 1 of [7]) that the sequence $\{v_k\}$ converges to a classical solution v of

$$\begin{aligned} \Delta v &= p(x)G(v),\ x \in \Omega, \\ v(x) &\to \infty,\ x \to \partial\Omega. \end{aligned}$$

It then follows that $w_1 \leq u_{k-1} \leq u_k \leq v$. Hence, the sequence $\{u_k\}$ converges on Ω to some function u. A standard regularity argument for elliptic equations (See, e.g., the proof of Lemma 3 in [9].) then shows that u is a classical solution to (1.1). By construction, u is clearly a large solution.

Now suppose that f does not satisfy (1.3); i.e. f satisfies

$$\int_1^\infty \left[\int_0^s f(t)\,dt\right]^{-1/2} ds = \infty \tag{2.7}$$

and assume, for contradiction, that u is a nonnegative large solution of (1.1). Let v_k be a nonnegative classical solution of

$$\begin{aligned} \Delta v_k &= p(x)H(v_k),\ x \in \Omega, \\ v_k(x) &= k,\ x \in \partial\Omega. \end{aligned} \tag{2.8}$$

Then the sequence $\{v_k\}$ is nondecreasing and $v_k \leq u$ on Ω. It follows that $\{v_k\}$ converges to a nonnegative function v on Ω. Another standard regularity argument will show that v is a classical solution of the system

$$\begin{aligned} \Delta v &= p(x)H(v), \quad x \in \Omega \\ v(x) &\to \infty, \qquad\quad x \to \partial\Omega. \end{aligned}$$

This problem, however, has no solution because, as a consequence of (2.7), H satisfies

$$\int_1^\infty [\int_0^s H(t)dt]^{-1/2} ds = \infty$$

(see Theorem 1 of [7]). Hence, equation (1.1) has no nonnegative large solution on Ω. This completes the proof. ∎

3. EXISTENCE OF SOLUTIONS ON UNBOUNDED DOMAINS

We now consider the case where Ω is unbounded and begin by letting $\Omega = \mathbb{R}^n$. Consistent with results for nondecreasing f, we require

$$\int_0^\infty r\phi(r)dr < \infty, \tag{3.1}$$

where $\phi(r) = \max_{|x|=r} p(x)$.

Theorem 3.1. *Suppose p is a nonnegative c-positive $C^\alpha_{loc}(\mathbf{R}^n)$ function which satisfies (3.1), f is Lipschitz continuous on $[0,\infty)$, $f(0) = 0$, $f(s) > 0$ for $s > 0$, and f satisfies (1.2). Then (1.1) has a nonnegative entire large solution provided f satisfies (1.3)*

Proof. Using a proof similar to that of Theorem 2.2, it is a straightforward exercise to prove the existence of nonnegative solutions v_k and w_k to the following boundary value problems

$$\begin{aligned} \Delta v_k &= p(x)G(v_k),\ |x| < k, \\ v_k(x) &\to \infty \text{ as } |x| \to k, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \Delta w_k &= p(x)H(w_k),\ |x| < k, \\ w_k(x) &\to \infty \text{ as } |x| \to k, \end{aligned} \tag{3.3}$$

which satisfy $w_k \le v_k$ on $|x| \le k$. It is clear, by the maximum principle, that $v_k(x) \ge v_{k+1}(x)$ on $|x| \le k$, for each k. By defining $v_k(x) = \infty$ for $|x| \ge k$, we have that the sequence v_k is monotonely deceasing on R^n. Furthermore, we can employ the same method to produce a nonnegative solution u_k to the boundary value problem

$$\begin{aligned} \Delta u_k &= p(x)f(u_k),\ |x| < k, \\ u_k(x) &\to \infty \text{ as } |x| \to k, \end{aligned} \tag{3.4}$$

with $w_k \le u_k \le v_k$. If we can show that the sequence $\{u_k\}$ is uniformly bounded and equicontinuous on bounded subsets, then the Ascoli-Arzela Theorem will allow us to prove that $\{u_k\}$ has a convergent subsequence on $\mathbb{R}^n$ which is uniformly convergent on compact sets. To do this, we let $B(0,1) \subseteq \Omega = \mathbb{R}^n$ be the ball centered at zero with radius one. Notice that $u_k \le v_k$, and that the sequence $\{v_k\}$ is decreasing. Then we have that $u_k \le v_2$ on $B(0,1)$ for all $k \ge 2$. Hence, the sequence u_k is uniformly bounded on $\overline{B(0,1)}$. We also have that u_k is a solution to (3.4) on $B(0,1)$, and $u_k \in C^{2,\alpha}(B(0,1))$. Thus, by Theorem 3.9 of [4], we have, for $k \ge 3$, the gradient bound

$$\sup_{|x|<2} d_x|\nabla u_k(x)| \le C(\sup_{|x|<2} |u_k| + \sup_{|x|<2} d_x^2|p(x)f(u_k(x))|), \tag{3.5}$$

where $C = C(n)$ and $d_x = dist(x, \partial B(0,2))$. Furthermore, since $d_x \ge 1$ for $|x| \ge 1$ we have

$$\sup_{|x|<1} |\nabla u_k(x)| \le \sup_{|x|<1} d_x|\nabla u_k(x)| \le \sup_{|x|<2} d_x|\nabla u_k(x)|, \tag{3.6}$$

implying the sequence $\{u_k\}$, $k \geq 3$, is equicontinuous on $\overline{B(0,1)}$. Hence there exists a subsequence $\{u_k^1\}$ of $\{u_k\}$ which converges to a nonnegative function u^1 on the ball $B(0,1) \subseteq \Omega$.

Now, consider the subsequence $\{u_k^1\}$ on the ball $B(0,2) \subseteq \Omega = \mathbb{R}^n$ centered at 0 with radius two. It is clear that the subsequence $\{u_k^1\}$ is uniformly bounded on $\overline{B(0,2)}$. Furthermore, u_k^1 is a solution to equation (3.4) on $B(0,2)$, and therefore $u_k^1 \in C^{2,\alpha}(B(0,2))$. Thus, we have the gradient bound

$$\sup_{|x|<3} d_x|\nabla u_k^1(x)| \leq C(\sup_{|x|<3} |u_k^1| + \sup_{|x|<3} d_x^2|p(x)f(u_k^1(x))|), \tag{3.7}$$

where $C = C(n)$ and $d_x = dist(x, \partial B(0,3))$. Again, since $d_x \geq 1$ we have

$$\sup_{|x|<2} |\nabla u_k^1(x)| \leq \sup_{|x|<2} d_x|\nabla u_k^1(x)| \leq \sup_{|x|<3} d_x|\nabla u_k^1(x)|, \tag{3.8}$$

so that the subsequence $\{u_k^1\}$ is also equicontinuous on $\overline{B(0,2)}$. So, there exists a subsequence $\{u_k^2\}$ of $\{u_k^1\}$ which converges to a nonnegative function u^2 on the ball $B(0,2) \subseteq \Omega$.

Continuing this line of reasoning, we have that there exist nonnegative large solutions u^3, u^4, u^5, ... on the balls $B(0,3)$, $B(0,4)$, $B(0,5)$, ..., respectively. Furthermore we note that

$$u^1 = u^2 = u^3 = u^4 = u^5 = ..., \text{ on } B(0,1) \tag{3.9}$$

and, more generally,

$$u^m = u^{m+1} = u^{m+2} = ..., \text{ on } B(0,m).$$

Now we define the function u on $\mathbb{R}^n$ as $u(x) = u^i(x)$ for $|x| < i$. Thus $u^i(x) \to u(x)$ as $i \to \infty$ for all $x \in \mathbb{R}^n$ and the convergence is uniform on compact sets. Once again, a standard regularity argument will show that u is a solution to (1.1) on $\Omega = \mathbb{R}^n$. It is easy to see that u is, in fact, a large solution since $w \equiv \lim_{k\to\infty} w_k$ satisfies $w \leq u$, and w is large by virtue of (3.1) (see Theorem 2 of [7]). ∎

We now extend this result to somewhat arbitrary unbounded domains.

Theorem 3.2. *Suppose Ω is an unbounded domain in $\mathbb{R}^n$, $n \geq 3$, with compact $C^{2,\gamma}$ boundary and suppose there exists a sequence of bounded domains $\{\Omega_k\}$, each with smooth boundary, such that $\Omega_k \subseteq \Omega_{k+1}$ for all $k = 1, 2, ...$ and $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$. Suppose p is a nonnegative c-positive $C^{\alpha}_{loc}(\mathbb{R}^n)$ function with $\phi(r) \equiv \max\{p(x) : |x| = r,\ x \in \Omega\}$ and assume that it satisfies inequality (3.1). Assume that f is Lipschitz continuous on $[0,\infty)$, $f(0) = 0$, $f(s) > 0$ for $s > 0$, and f satisfies (1.2). Then (1.1) has a nonnegative large solution provided f satisfies (1.3).*

Proof. We replace the functions v_k and w_k in the proof of Theorem 3.1 with the solutions to

$$\begin{aligned} \Delta v_k &= p(x)G(v_k),\ x \in \Omega_k, \\ v_k(x) &\to \infty,\ x \to \partial\Omega_k, \end{aligned}$$

$$\begin{aligned} \Delta w_k &= p(x)H(w_k),\ x \in \Omega_k, \\ w_k(x) &\to \infty,\ x \to \partial\Omega_k, \end{aligned}$$

for each k. The proof now follows an analogous approach to that of Theorem 3.1. We omit the details. ∎

We now give a partial converse to Theorem 3.2.

Theorem 3.3. *Let p, f, ϕ, and Ω be as in Theorem 3.3. In addition, suppose there exists a nonnegative function h continuous on $[0,\infty)$ and differentiable on $(0,\infty)$ such that $0 \leq \phi(r) \leq h^2(r)$ for all $r \geq 0$ and h satisfies one of the following: (a) there exists a constant C such that*

$0 \leq r^{2n-2}\phi(r) \leq C, \ \forall r \geq 0$*; or (b)* $\lim_{r\to\infty} r^{n-1}h(r) = \infty$ *and* $\int_0^\infty h(r)dr < \infty$*. If (1.1) has a nonnegative large solution on* Ω*, then* f *satisfies inequality (1.3).*

Proof. Let u be a large solution of (1.1). We can now construct a proof, very similar to the proof of Theorem 5 in [7], using the equation $\Delta v = \phi(r)H(v)$ in place of $\Delta v = \phi(r)f(v)$ in [7] to obtain a contradiction. We omit the details. ∎

Our final result provides necessary and sufficient conditions to ensure the existence of a large solution of (1.1) on an unbounded domain. It closely follows the corollary of [7] and therefore stated without proof.

Corollary 3.4. *Let* f *and* Ω *be as in Theorem 3.2, and assume* p *is a nonnegative c-positive* $C^\alpha_{loc}(\Omega)$ *function for which there exists a constant* K *such that*

$$p(x) \leq K|x|^{-\alpha}, \quad \alpha > 2. \tag{3.10}$$

for $|x|$ *large and* $x \in \Omega$*, Then a necessary and sufficient condition for (1.1) to have a nonnegative large solution on* Ω *is that* f *satisfy inequality (1.3).*

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