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NORMALIZED TRUNCATED LEVY MODELS APPLIED TO THE STUDY OF FINANCIAL MARKETS

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ABSTRACT. This work is devoted to the study of the statistical properties of financial instruments from developed markets. We performed a new analysis of the behavior of companies corresponding to the DJIA index, and of the index itself, by using a normalized Truncated Levy walk model. We conclude that the Truncated Levy distribution describes perfectly the evolution of the companies and of the index near a crash.

Key words and phrases: Levy flight, Financial Mathematics, Stock Market Prices, Financial Indices.

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1. INTRODUCTION

The Statistical Mechanics theory, like phase transitions and critical phenomena models, have been applied by many authors to the study of the speculative bubbles preceding a financial crash

Over the past years it has been an increasing interest in problems arising in Economy and Finance and many challenging mathematical problems have been motivated.

At the same time a new discipline: Econophysics, has been developed. This discipline was introduced in 1995, see *Stanley et all* [15] and [19]. It studies the application of mathematical tools that are usually applied to physical models, to the study of financial models. Simultaneously, there has been a growing literature in financial economics analyzing the behavior of major Stock Indices, see [2, 3, 7, 8, 16, 20] and the references there in.

The Statistical Mechanics theory, like phase transitions and critical phenomena have been applied by many authors to the study of the speculative bubbles preceding a financial crash (see for example [5, 9]). In these works the main assumption is the existence of log-periodic oscillation in the data. The scale invariance in the behavior of financial indices near a crash has been studied in [1, 4].

The statistical properties of the temporal series analyzing the evolution of the different markets have been of a great importance in the study of financial indices. The empirical characterization of stochastic processes usually requires the study of temporal correlations and the determination of asymptotic probability density functions (*pdf*). The first model describing option price evolution is Brownian motion. This model assumes that the increment in the logarithm of prices follows a diffusive process with Gaussian distribution [21]. However, the empirical study of the temporal series associated to some of the most important financial indices shows that in short time intervals the associated *pdf* has greater kurtosis than a Gaussian distribution [15]. The first step in order to explain this behavior was done in 1963 by *Mandelbrot* [14]. He developed a model for the evolution of the cotton price by a stable stochastic non gaussian Levy process [12]. However, these distributions are not appropriated for working in long-range correlation scales. These problems can be avoided considering that the temporal evolution of financial markets is described by a Truncated Levy Flight (*TLF*) [11, 17].

Most of the studies cited above have been done with financial indices of developed markets that have a great volume of transactions. In this work we analyze the evolution of individual companiesŠ stock prices corresponding to developed markets. Specifically, we perform a new analysis of the behavior of stock prices comprising the Dow Jones Industrial Average (DJIA) index, and of the index itself.

Our main interest is to verify that the Truncated Levy Flight distribution describes accurately the behavior of individual companies stock prices, as well as the behavior of the financial index.

The presentation is organized as follows: In the second section we give a short introduction to the Levy distributions. In the third section we present the companies that will be analyzed and the normalized Levy model that we will use for our numerical analysis. Finally, in the last section we conclude that our results are perfectly compatible with a *TLF* distribution.

2. THE TRUNCATED LEVY FLIGHT

Levy [13] and *Khintchine* [10] solved the problem of determining the functional form that all the stable distributions must follow. They found that the most general representation is through the characteristic functions $\varphi(q)$, that are defined by the following equation:

$$\ln(\varphi(q)) = \begin{cases} i\mu q - \gamma|q|^2[1 - i\beta\frac{q}{|q|}\tan(\frac{\pi\alpha}{2})] & \alpha \neq 1 \\ \mu q - \gamma|q|[1 + i\beta\frac{q}{|q|}\frac{2}{\pi}\log(q)] & \alpha = 1 \end{cases}$$

where $0 < \alpha \leq 2$ γ is a positive scale factor, μ is a real number and β is an asymmetry parameter ranging from -1 to 1.

The analytical form of the Levy stable distribution is known only for a few values of α and β:

$\cdot$ $\alpha = \frac{1}{2}$, $\beta = 1$ (Levy - Smirnov)
$\cdot$ $\alpha = 1$, $\beta = 0$ (Lorentzian)
$\cdot$ $\alpha = 2$ (Gaussian).

We consider the symmetric distribution ($\beta = 0$) with a zero mean ($\mu = 0$). In this case the characteristic function takes the form:

$$\varphi(q) = \exp(-\gamma|q|^{\alpha}).$$

As the characteristic function of a distribution is its Fourier transform, the stable distribution of index α and scale factor γ is

$$P_L(x) = \frac{1}{\pi}\int_0^{\infty} \exp(-\gamma|q|^{\alpha})\cos(qx)dq.$$

The asymptotic behavior of the distribution for big values of the absolute value of x is given by:

$$P_L(x) \simeq \frac{\gamma\Gamma(1+\alpha)\sin\frac{\pi\alpha}{2}}{\pi|x|^{1+\alpha}} \simeq |x|^{-(1+\alpha)}$$

and the value at zero $P_L(x=0)$ by:

$$P_L(x=0) = \frac{\Gamma(1/\alpha)}{\pi\alpha\gamma^{1/\alpha}}.$$

The fact that the asymptotic behavior for big values of x is a power law has as a consequence that the stable Levy processes have infinite variance. In order to avoid the problems arising in the infinite second moment *Mantegna et all* considered a stochastic process with finite variance that follows scale relations called Truncated Levy flight (*TLF*) [17]. The *TLF* distribution is defined by

$$T(x) = \begin{cases} 0 & x > l \\ cP(x) & -l < x < l \\ 0 & x < -l \end{cases}$$

with $P(x)$ a symmetric Levy distribution.

Another property of the *TLF* is that the cut that it presents in its tails is very abrupt. *Koponen* [11] considered a *TLF* in which the cut function is a decreasing exponential characterized by a parameter l.

The characteristic function of this distribution is defined as:

$$\varphi(q) = \exp\{c_0 - c_1\frac{(q^2+1/l^2)^{\frac{\alpha}{2}}}{\cos(\pi\alpha/2)}\cos(\alpha\arctan(l|q|))\}$$

with c_1 a scale factor:

$$c_1 = \frac{2\pi\cos(\pi\alpha/2)}{\alpha\Gamma(\alpha)\sin(\pi\alpha)}At$$

and

$$c_0 = \frac{l^{-\alpha}}{\cos(\pi\alpha/2)}c_1 = \frac{2\pi}{\alpha\Gamma(\alpha)\sin(\pi\alpha)}Al^{-\alpha}t.$$

The variance can be calculated from the characteristic function:

$$\sigma^2(t) = \frac{\partial^2 \varphi(q)}{\partial q^2}|_{k=0} = t\frac{2A\pi(1-\alpha)}{\Gamma(\alpha)\sin(\pi\alpha)}l^{2-\alpha}$$

If we discretize in time with steps Δt, we obtain that $T = N\Delta t$. At the end of each interval we must calculate the sum of N stochastic variables that are independent and identically distributed. Therefore, the new characteristic function will be:

$$\varphi(q, N) = \exp\{c_0 N - c_1 \frac{N(q^2 + 1/l^2)^{\alpha/2}}{\cos(\pi\alpha/2)} \cos(\alpha \arctan(l|q|))\}.$$

For small values of N the return probability will be very similar to the stable Levy distribution:

$$P_L(x = 0) = \frac{\Gamma(1/\alpha)}{\pi\alpha(\gamma N)^{1/\alpha}}.$$

3. METHODS AND DATA ANALYSIS

We studied the behavior of stock prices comprising the Dow Jones Industrial Average index, along with the index itself.

Specifically, we studied the behavior of Intel, Pfizer, Exxon and Coca-Cola from Jan. 2, 1985 to Oct. 2, 1987. We chose to apply the Levy Walk to individual companies to determine if this technique is not only applicable to predicting a correction of indices but also corrections to specific company stock prices.

In this work, the analyzed stochastic variable is the return G_t, defined as the difference of the logarithm of two consecutive index prices:

$$G_t = \ln I_t - \ln I_{t-T}$$

where T is the difference (in labor days) between two values of the index. In order to compare the returns for different values of T we define the normalized return as:

$$g \equiv \frac{G - \langle G \rangle_T}{\sigma}$$

where

$$\sigma^2 \equiv \langle G^2 \rangle_T - \langle G \rangle_T^2$$

where

$$\sigma^2 \equiv \langle G^2 \rangle_T - \langle G \rangle_T^2$$

denotes the arithmetic average over the temporal series of scale T.

In order to use Koponen model to describe the normalized returns of the indices, we need to normalize the model. Given that

$$\sigma^2 = -\frac{\partial^2 \varphi(q)}{\partial q^2}|_{k=0}$$

we have that

$$-\frac{\partial^2 \varphi(q/\sigma)}{\partial q^2}|_{k=0} = -\frac{1}{\sigma^2}\frac{\partial^2 \varphi(q)}{\partial q^2}|_{k=0} = 1$$

Therefore, a normalized model is:

Stocks	**DJIA Index**	**Intel**	**Pfizer**	**Exxon**	**Coca-Cola**
T=1	1.05	1.60	1.20	1.35	1.20
T=4	0.80	1.30	1.20	1.25	1.20
T=8	0.70	0.80	0.80	1.90	1.80
T=16	0.40	0.30	0.80	1.40	1.70

Table 4.1: Values of the exponent α for 4 different values of the time scale T

$$\ln \varphi_N(q) = \ln \varphi(\frac{q}{\sigma}) = c_0 - c_1 \frac{((q/\sigma)^2 + 1/l^2)^{\alpha/2}}{\cos(\pi\alpha/2)} \cos(\alpha \arctan(l\frac{|q|}{\sigma}))$$
$$= \frac{2\pi A l^{-\alpha} t}{\alpha\Gamma(\alpha)\sin(\pi\alpha)}[1 - ((\frac{ql}{\sigma})^2 + 1)^{\alpha/2} \cos(\alpha \arctan(\frac{ql}{\sigma}))].$$

This is the normalized Levy model that will be used for the numerical analysis. We will fix l at some arbitrary reasonable number, and then adjust A and the characteristic exponent α simultaneously in order to fit the cumulative function.

4. DISCUSSION AND CONCLUSIONS

This work offers a new way of analyzing financial companies. We did an empirical study of the statistical behavior of a financial index along with the rate of return of specific companies within the index, by using a normalized Truncated Levy Flight model. In all the cases we obtained that the evolution of the financial index along with the specific companies can be reasonably described by the model. In table 4.1 we can see that all the values obtained for the exponent α are lower than 2. The excellent numerical results validate the model.

We want to recall that in a previous work ([7]) it was found that the exponents α calculated for emergent market indices were strictly greater than 2. This behavior was compatible with a slow convergence to a Gaussian distribution but we could not conclude that the Levy distribution was the appropriate stochastic process for explaining the evolution of the financial indices. We believe that the normalized Levy model that we used in this work allowed us making a more accurately numerical analysis.

We want to remark that in the cumulative distribution curve of each company there are some outlying points. Those outlying points correspond to the beginning of a significant decrease in the stock price within a short period of time. This is exactly the way in which we define a market crash: *A market crash is an outlying point in the cumulative probability distribution of the stochastic process described by the Levy model*.

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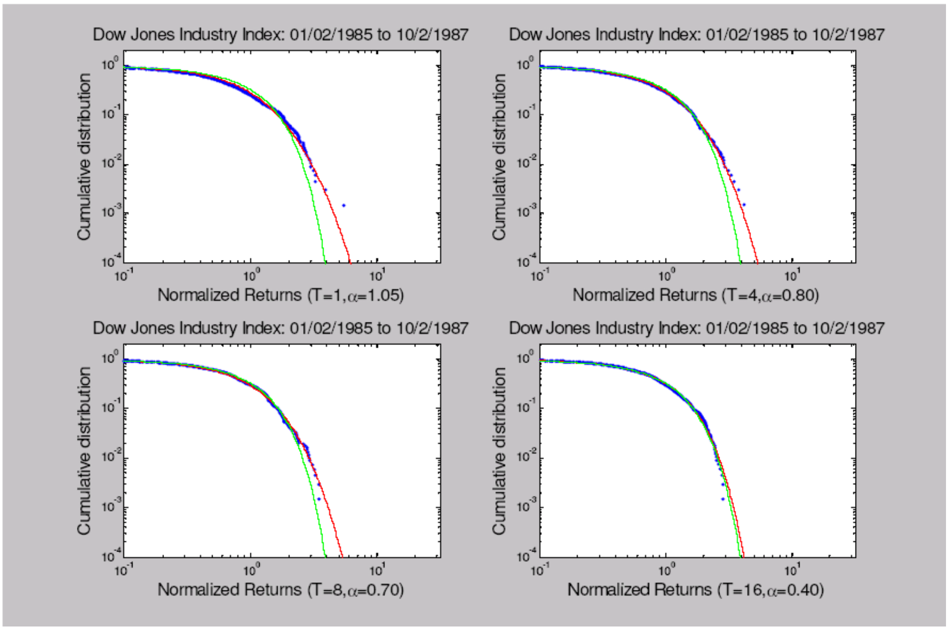

Figure 1: Log-log plot of the cumulative distribution of the normalized returns of DJIA Index for four different values of time scale T. The red line is the best fit of the Levy distribution. The green line indicates the Gaussian distribution. We use data from Jan. 2, 1985 to Oct. 2, 1987.

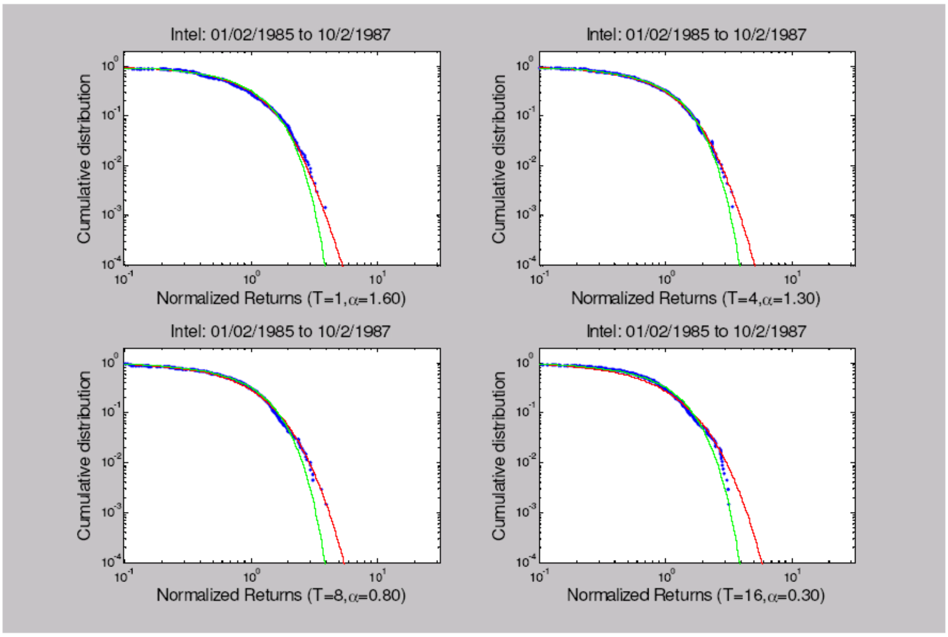

Figure 2: Log-log plot of the cumulative distribution of the normalized returns of Intel stock prices for four different values of time scale T. The red line is the best fit of the Levy distribution. The green line indicates the Gaussian distribution. We use data from Jan. 2, 1985 to Oct. 2, 1987.

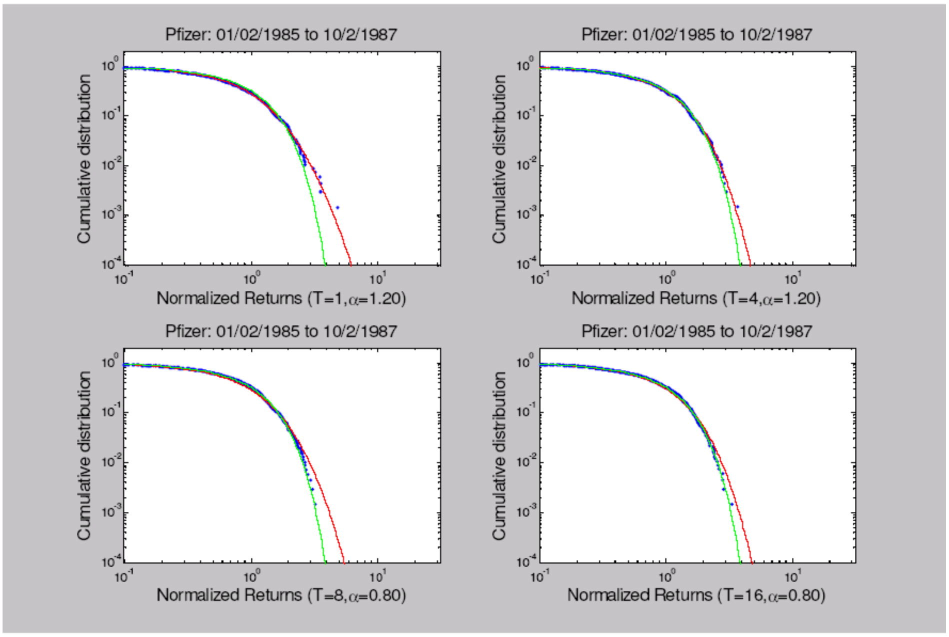

Figure 3: Log-log plot of the cumulative distribution of the normalized returns of Pfizer stock prices for four different values of time scale T. The red line is the best fit of the Levy distribution. The green line indicates the Gaussian distribution. We use data from Jan. 2, 1985 to Oct. 2, 1987.

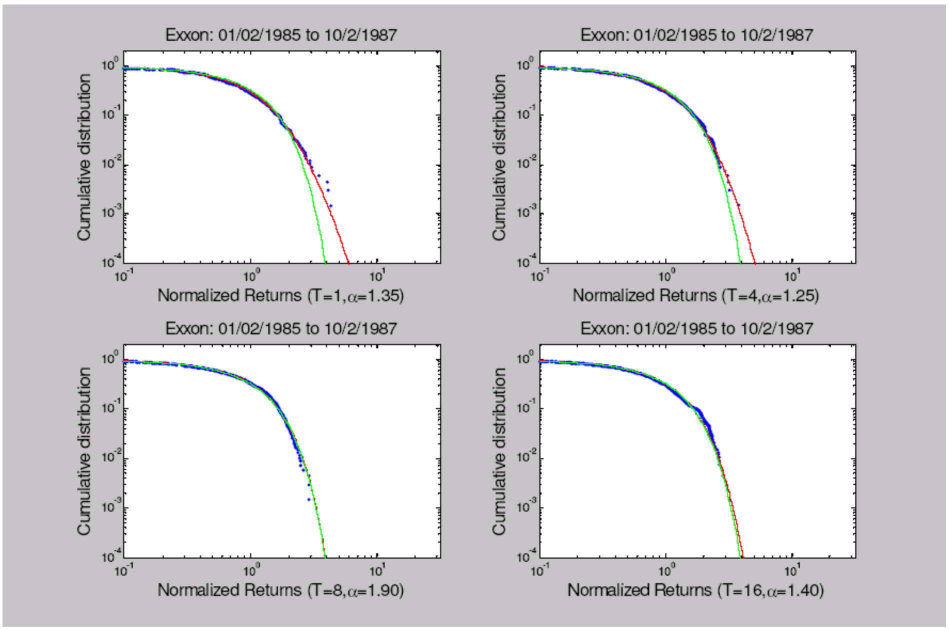

Figure 4: Log-log plot of the cumulative distribution of the normalized returns of Exxon stock prices for four different values of time scale T. The red line is the best fit of the Levy distribution. The green line indicates the Gaussian distribution. We use data from Jan. 2, 1985 to Oct. 2, 1987.

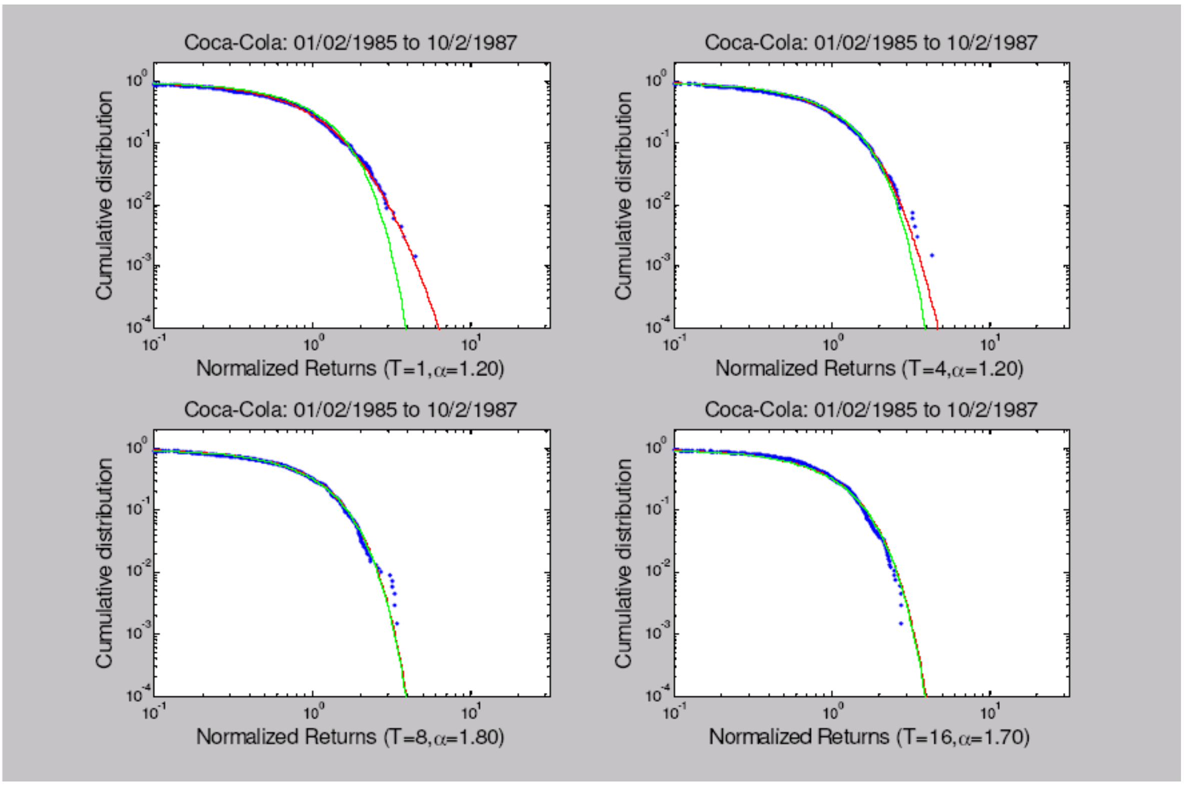

Figure 5: Log-log plot of the cumulative distribution of the normalized returns of Coca-Cola stock prices for four different values of time scale T. The red line is the best fit of the Levy distribution. The green line indicates the Gaussian distribution. We use data from Jan. 2, 1985 to Oct. 2, 1987.