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TWO CLASSES OF COMPLETELY MONOTONIC FUNCTIONS INVOLVING GAMMA AND POLYGAMMA FUNCTIONS

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ABSTRACT. The function

$$\frac{[\Gamma(x+1)]^{1/x}}{x^c}\left(1+\frac{1}{x}\right)^x$$

is logarithmically completely monotonic in $(0,\infty)$ if and only if $c \geq 1$ and its reciprocal is logarithmically completely monotonic in $(0,\infty)$ if and only if $c \leq 0$. The function

$$\psi''(x) + \frac{2+(6+c)x+(4+3c)x^2+(2+3c)x^3+cx^4}{x^3(x+1)^3}$$

is completely monotonic in $(0,\infty)$ if and only if $c \geq 1$ and its negative is completely monotonic in $(0,\infty)$ if and only if $c \leq 0$.

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1. INTRODUCTION

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \ge 0 \tag{1.1}$$

for $x \in I$ and $n \ge 0$. The set of completely monotonic functions is denoted by $\mathcal{C}[I]$.

A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^k [\ln f(x)]^{(k)} \ge 0 \tag{1.2}$$

for $k \in \mathbb{N}$ on I. The set of logarithmically completely monotonic functions is denoted by $\mathcal{L}[I]$.

A function f is called a Stieltjes transform if it can be of the form

$$f(x) = a + \int_0^\infty \frac{\mathrm{d}\,\mu(s)}{s+x}, \tag{1.3}$$

where $a \ge 0$ and μ is a nonnegative measure on $[0,\infty)$ satisfying

$$\int_0^\infty \frac{1}{1+s}\,\mathrm{d}\,\mu(s) < \infty. \tag{1.4}$$

The set of Stieltjes transforms is denoted by $\mathcal{S}$.

The notion or terminology "logarithmically completely monotonic function" was explicitly introduced in [9], formally published in [8], and immediately studied or cited in [2, 4, 5, 10, 11]. Among other things, it is implicitly or explicitly proved in [2, 3, 8, 9, 12] that $\mathcal{L}[I] \subset \mathcal{C}[I]$, but not conversely [9]. Among others, it is also showed in [2, 12] that

$$\mathcal{S} \setminus \{0\} \subset \mathcal{L}[(0,\infty)] \subset \mathcal{C}[(0,\infty)]. \tag{1.5}$$

In [2, Theorem 1.1] and [4, 10] it is pointed out that the logarithmically completely monotonic functions on $(0,\infty)$ can be characterized as the infinitely divisible completely monotonic functions studied by Horn in [6, Theorem 4.4]. For more information on the logarithmically completely monotonic functions, please refer to [2, 7, 12] and the references therein.

In [10, 11], it is proved that

$$\Phi(x) = \frac{[\Gamma(x+1)]^{1/x}}{x}\left(1+\frac{1}{x}\right)^x \in \mathcal{L}[(0,\infty)], \tag{1.6}$$

where $\Gamma(x)$ is the well known classical Euler gamma function, which is one of the most important special functions and has much extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences.

Motivated by [9, 11], the paper [2] proved that $\Phi(x) \in \mathcal{S}$ and $\ln \Phi(x) \in \mathcal{S}$ and obtained the integral representations for $\Phi(x)$ and $\ln \Phi(x)$ respectively.

Define

$$\Phi_c(x) = \frac{[\Gamma(x+1)]^{1/x}}{x^c}\left(1+\frac{1}{x}\right)^x \tag{1.7}$$

for $x \in (0,\infty)$. It is clear that $\Phi_1(x) = \Phi(x)$.

The main purpose of this paper is to find the range of $c \in \mathbb{R}$ such that $\Phi_c(x) \in \mathcal{L}[(0,\infty)]$. Our main results are as follows.

Theorem 1.1. *The function*

$$\phi(x) = \psi''(x) + \frac{2+(6+c)x+(4+3c)x^2+(2+3c)x^3+cx^4}{x^3(x+1)^3} \in \mathcal{C}[(0,\infty)] \tag{1.8}$$

if and only if $c \geq 1$ *and* $-\phi(x) \in \mathcal{C}[(0,\infty)]$ *if and only if* $c \leq 0$.

Theorem 1.2. *The function* $\Phi_c(x) \in \mathcal{L}[(0,\infty)]$ *if and only if* $c \geq 1$ *and* $[\Phi_c(x)]^{-1} \in \mathcal{L}[(0,\infty)]$ *if and only if* $c \leq 0$.

Remark 1.1. Since $\Phi_1(x)$ and $\ln \Phi_1(x)$ are both Stieltjes transforms, it is natural to ask whether the functions $\Phi_c(x)$ and $\ln \Phi_c(x)$ are Stieltjes transforms for $c \neq 1$.

2. PROOFS OF THEOREM 1.1 AND THEOREM 1.2

Proof of Theorem 1.1. It is well known [1] that, for $x > 0$, $r > 0$, and $k \in \mathbb{N}$,

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-xt} \,\mathrm{d}t, \tag{2.1}$$

$$\psi^{(k)}(x) = (-1)^{k+1} k! \sum_{i=0}^\infty \frac{1}{(x+i)^{k+1}}, \tag{2.2}$$

$$\psi^{(k)}(x) = (-1)^{k+1} \int_0^\infty \frac{t^k e^{-xt}}{1-e^{-t}} \,\mathrm{d}t, \tag{2.3}$$

$$\psi^{(k-1)}(x+1) = \psi^{(k-1)}(x) + \frac{(-1)^{k-1}(k-1)!}{x^k}. \tag{2.4}$$

From formulas (2.1), (2.2), (2.3) and (2.4), for $x \in (0,\infty)$ and any nonnegative integer i, it follows that

$$\begin{aligned}
\phi(x) &\triangleq \psi''(x) + g_2(x) + h_2(x) \\
&\triangleq \psi''(x) + \frac{2 + cx - 2x^2}{x^3} + \frac{2(3+3x+x^2)}{(x+1)^3} \\
&= \psi''(x) + \frac{2}{x^3} + \frac{c}{x^2} - \frac{2}{x} + \frac{2}{(1+x)^3} + \frac{2}{(1+x)^2} + \frac{2}{1+x} \\
&= \frac{c}{x^2} - \frac{2}{x} + \frac{2}{(1+x)^2} + \frac{2}{1+x} - 2\sum_{i=2}^\infty \frac{1}{(x+i)^3} \\
&= \psi''(x+2) + \frac{c}{x^2} - \frac{2}{x} + \frac{2}{(1+x)^2} + \frac{2}{1+x} \\
&= c\int_0^\infty te^{-xt}\,\mathrm{d}t - 2\int_0^\infty e^{-xt}\,\mathrm{d}t + 2\int_0^\infty te^{-(x+1)t}\,\mathrm{d}t \\
&\quad + 2\int_0^\infty e^{-(x+1)t}\,\mathrm{d}t - \int_0^\infty \frac{t^2 e^{-(x+2)t}}{1-e^{-t}}\,\mathrm{d}t \\
&= \int_0^\infty \big[(ct-2)e^{2t} + (2t - ct + 4)e^t - (t^2+2t+2)\big] \frac{e^{-(x+2)t}}{1-e^{-t}}\,\mathrm{d}t \\
&\triangleq \int_0^\infty q(t) \frac{e^{-(x+2)t}}{1-e^{-t}}\,\mathrm{d}t
\end{aligned}$$

and

$$\phi^{(i)}(x) = (-1)^i \int_0^\infty t^i q(t) \frac{e^{-(x+2)t}}{1-e^{-t}}\,\mathrm{d}t. \tag{2.5}$$

Standard argument shows that $q(t) \lesseqgtr 0$ is equivalent to

$$c \lesseqgtr \frac{2e^{2t} - 2(t+2)e^t + t^2 + 2t + 2}{te^t(e^t-1)} = \varphi(t) \tag{2.6}$$

for $t \geq 0$. Straightforward computing yields

$$\begin{aligned}
\varphi'(t) &= \frac{2+2t+t^2+t^3-(6+4t+3t^2+2t^3)e^t+2(3+t+t^2)e^{2t}-2e^{3t}}{t^2e^t(e^t-1)^2} \\
&\triangleq \frac{\lambda_1(t)}{t^2e^t(e^t-1)^2}, \\
\lambda_1'(t) &= 2+2t+3t^2-(10+10t+9t^2+2t^3)e^t+2(7+4t+2t^2)e^{2t}-6e^{3t}, \\
\lambda_1''(t) &= 2+6t-(20+28t+15t^2+2t^3)e^t+4(9+6t+2t^2)e^{2t}-18e^{3t}, \\
\lambda_1'''(t) &= 6-54e^{3t}-(48+58t+21t^2+2t^3)e^t+16(6+4t+t^2)e^{2t}, \\
\lambda_1^{(4)}(t) &= -\big[106+100t+27t^2+2t^3+162e^{2t}-32(8+5t+t^2)e^t\big]e^t \\
&\triangleq \lambda_2(t), \\
\lambda_2'(t) &= 100+54t+6t^2-32(13+7t+t^2)e^t+324e^{2t}, \\
\lambda_2''(t) &= 6(9+2t)-32(20+9t+t^2)e^t+648e^{2t}, \\
\lambda_2'''(t) &= 4\big[3-8(29+11t+t^2)e^t+324e^{2t}\big], \\
\lambda_2^{(4)}(t) &= 32(81e^t-t^2-13t-40)e^t.
\end{aligned}$$

It is clear that $\lambda_2^{(4)}(t) > 0$ in $(0,\infty)$ and $\lambda_2^{(i)}(0) > 0$ for $0 \leq i \leq 3$. Therefore, the functions $\lambda_2^{(i)}(t)$ is increasing and positive for $0 \leq i \leq 3$ in $(0,\infty)$. This implies that $\lambda_1^{(4)}(t)$ is negative in $(0,\infty)$. Since $\lambda_1^{(i)}(0) = 0$ for $0 \leq i \leq 3$, it follows that $\lambda_1^{(i)}(t)$ is decreasing and negative for $0 \leq i \leq 3$ in $(0,\infty)$. This gives $\varphi'(t) < 0$ in $(0,\infty)$. Hence, the function $\varphi(t)$ is strictly decreasing in $(0,\infty)$.

Using the decreasingly monotonicity of $\varphi(t)$ and the fact that

$$\lim_{t\to0}\varphi(t) = 1 \quad \text{and} \quad \lim_{t\to\infty}\varphi(t) = 0$$

leads to $0 < \varphi(t) < 1$. If $c \geq 1$, then $q(t) \geq 0$; if $c \leq 0$, then $q(t) \leq 0$. This means that the function $\phi(x)$ is strictly completely monotonic in $(0,\infty)$ for $c \geq 1$ and $-\phi(x)$ is also strictly completely monotonic in $(0,\infty)$ for $c \leq 0$.

If $\phi(x)$ is completely monotonic in $(0,\infty)$, then by definition

$$\phi'(x) = \psi'''(x) - \frac{2(3+12x+17x^2+8x^3+3x^4)}{x^4(1+x)^4} - \frac{2c}{x^3} \leq 0 \tag{2.7}$$

which is equivalent to

$$c \geq \frac{x^3}{2}\left(\psi'''(x) - \frac{2(3+12x+17x^2+8x^3+3x^4)}{x^4(1+x)^4}\right) \to 1 \tag{2.8}$$

as $x \to \infty$ by using the asymptotic formula [1]:

$$(-1)^{n+1}\psi^{(n)}(x) = \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \frac{(n+1)!}{12x^{n+2}} + O\left(\frac{1}{x^{n+3}}\right), \quad x \to \infty. \tag{2.9}$$

Similarly, it is easy to see that the necessary condition of $-\phi(x)$ being completely monotonic in $(0,\infty)$ is $c \leq 0$. The proof of Theorem 1.1 is complete. ∎

The first proof of Theorem 1.2. Taking logarithm of $\Phi_c(x)$ gives

$$\ln\Phi_c(x) = x\ln\left(1+\frac{1}{x}\right) + \frac{\ln\Gamma(x+1)}{x} - c\ln x.$$

Differentiating yields

$$[\ln \Phi_c(x)]' = \ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1} + \frac{x\psi(x+1) - \ln\Gamma(x+1)}{x^2} - \frac{c}{x} \tag{2.10}$$

and

$$\begin{aligned}
[\ln \Phi_c(x)]^{(n)} &= (-1)^{(n-1)}(n-1)!x\left[\frac{1}{(x+1)^n} - \frac{1}{x^n}\right] + (-1)^n(n-1)!\frac{c}{x^n}\\
&\quad + (-1)^n(n-2)!n\left[\frac{1}{(x+1)^{n-1}} - \frac{1}{x^{n-1}}\right] + \frac{h_n(x)}{x^{n+1}}\\
&= (-1)^n(n-2)!\left[\frac{c(n-1)-x}{x^n} + \frac{x+n}{(x+1)^n}\right] + \frac{h_n(x)}{x^{n+1}},
\end{aligned}$$

where $n \geq 2$,

$$\psi^{(-1)}(x+1) = \ln\Gamma(x+1), \quad \psi^{(0)}(x+1) = \psi(x+1),$$

and

$$h_n(x) = \sum_{k=0}^{n} \frac{(-1)^{n-k} n! x^k \psi^{(k-1)}(x+1)}{k!}, \tag{2.11}$$

$$h_n'(x) = x^n \psi^{(n)}(x+1) \begin{cases} > 0 & \text{if } n \text{ is odd},\\ < 0 & \text{if } n \text{ is even}. \end{cases} \tag{2.12}$$

Therefore,

$$(-1)^n x^{n+1}[\ln \Phi_c(x)]^{(n)} + (-1)^{n+1}h_n(x) = (n-2)!\left\{c(n-1) - x + \frac{x^n(x+n)}{(x+1)^n}\right\}x$$

and, by (2.4),

$$\begin{aligned}
\frac{\mathrm{d}\left\{(-1)^n x^{n+1}[\ln \Phi_c(x)]^{(n)}\right\}}{\mathrm{d}x} &= (-1)^n x^n \psi^{(n)}(x+1) + (n-2)!\left\{c(n-1) - 2x\right.\\
&\quad \left. + \frac{x^n[n+n^2+(2+2n)x+2x^2]}{(x+1)^{n+1}}\right\}\\
&= x^n\left\{(-1)^n\psi^{(n)}(x+1) + (n-2)!\left[\frac{c(n-1)-2x}{x^n}\right.\right.\\
&\quad \left.\left. + \frac{n+n^2+(2+2n)x+2x^2}{(x+1)^{n+1}}\right]\right\}\\
&= x^n\left\{(-1)^n\psi^{(n)}(x) + \frac{n!}{x^{n+1}} + (n-2)!\left[\frac{c(n-1)-2x}{x^n}\right.\right.\\
&\quad \left.\left. + \frac{n+n^2+(2+2n)x+2x^2}{(x+1)^{n+1}}\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&= x^n\bigg\{(-1)^n\psi^{(n)}(x)+\frac{n!}{x^{n+1}}+(n-2)!\bigg[\frac{c(n-1)-2x}{x^n}\\
&\quad+\frac{n(n+1)+2(n+1)x+2x^2}{(x+1)^{n+1}}\bigg]\bigg\}\\
&= x^n\bigg\{(-1)^n\psi^{(n)}(x)+(n-2)!\bigg[\frac{n(n-1)+c(n-1)x-2x^2}{x^{n+1}}\\
&\quad+\frac{n(n+1)+2(n+1)x+2x^2}{(x+1)^{n+1}}\bigg]\bigg\}\\
&\triangleq x^n\big\{(-1)^n\psi^{(n)}(x)+(n-2)![g_n(x)+h_n(x)]\big\}
\end{aligned}
$$

with

$$g_n'(x)=-(n-1)g_{n+1}(x)\quad\text{and}\quad h_n'(x)=-(n-1)h_{n+1}(x)$$

which implies

$$g_2^{(n-2)}(x)=(-1)^n(n-2)!g_n(x)\quad\text{and}\quad h_2^{(n-2)}(x)=(-1)^n(n-2)!h_n(x)$$

by induction. Hence, by using Theorem 1.1, we have

$$\frac{\mathrm{d}\big\{(-1)^nx^{n+1}[\ln\Phi_c(x)]^{(n)}\big\}}{\mathrm{d}x}=(-1)^nx^n\phi^{(n-2)}(x)\begin{cases}>0 & \text{if and only if } c\ge 1,\\ <0 & \text{if and only if } c\le 0,\end{cases}$$

and the function $(-1)^nx^{n+1}[\ln\Phi_c(x)]^{(n)}$ is increasing (or decreasing) if and only if $c\ge 1$ (or $c\le 0$) in $(0,\infty)$. From $\lim_{x\to 0}\big\{(-1)^nx^{n+1}[\ln\Phi_c(x)]^{(n)}\big\}=0$, it is deduced that

$$(-1)^nx^{n+1}[\ln\Phi_c(x)]^{(n)}\begin{cases}>0 & \text{if and only if } c\ge 1\\ <0 & \text{if and only if } c\le 0\end{cases}$$

and

$$(-1)^n[\ln\Phi_c(x)]^{(n)}\begin{cases}>0 & \text{if and only if } c\ge 1\\ <0 & \text{if and only if } c\le 0\end{cases}$$

for $n\ge 2$ in $(0,\infty)$. This implies the function $[\ln\Phi_c(x)]'$ is increasing (or decreasing) if and only if $c\ge 1$ (or $c\le 0$) in $(0,\infty)$. It is ready to obtain $\lim_{x\to\infty}[\ln\Phi_c(x)]'=0$, so

$$[\ln\Phi_c(x)]'\begin{cases}<0 & \text{if and only if } c\ge 1\\ >0 & \text{if and only if } c\le 0\end{cases}$$

and $\ln\Phi_c(x)$ is decreasing (or increasing) if and only if $c\ge 1$ (or $c\le 0$) in $(0,\infty)$. The first proof of Theorem 1.2 is complete. ∎

The second proof of Theorem 1.2. Write $\Phi_c(x)=\frac{\Phi(x)}{x^{c-1}}$. Hence

$$f(x)\equiv\ln[\Phi_c(x)]=-(c-1)\ln x+\ln[\Phi(x)].$$

By applying one of the results in [10] that $\Phi(x)$ is logarithmically completely monotonic in $(0,\infty)$, it is easy to show $(-1)^nf^{(n)}(x)\ge 0$ in $(0,\infty)$ for all $n\in\mathbb{N}$ if $c\ge 1$.

For the part of $c<1$, the second part of Theorem 1.2 is proved if one uses $\ln\frac{1}{\Phi_c(x)}=-\ln[\Phi_c(x)]$.

If the function $\Phi_c(x)$ is logarithmically completely monotonic in $(0,\infty)$, then by definition $[\ln\Phi_c(x)]'\le 0$ which is equivalent to

$$c\ge x\ln\left(1+\frac{1}{x}\right)-\frac{x}{x+1}+\frac{x\psi(x+1)-\ln\Gamma(x+1)}{x}\triangleq\vartheta(x) \tag{2.13}$$

from (2.10). If $\frac{1}{\Phi_c(x)}$ is logarithmically completely monotonic in $(0, \infty)$, then by definition $[\ln \Phi_c(x)]' \geq 0$ which is equivalent to the reversed inequality of (2.13). By L'Hôspital's rule, it is easy to obtain that $\lim_{x\to 0} \vartheta(x) = 0$. Utilizing directly the following formulas

$$\ln \Gamma(x = \left(x - \frac{1}{2}\right) \ln x - x + \frac{\ln(2\pi)}{2} + \frac{1}{12x} + O\left(\frac{1}{x^2}\right), \quad x \to \infty \tag{2.14}$$

and

$$\psi(x) = \ln x - \frac{1}{2x} - \frac{1}{12x^2} + O\left(\frac{1}{x^3}\right), \quad x \to \infty \tag{2.15}$$

yields $\lim_{x\to\infty} \vartheta(x) = 1$. Therefore, the necessary condition of $\Phi_c(x)$ being logarithmically completely monotonic in $(0, \infty)$ is $c \geq 1$ and the necessary condition of $\frac{1}{\Phi_c(x)}$ being logarithmically completely monotonic in $(0, \infty)$ is $c \leq 0$. The second proof of Theorem 1.2 is complete. ∎

REFERENCES

[1] M. ABRAMOWITZ AND I.A. STEGUN (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Reprint of the 1972 edition. A Wiley-Interscience Publication. Selected Government Publications. John Wiley & Sons, Inc., New York; National Bureau of Standards, Washington, DC, 1984.

[2] C. BERG, Integral representation of some functions related to the gamma function, *Mediterr. J. Math.*, **1** (2004), no. 4, pp. 433–439.

[3] S. BOCHNER, *Harmonic Analysis and the Theory of Probability*, California Monographs in Mathematical Sciences, University of California Press, Berkeley and Los Angeles, 1955.

[4] A. Z. GRINSHPAN AND M. E. H. ISMAIL, Completely monotonic functions involving the Gamma and q-gamma functions, *Proc. Amer. Math. Soc.*, **134** (2006), no. 4, pp. 1153–1160.

[5] B.-N. GUO AND F. QI, Two classes of completely monotonic functions involving gamma and polygamma functions, *RGMIA Res. Rep. Coll.*, **8** (2005), no. 3, Art. 16, pp. 511–519. [Online: `http://rgmia.vu.edu.au/v8n3.html`].

[6] R. A. HORN, On infinitely divisible matrices, kernels and functions, *Z. Wahrscheinlichkeitstheorie und Verw. Geb*, **8** (1967), pp. 219–230.

[7] F. QI, Certain logarithmically N-alternating monotonic functions involving gamma and q-gamma functions, *Nonlinear Funct. Anal. Appl.*, **13** (2008), no. 1, in press.. *RGMIA Res. Rep. Coll.*, **8** (2005), no. 3, Art. 5, pp. 413–422; [Online: `http://rgmia.vu.edu.au/v8n3.html`].

[8] F. QI AND CH.-P. CHEN, A complete monotonicity property of the gamma function, *J. Math. Anal. Appl.*, **296** (2004), no. 2, pp. 603–607.

[9] F. QI AND B.-N. GUO, Complete monotonicities of functions involving the gamma and digamma functions, *RGMIA Res. Rep. Coll.*, **7** (2004), no. 1, Art. 8, pp. 63–72. [Online: `http://rgmia.vu.edu.au/v7n1.html`].

[10] F. QI, B.-N. GUO, AND CH.-P. CHEN, Some completely monotonic functions involving the gamma and polygamma functions, *J. Austral. Math. Soc.*, **80** (2006), pp. 81–88.

[11] F. QI, B.-N. GUO, AND CH.-P. CHEN, Some completely monotonic functions involving the gamma and polygamma functions, *RGMIA Res. Rep. Coll.*, **7** (2004), no. 1, Art. 5, pp. 31–36. [Online: `http://rgmia.vu.edu.au/v7n1.html`].

[12] H. VAN HAERINGEN, *Completely Monotonic and Related Functions*, Report 93-108, Faculty of Technical Mathematics and Informatics, Delft University of Technology, Delft, The Netherlands, 1993.