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# ON AN INTEGRAL INEQUALITY OF THE HARDY-TYPE

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ABSTRACT. In this paper, we obtain an integral inequality which extends Shum's and Imoru's generalization of Hardy's Inequality. Our main tool is Imoru's adaptation of Jensen's Inequality for convex functions.

*Key words and phrases:* Integral inequality, Jensen's inequality.

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## 1. INTRODUCTION

In [2], Shum obtained the inequalities

$$\int_0^b x^{-r}F(x)^p dx \ + \ \frac{p}{r-1}b^{1-r}F(b)^p \leq \left(\frac{p}{r-1}\right)^p \int_0^b x^{-r}[\,xf(x)\,]^p dx \ (r>1) \tag{1.1}$$

and

$$\int_a^\infty x^{-r}F(x)^p dx \ + \ \frac{p}{1-r}a^{1-r}F(a)^p \leq \left(\frac{p}{1-r}\right)^p \int_a^\infty x^{-r}[\,xf(x)\,]^p dx, \ (r<1), \tag{1.2}$$

where $p \geq 1$ and

$$F(x) = \begin{cases} \int_x^\infty f(t)dt\ , \ (\ r \ < \ 1\ ) \\ \int_0^x f(t)dt\ , \ (\ r \ > \ 1\ ) \end{cases} \tag{1.3}$$

as a generalization of Hardy's inequality. In [1], Imoru obtained the following generalization of Shum's inequalities:

$$\begin{aligned} &\int_0^b g(x)^{-r}F^p(x)dg(x) \ + \ \frac{p}{r-1}g(b)^{1-r}F^p(b) \\ &\leq \left[\ \frac{p}{r-1}\ \right]^p \int_0^b g(x)^{-r}[\,g(x)f(x)\,]^p dg(x)\ ,(r>1) \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} &\int_a^\infty g(x)^{-r}F^p(x)dg(x) \ + \ \frac{p}{1-r}g(a)^{1-r}F^p(a) \\ &\leq \left[\ \frac{p}{1-r}\ \right]^p \int_a^\infty g(x)^{-r}[\,g(x)f(x)\,]^p dg(x)\ ,(r<1) \end{aligned} \tag{1.5}$$

with both inequalities reversed if $0 < p \leq 1$.
In this paper, we obtain an extension of these inequalities. The result is established mainly by using the Jensen's inequality for convex functions.

## 2. THE MAIN RESULT

The main result of this paper is the following.

**Theorem 2.1.** *Let $g$ be continuous and nondecreasing on $[a,b]$, $0 \leq a < b \leq \infty$, with $g(x) > 0,\ x > 0$. Let $p \geq 1,\ r \neq 1$ and $f(x)$ be nonnegative and Lebesgues-Stieltjes integrable with respect to $g(x)$ on $[a,b]$. Suppose*

$$F_a(x) = \int_a^x f(t)dg(t), \tag{2.1}$$

$$F_b(x) = \int_x^b f(t)dg(t), \tag{2.2}$$

*and*

$$\delta = \frac{1-r}{p},\ r \neq 1. \tag{2.3}$$

*Then*

$$\int_a^b g(x)^{\delta-1}\left[\, g(x)^{-\delta} \;-\; g(a)^{-\delta}\,\right]^{1-p} F_a(x)^p dg(x) + \frac{p}{r-1} g(b)^{\delta}\left[\, g(b)^{-\delta} \;-\; g(a)^{-\delta}\,\right]^{1-p} F_a(b)^p \leq \left[\frac{p}{r-1}\right]^p \int_a^b g(x)^{\delta p-1}\left[g(x)f(x)\right]^p dg(x),\ (r>1) \tag{2.4}$$

*and*

$$\int_a^b g(x)^{\delta-1}\left[\, g(x)^{-\delta} \;-\; g(b)^{-\delta}\,\right]^{1-p} F_b(x)^p dg(x) + \frac{p}{1-r} g(a)^{\delta}\left[g(a)^{-\delta} \;-\; g(b)^{-\delta}\right]^{1-p} F_b(a)^p \leq \left[\,\frac{p}{1-r}\,\right]^p \int_a^b g(x)^{\delta p-1}[\, g(x)f(x)\,]^p dg(x),\ (r<1). \tag{2.5}$$

*The inequalities are reversed if* $0 < p \leq 1$.

Equality holds in either of the two inequalities when either $p = 1$ or $f = 0$. The constant $\left[\,\frac{p}{r-1}\,\right]^p$ or $\left[\,\frac{p}{1-r}\,\right]^p$ is the best possible when the left sides of (2.4) and (2.5) are unchanged. Note that the left sides of (2.4) and (2.5) exist if the right sides do.
The following results are needed for the proof of the Theorem.

**Lemma 2.2.** *Let*

$$\theta_a(x) = \int_a^x g(t)^{-(1-p)(1+\delta)} f^p(t) dg(t)\ , \tag{2.6}$$

$$\theta_b(x) = \int_x^b g(t)^{-(1-p)(1+\delta)} f^p(t) dg(t)\ . \tag{2.7}$$

*Then*

$$g(x)^{\delta}\theta_a(x) \geq (-\delta^{-1})^{1-p} g(x)^{\delta}[\, g(x)^{-\delta} \;-\; g(a)^{-\delta}\,]^{1-p} F_a(x)^p\ ,\ (r>1) \tag{2.8}$$

*and*

$$g(x)^{\delta}\theta_b(x) \geq (-\delta^{-1})^{1-p} g(x)^{\delta}[\, g(b)^{-\delta} \;-\; g(x)^{-\delta}\,]^{1-p} F_b^p(x)\ ,\ (r<1). \tag{2.9}$$

*Proof.* Our main tool is the Jensen's Inequality for convex functions. Let $g$ be continuous and nondecreasing on $[a,b]$ where $0 \leq a < b \leq \infty$. Let $h(x,t)$ be nonnegative, $x \geq 0,\ t \geq 0$ and $\lambda$ be nondecreasing. Then, if $p \geq 1$

$$\int_a^x h(x,t)d\lambda(t) \geq \left[\int_a^x d\lambda(t)\right]^{1-p} \left[\int_a^x h(x,t)^{\frac{1}{p}} d\lambda(t)\right]^p \tag{2.10}$$

and

$$\int_x^b h(x,t)d\,\lambda(t) \geq \left[\int_x^b d\,\lambda(t)\right]^{1-p} \left[\int_x^b h(x,t)^{\frac{1}{p}}\, d\lambda(t)\right]^p. \tag{2.11}$$

So letting

$$h(x,t) = g(x)^{\delta} g(t)^{p(1+\delta)} f(t)^p \tag{2.12}$$

and

$$d\lambda(t) = g(t)^{-(1+\delta)} dg(t) \tag{2.13}$$

we obtain

$$g(x)^{\delta}\theta_a(x) = \int_a^x h(x,t)d\lambda(t) \tag{2.14}$$

and

$$g(x)^{\delta}\theta_b(x) = \int_x^b h(x,t)d\lambda(t) \tag{2.15}$$

whenever $r > 1$ and $r < 1$ respectively. Hence from (2.10), for $r > 1$ we have

$$\begin{aligned} \int_a^x h(x,t)d\lambda(t) &\geq \left[\int_a^x d\lambda(t)\right]^{1-p} \left[\int_a^x h(x,t)^{\frac{1}{p}}\, d\lambda(t)\right]^p \\ &= \left[\int_a^x g(t)^{-(1+\delta)}\, dg(t)\right]^{1-p} \left[\int_a^x g(x)^{\frac{\delta}{p}}\, f(t)dg(t)\right]^p \\ &= (-\delta^{-1})^{1-p}\, g(x)^{\delta}\, [g(x)^{-\delta} - g(a)^{-\delta}]^{1-p} F_a(x) \end{aligned} \tag{2.16}$$

Hence, by the definition of $\theta_a(x)$, the above inequality is inequality (2.8). A similar argument yields inequality (2.9). If we replace $x$ by $b$ in (2.8) and $x$ by $a$ in (2.9) we obtain respectively

$$g(b)^{\delta}\theta_a(b) \geq (-\delta^{-1})^{1-p} g(b)^{\delta} \left[g(b)^{-\delta} - g(a)^{-\delta}\right]^{1-p} F_a(b)^p\ ,\ (\ r > 1\ ) \tag{2.17}$$

and

$$g(a)^{\delta}\theta_b(a) \geq (-\delta^{-1})^{1-p} g(a)^{\delta} \left[g(b)^{-\delta} - g(a)^{-\delta}\right]^{1-p} F_b(a)^p\ ,\ (\ r < 1\ ). \tag{2.18}$$

Multiplying both sides of inequality (2.8) and inequality (2.9) by $g(x)^{-1}dg(x)$, we obtain

$$\begin{aligned} &\int_a^b g(x)^{\delta-1}\theta_a(x)\, dg(x) \\ \geq\ & (-\delta^{-1})^{1-p} \int_a^b g(x)^{\delta-1} \left[g(x)^{-\delta} - g(a)^{-\delta}\right]^{1-p} F_a(x)^p dg(x)\ ,\ (\ r > 1\ ) \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &\int_a^b g(x)^{\delta-1}\theta_b(x)\, dg(x) \\ \geq\ & (-\delta^{-1})^{1-p} \int_a^b g(x)^{\delta-1} \left[g(b)^{-\delta} - g(x)^{-\delta}\right]^{1-p} F_b(x)^p\ ,\ (\ r < 1\ ). \end{aligned} \tag{2.20}$$

∎

We shall now give the proof of Theorem 2.1.

*Proof.* From the definitions of $\theta_a(x)$ and $\theta_b(x)$, we have

$$\lim_{x\to a} g(x)^{\delta}\theta_a(x) = 0 \tag{2.21}$$

and

$$\lim_{x\to b} g(x)^{\delta}\theta_b(x) = 0. \tag{2.22}$$

Thus, integrating the left side of (2.19) and the left side of (2.20), by parts we obtain

$$\begin{aligned} \int_a^b g(x)^{\delta-1}\theta_a(x)\, dg(x) &= \delta^{-1} g(b)^{\delta}\theta_a(b) \\ -\delta^{-1} \int_a^b g(x)^{\delta p-1}\, [g(x)f(x)]^p\, dg(x)\ &,\ (\ r > 1\ ) \end{aligned} \tag{2.23}$$

and

$$\int_a^b g(x)^{\delta-1}\theta_b(x)\, dg(x) = -\delta^{-1}g(a)^\delta\theta_b(a) + \delta^{-1}\int_a^b g(x)^{\delta p-1}\,[g(x)f(x)]^p\, dg(x)\,,\ (\ r<1\ ). \tag{2.24}$$

In the case when $r>1$, multiply the inequality (2.17) through by $\delta^{-1}$ and use this in (2.23) to obtain

$$\int_a^b g(x)^{\delta-1}\theta_a(x)\, dg(x) \le -(-\delta^{-1})^{2-p}g(b)^\delta \times \left[g(b)^{-\delta} - g(a)^{-\delta}\right]^{1-p} F_a(b)^p\delta^{-1}\int_a^b g(x)^{\delta p-1}[g(x)f(x)]^p\, dg(x). \tag{2.25}$$

Now combine (2.25) with (2.19) to yield the assertion (2.4) of the theorem.
When $r<1$, multiply inequality (2.18) through by $-\delta^{-1}$ and use the result in (2.24) to obtain

$$\int_a^b g(x)^{\delta-1}\theta_b(x)dg(x) \le (-\delta^{-1})^{2-p}g(a)^\delta[g(b)^{-\delta} - g(a)^{-\delta}]^{1-p}F_b(a) +\delta^{-1}\int_a^b g(x)^{\delta p-1}[g(x)f(x)]^p dg(x). \tag{2.26}$$

Now combine (2.26) with (2.20) to yield the assertion (2.5) of the theorem. ∎

**Remark 2.1.** If

$$\lim_{a\to 0} g(a) = 0 \tag{2.27}$$

and

$$\lim_{b\to\infty} g(b) = \infty, \tag{2.28}$$

then (2.4) and (2.5) reduce to (1.4) and (1.5) respectively.

## References


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