The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 4, Issue 2, Article 1, pp. 1-7, 2007

A DIFFERENT PROOF FOR THE NON-EXISTENCE OF HILBERT-SCHMIDT HANKEL OPERATORS WITH ANTI-HOLOMORPHIC SYMBOLS ON THE BERGMAN SPACE

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Received 28 May, 2006; accepted 06 February, 2007; published 30 July, 2007.

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ABSTRACT. We show that there are no (non-trivial) Hilbert-Schmidt Hankel operators with anti-holomorphic symbols on the Bergman space of the unit-ball $B^2(\mathbb{B}^l)$ for $l \geq 2$. The result dates back to [6]. However, we give a different proof. The methodology can be easily applied to other more general settings. Especially, as indicated in the section containing generalizations, the new methodology allows to prove some robustness results for existing ones.

Key words and phrases: Hankel operator, Weighted Bergman spaces, Bergman projection.

2000 *Mathematics Subject Classification.* Primary: 32A36, Secondary: 47B35, 32A15.

ISSN (electronic): 1449-5910

1. INTRODUCTION

The Bergman space $B^2(\Omega)$ of the domain Ω in $\mathbb{C}^l$ is defined as

$$B^2(\Omega) := \left\{ f : \quad \text{holomorphic in } \Omega \text{ and } \int_\Omega |f(z)|^2 \, d\lambda(z) < \infty \right\}, \tag{1.1}$$

where λ denotes the Lebesgue-measure in $\mathbb{C}^l$. Remember, that the Hankel operator with symbol g is given by

$$H_g(f) : B^2(\Omega) \longrightarrow L^2(\Omega) : H_g(f) = (I - P)(gf),$$

where

$$L^2(\Omega) := \left\{ f : \quad \text{measurable in } \Omega \text{ and } \int_\Omega |f(z)|^2 \, d\lambda(z) < \infty \right\} \tag{1.2}$$

and P is the orthogonal projection onto $B^2(\Omega)$ (the Bergman projection). In the following we will restrict our attention to the unit-ball $\mathbb{B}^l$ in $\mathbb{C}^l$.

We define

$$c_n^2 = \int_{\mathbb{B}^l} |z^n|^2 \, d\lambda(z). \tag{1.3}$$

Here $n = (n_1, \ldots, n_l)$ is a multi-index. Note, that the set

$$\left\{ \frac{z^n}{c_n}; n \in \mathbb{N}^l \right\} \tag{1.4}$$

is a complete orthonormal-system of $B^2(\mathbb{B}^l)$.

The aim of this paper is to give a new proof for the fact that there are no (non-trivial) Hankel operators with anti-holomorphic symbols $\overline{g}$ if $l \geq 2$. Such a symbol can be written as $\overline{g}(z) = \sum_j a_j \overline{z}^j$, where the summation is over all possible multi-indices $j = (j_1, \ldots, j_l)$. The following section reviews some related literature and Section 3 gives the new proof. Section 4 considers some generalizations. Especially, it is shown that the new methodology allows to prove robustness of the mentioned result. This shows the usefulness of the new approach. In addition some open problems are indicated.

2. RELATED LITERATURE

In the following section we will prove that there are no (non-trivial) Hilbert-Schmidt Hankel operators with anti-holomorphic symbols on the Bergman space of the unit-ball $B^2(\mathbb{B}^l)$ for each $l \in \mathbb{N}$ and $l \geq 2$. The first proof of this result is due to [6]. In the following years there has been quite a lot of work in this field. In [2] it is shown that on the weighted spaces $B^2_\alpha(\mathbb{B}^l)$ the operators H_f and $H_{\overline{f}}$ are in the Schatten-p-class S_p if and only if $MO(f) \in L^p(\mathbb{B}^l)$ for $2 \leq p < \infty$. (This extends the work of [7]. Here the special case $\alpha = 0$ is considered.) Some of the mentioned definitions and concepts need to be explained. First,

$$B^2_\alpha(\mathbb{B}^l) := \left\{ f : f \text{is holomorphic in } \mathbb{B}^l \text{ and } \int_{\mathbb{B}^l} |f(z)|^2 \, d\mu_\alpha(z) < \infty \right\}. \tag{2.1}$$

Here $d\mu_\alpha(z) = (1 - |z|^2)^\alpha d\lambda(z)$. The mean oscillation is given by

$$MO(f)(z) = (\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2)^{1/2} \tag{2.2}$$

and $\tilde{f}$ is the Berezin transform of f given by $\tilde{f}(z) = (fk_z, k_z)$. (Here k_z is the normalized reproducing kernel of $B^2_\alpha(\mathbb{B}^l)$). There have also been results in this context for the case $1 \leq p < 2$.

For more on this see (for the special case $\alpha = 0$) [5]. Further results in this context for the spaces $B^2_\alpha(\mathbb{B}^l)$ can be found in [3].

In this work, we want to take a different approach than the mentioned papers. It is more functional analytical and the crucial results only depend on the Hilbert space structure of the space $B^2_\alpha(\mathbb{B}^l)$ (as will be indicated in the following). The used approach has been (partially) developed in [4] and [1]. To indicate, how it can be used to (easily) yield some results, we will prove that there are no non-trivial Hilbert-Schmidt Hankel operators with anti-holomorphic symbols in the following section. For some possible generalizations see Section 4.

3. NON-EXISTENCE OF HILBERT-SCHMIDT HANKEL OPERATORS ON THE BERGMAN SPACE

In the following proposition we show that there are no non-trivial Hilbert-Schmidt Hankel operators with anti-holomorphic symbols.

Proposition 3.1. *There are no non-trivial Hilbert-Schmidt Hankel operators with anti-holomorphic symbols on the Bergman space of the unit-ball $B^2(\mathbb{B}^l)$ for each $l \in \mathbb{N}$ and $l \geq 2$. That is, the only Hilbert-Schmidt Hankel operators with anti-holomorphic symbols have constant symbols.*

Remark 3.1. A Hankel operator with constant symbol satisfies

$$H_c(f) = c(f - f) = 0 \qquad \forall f \in B^2(\mathbb{B}^l). \tag{3.1}$$

As in previous work (concerning Hankel operators on generalized Fock-spaces) the limiting behavior of the sequence $\left\{c^2_{n+k}/c^2_n - c^2_n/c^2_{n-k}\right\}_{n\in\mathbb{N}^l}$ will play an important role. The following lemma describes the limiting behavior of the sequence $\left\{c^2_{n+k}/c^2_n - c^2_n/c^2_{n-k}\right\}_{n\in\mathbb{N}^l}$. For simplicity, we only consider the case $l = 2$.

Lemma 3.2. *Let $n = (n_1, n_2)$ and $k = (k_1, k_2) \neq (0, 0)$. We have*

$$\frac{c^2_{n+k}}{c^2_n} - \frac{c^2_n}{c^2_{n-k}} \approx C \frac{n_1^{k_1} n_2^{k_2}}{(n_1 + n_2)^{k_1+k_2}} E(n_1, n_2). \tag{3.2}$$

Here, we abbreviated $E(n_1, n_2) := \left(C_1 \frac{1}{n_1} + C_2 \frac{1}{n_2}\right)$.

Proof. Direct calculation shows (for $n = (n_1, n_2)$) that (see also [8])

$$c^2_n = D \frac{n_1!\, n_2!}{(n_1 + n_2 + 2)!}, \tag{3.3}$$

where D is a constant. Therefore, we have

$$\begin{aligned} &\frac{c^2_{n+k}}{c^2_n} - \frac{c^2_n}{c^2_{n-k}} \\ &= \frac{(n_1 + k_1)!(n_2 + k_2)!(n_1 + n_2 + 2)!}{(n_1)!(n_2)!(n_1 + n_2 + k_1 + k_2 + 2)!} - \frac{(n_1)!(n_2)!(n_1 + n_2 - k_1 - k_2 + 2)!}{(n_1 - k_1)!(n_2 - k_2)!(n_1 + n_2 + 2)!} \\ &\approx C \frac{n_1^{k_1} n_2^{k_2}}{(n_1 + n_2)^{k_1+k_2}} \left(C_1 \frac{1}{n_1} + C_2 \frac{1}{n_2}\right) \\ &:= C \frac{n_1^{k_1} n_2^{k_2}}{(n_1 + n_2)^{k_1+k_2}} E(n_1, n_2). \end{aligned} \tag{3.4}$$

This finishes the proof. ∎

Before we start the proof of Proposition 3.1, we prove the following proposition. It is the main ingredient of the proof of Proposition 3.1 and reflects the functional analytic nature of the approach.

Proposition 3.3. *Assume that the Hankel operator $H_{\overline{f}}$ is Hilbert-Schmidt, where $\overline{f} = \sum_k b_k \overline{z}^k$. Then all Hankel operators $H_{\overline{z}^k}$, where k satisfies $b_k \neq 0$, have to be Hilbert-Schmidt.*

Proof. For simplicity, we prove the result for $l = 2$. It can be shown that (see [1])

$$\left\| H_{\overline{f}}\left(\frac{z^n}{c_n}\right)\right\|^2 = \sum_{k\leq n} |b_k|^2 \left[\frac{c_{n+k}^2}{c_n^2} - \frac{c_n^2}{c_{n-k}^2}\right] + \sum_{k\nleq n} |b_k|^2 \frac{c_{n+k}^2}{c_n^2}, \tag{3.5}$$

where $\overline{f} = \sum_k b_k \overline{z}^k$. Here $k = (k_1, k_2)$ is a multi-index and $k \leq n$ means $k_1 \leq n_1$ and $k_2 \leq n_2$. (Summation of multi-indices will also be defined component-wise.) Especially, we have for $\overline{f} = \overline{z}^k$ that

$$\left\| H_{\overline{f}}\left(\frac{z^n}{c_n}\right)\right\|^2 = \frac{c_{n+k}^2}{c_n^2} \tag{3.6}$$

if $k \nleq n$ and

$$\left\| H_{\overline{f}}\left(\frac{z^n}{c_n}\right)\right\|^2 = \frac{c_{n+k}^2}{c_n^2} - \frac{c_n^2}{c_{n-k}^2} \tag{3.7}$$

if $k \leq n$. The operator $H_{\overline{f}}$ is Hilbert-Schmidt if and only if

$$\sum_n \left\| H_{\overline{f}}\left(\frac{z^n}{c_n}\right)\right\|^2 < \infty. \tag{3.8}$$

Therefore, if $H_{\overline{f}}$ is Hilbert-Schmidt for $\overline{f} = \sum_k b_k \overline{z}^k$ then $H_{\overline{z}^k}$ must be Hilbert-Schmidt for all k with $b_k \neq 0$. ∎

Remark 3.2. Note, that equation 3.5 reflects the functional analytic nature of the approach. It can be verified by direct calculation using the expansion of $\overline{f}$ and rewriting the norm in terms of the corresponding inner product. Furthermore, one has to make use of the fact, that the holomorphic functions z^n are pairwise orthogonal.

Now we give a proof for Proposition 3.1.

Proof. (**Proposition 3.1**) Remember, that it follows from Proposition 3.3 that if $H_{\overline{f}}$ is Hilbert-Schmidt for $\overline{f} = \sum_k b_k \overline{z}^k$ then $H_{\overline{z}^k}$ must be Hilbert-Schmidt for all k with $b_k \neq 0$. Therefore, it is enough to show that $H_{\overline{z}^k}$ is not Hilbert-Schmidt for all $k \neq 0$. Direct calculation shows that (see also [8])

$$c_n^2 = \pi^2 \frac{n_1!\, n_2!}{(n_1 + n_2 + 2)!}. \tag{3.9}$$

If $H_{\overline{z}^k}$ were Hilbert-Schmidt, then we would (at least) have to have

$$\sum_{n_1,n_2} \frac{c_{n+k}^2}{c_n^2} - \frac{c_n^2}{c_{n-k}^2} < \infty. \tag{3.10}$$

However

$$\frac{c_{n+k}^2}{c_n^2} - \frac{c_n^2}{c_{n-k}} \approx C \frac{n_1^{k_1} n_2^{k_2}}{(n_1+n_2)^{k_1+k_2}} \left(C_1 \frac{1}{n_1} + C_2 \frac{1}{n_2} \right) \tag{3.11}$$
$$:= C \frac{n_1^{k_1} n_2^{k_2}}{(n_1+n_2)^{k_1+k_2}} E(n_1, n_2).$$

Clearly, we have

$$\begin{aligned} \sum_{n_1,n_2} \frac{c_{n+k}^2}{c_n^2} - \frac{c_n^2}{c_{n-k}^2} &= \sum_{n_1,n_2} C \frac{n_1^{k_1} n_2^{k_2}}{(n_1+n_2)^{k_1+k_2}} E(n_1,n_2) \\ &\geq C' \sum_{n_1,n_2} \frac{n_1^{k_1} n_2^{k_2}}{(n_1+n_2)^{k_1+k_2+1}} \\ &= \sum_{n=0}^{\infty} \sum_{\substack{n_1+n_2=n \\ n_1,n_2\geq 0}} \frac{n_1^{k_1} n_2^{k_2}}{(n_1+n_2)^{k_1+k_2+1}} \\ &= \sum_{n=0}^{\infty} \sum_{n_1=0}^{n} \frac{n_1^{k_1}(n-n_1)^{k_2}}{n^{k_1+k_2+1}}. \end{aligned} \tag{3.12}$$

In addition,

$$\sum_{n_1=0}^{n} n_1^{k_1}(n-n_1)k_2 \geq \left[\frac{n}{2}\right]^{k_1} \sum_{n_1=\left[\frac{n}{2}\right]}^{n} (n-n_1)^{k_2} \approx C'' n^{k_1+k_2+1}. \tag{3.13}$$

Here $[x]$ denotes the largest integer smaller or equal to x. The above calculation follows from Euler's summation formula

$$f(0) + f(1) + \ldots + f(n) = \int_0^n f(x)\,dx + \frac{f(0)+f(n)}{2} + \int_0^n B_1(x) f'(x)\,dx, \tag{3.14}$$

where $B_1(x)$ is the first Bernoulli polynomial. That is $B_1(x) = x - \frac{1}{2}$ for $x \in [0,1]$ and $B_1(x+k) = B_1(x)\ \forall k \in \mathbb{Z}$. Therefore, the above sum cannot converge. ∎

Remark 3.3. The same methodology applies if we replace the Bergman Space $B^2(\mathbb{B}^2)$ by the following weighted Bergman spaces

$$B_\alpha^2(\mathbb{B}^2) := \left\{ f : f \text{ is holomorphic in } \mathbb{B}^2 \text{ and } \int_{\mathbb{B}^2} |f(z_1,z_2)|^2\, d\mu_\alpha(z_1,z_2) < \infty \right\}. \tag{3.15}$$

Here $d\mu_\alpha(z_1,z_2) = (1 - |z_1|^2 - |z_2|^2)^\alpha d\lambda(z_1,z_2)$. In this case the moments $c_{n,\alpha}^2$ are given by the formula

$$c_{n,\alpha}^2 = D \frac{\Gamma(n_1+1)\Gamma(n_2+1)}{\Gamma(n_1+n_2+\alpha+3)}, \tag{3.16}$$

where D is a constant and $n = (n_1, n_2)$.

4. GENERALIZATIONS

In this section we want to emphasize the advantage of the approach, which stems from the fact that - as mentioned above - the proof is a functional analytic one. Concretely, only some Hilbert space properties of the Bergman space have been used. As an example, we want to consider certain radial-symmetric perturbations of the weight-functions $(1-|z_1|^2-|z_2|^2)^\alpha$. Concretely, consider some $0<R<1$ and let $\rho_\alpha : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a continuous function with $\rho_\alpha(r) = (1-r_1^2-r_2^2)^\alpha$, where $r = |(r_1, r_2)|$ for $r > R$. We call the corresponding spaces perturbed Bergman spaces $B^2_{\rho_\alpha}(\mathbb{B}^2)$. Concretely, they are defined as follows:

$$B^2_{\rho_\alpha}(\mathbb{B}^2) := \left\{ f : f \text{ is holomorphic in } \mathbb{B}^2 \text{ and } \int_{\mathbb{B}^2} |f(z_1,z_2)|^2 \, d\rho_\alpha(z_1,z_2) < \infty \right\}. \tag{4.1}$$

Here $d\rho_\alpha(z_1,z_2) = \rho_\alpha(|(z_1,z_2)|)d\lambda(z_1,z_2)$. The following proposition holds and can be seen as a robustness result. For simplicity, we only consider the special case $\alpha = 0$.

Proposition 4.1. *There are no non-trivial Hilbert-Schmidt Hankel operators with anti-holomorphic symbols on the perturbed Bergman spaces of the unit-ball $B^2_{\rho_\alpha}(\mathbb{B}^l)$ for each $l \in \mathbb{N}$ and $l \geq 2$. That is, the only Hilbert-Schmidt Hankel operators with anti-holomorphic symbols have constant symbols.*

Proof. For sake of simplicity, we only explicitly consider the special case $\alpha = 0$ and $l = 2$. To show the result, we first note that an analogue to Proposition 3.3 holds. Inspection of the proof of Proposition 3.1 shows that it is enough to show that the limiting behavior of the moments

$$c^2_{n,\rho_\alpha} = \int_{\mathbb{B}^2} |z^n|^2 \, d\rho_\alpha(z), \tag{4.2}$$

is the same as the one of the moments c_n^2. (For a definition of c_n^2 see the introduction.) For each $0 < R < 1$ we consider ($\mathbb{B}^2_R := \{z | R < |z| < 1\}$)

$$1 \leq \frac{\int_{\mathbb{B}^2} |z^n|^2 \, d\mu_0(z)}{\int_{\mathbb{B}^2_R} |z^n|^2 \, d\mu_0(z)} = \frac{\int_{\mathbb{B}^2 - \mathbb{B}^2_R} |z^n|^2 \, d\mu_0(z) + \int_{\mathbb{B}^2_R} |z^n|^2 \, d\mu_0(z)}{\int_{\mathbb{B}^2_R} |z^n|^2 \, d\mu_0(z)} \tag{4.3}$$

and want to show that the above quotient converges to 1 if $|n| = n_1 + n_2 \to \infty$. Using polar coordinates we see ($\mathbb{B}^2_{R,L} := \{z | R < |z| < L\}$)

$$\begin{aligned} \int_{\mathbb{B}^2_{R,L}} |z^n|^2 \, d\mu_0(z) &= C \int_R^L r^{2|n|+3} \, d\lambda(r) \int_{\mathbb{S}_n} |\zeta^n|^2 \, d\sigma(\zeta) \\ &= C \int_R^L r^{2|n|+3} \, d\lambda(r) \frac{n_1! n_2!}{(|n|+1)!}. \end{aligned} \tag{4.4}$$

Consequently ($R' > R$)

$$\begin{aligned} \frac{\int_{\mathbb{B}^2 - \mathbb{B}^2_R} |z^n|^2 \, d\mu_0(z)}{\int_{\mathbb{B}^2_R} |z^n|^2 \, d\mu_0(z)} &\leq \frac{\int_{\mathbb{B}^2 - \mathbb{B}^2_R} |z^n|^2 \, d\mu_0(z)}{\int_{\mathbb{B}^2_{R'}} |z^n|^2 \, d\mu_0(z)} \\ &= \frac{\int_0^R r^{2|n|+3} \, d\lambda(r)}{\int_{R'}^1 r^{2|n|+3} \, d\lambda(r)} \\ &\leq \frac{R}{1-R'} \frac{R^{2|n|+3}}{R'^{2|n|+3}} \to 0 \end{aligned} \tag{4.5}$$

as $|n| \to 0$ and therefore

$$\frac{\int_{\mathbb{B}^2} |z^n|^2 \, d\mu_0(z)}{\int_{\mathbb{B}^2_R} |z^n|^2 \, d\mu_0(z)} \to 1 \tag{4.6}$$

as $|n| \to 0$. A similar argument is valid for the quotients

$$\frac{\int_{\mathbb{B}^2} |z^n|^2 \, d\rho_0(z)}{\int_{\mathbb{B}^2_R} |z^n|^2 \, d\rho_0(z)}, \tag{4.7}$$

since ρ_0 must be bounded. However, for suitable values of R

$$\int_{\mathbb{B}^2_R} |z^n|^2 \, d\rho_0(z) = \int_{\mathbb{B}^2_R} |z^n|^2 \, d\mu_0(z) \tag{4.8}$$

and consequently the limiting behavior of the moments c^2_{n,ρ_0} is the same as the one of the moments c^2_n. ∎

Remark 4.1. In the case $\alpha > 0$ one has to be a little more precise with the estimates since $(1 - |z|)^\alpha$ vanishes at the boundary of the unit-ball.

Remark 4.2. As mentioned above, Proposition 4.1 can be seen as a robustness result for Proposition 3.1. However, there are some open research questions.

(1) In connection with some of the literature presented in Section 2 it would be interesting to investigate if the presented methodology can be adopted in order to investigate Schatten-class membership of Hankel operators with anti-holomorphic functions.
(2) It would be of interest if the functional analytic approach can also (after some modification) be applied to general L^2-symbols.
(3) Some different spaces of holomorphic functions could be considered. As mentioned above, there are already some existing results for generalized Fock spaces. (See [4] and [1].)

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