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1-TYPE PSEUDO-CHEBYSHEV SUBSPACES IN GENERALIZED 2-NORMED SPACES

SH. REZAPOUR

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DEPARTMENT OF MATHEMATICS, AZARBAIJAN UNIVERSITY OF TARBIAT MOALLEM, AZARSHAHR, TABRIZ, IRAN.
sh.rezapour@azaruniv.edu
URL: http://www.azaruniv.edu/~rezapour

ABSTRACT. We construct a generalized 2-normed space from every normed space. We introduce 1-type pseudo-Chebyshev subspaces in generalized 2-normed spaces and give some results in this field.

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1. INTRODUCTION

The concept of linear 2-normed spaces has been investigated by G$\ddot{a}$hler in 1965 ([3]) and has been developed extensively in different subjects by others. Lewandowska published a series of papers on 2-normed sets and generalized 2-normed spaces in 1999-2003 ([5]-[9]). There are some works on characterization of 2-normed spaces, extension of 2-functionals and approximation in 2-normed spaces ([1], [2] and [4]). Also, there are some works in approximation theory (for example, [10]-[12]).

Let X be a linear space of dimension greater than 1 over K, where K is the real or complex numbers field. Suppose $\|.,.\|$ be a non-negative real-valued function on $X \times X$ satisfying the following conditions:

(i) $\|x,y\| = 0$ if and only if x and y are linearly dependent vectors.

(ii) $\|x,y\| = \|y,x\|$ for all $x, y \in X$.

(iii) $\|\lambda x, y\| = |\lambda| \, \|x,y\|$ for all $\lambda \in K$ and all $x, y \in X$.

(iv) $\|x+y, z\| \leq \|x,z\| + \|y,z\|$ for all $x, y, z \in X$.

Then $\|.,.\|$ is called a 2-norm on X and $(X, \|.,.\|)$ is called a linear 2-normed space.

Every 2-normed space is a locally convex topological vector space. In fact for a fixed $b \in X$, $p_b(x) = \|x, b\|$, $x \in X$, is a seminorm and the family $P = \{p_b : b \in X\}$ of seminorms generates a locally convex topology on X. But, there are no remarkable relations between normed spaces and 2-normed spaces.

We couldn't construct any 2-norm on X by a normed space $(X, \|.\|)$, and this could be a motive for definition of generalized 2-normed spaces.

Definition 1.1. ([5]-[7]) Let X and Y be linear spaces, D be a non-empty subset of $X \times Y$ such that for every $x \in X$, $y \in Y$ the sets

$$D_x = \{y \in Y : (x,y) \in D\}, D^y = \{x \in X : (x,y) \in D\}$$

are linear subspaces of the spaces Y and X, respectively. A function $\|.,.\| : D \longrightarrow [0, \infty)$ is called a generalized 2-norm on D if it satisfies the following conditions:

(N_1) $\|x, \alpha y\| = |\alpha| \|x,y\| = \|\alpha x, y\|$, for all $(x,y) \in D$ and every scalar α.
(N_2) $\|x, y+z\| \leq \|x,y\| + \|x,z\|$, for all $(x,y), (x,z) \in D$.
(N_3) $\|x+y, z\| \leq \|x,z\| + \|y,z\|$, for all $(x,z), (y,z) \in D$.

Then, $(D, \|.,.\|)$ is called a 2-normed set. In particular, if $D = X \times Y$, $(X \times Y, \|.,.\|)$ is called a generalized 2-normed space. Moreover, if $X = Y$, then the generalized 2-normed space is denoted by $(X, \|.,.\|)$.

Definition 1.2. ([5]-[7]) Let X be a linear space, χ be a non-empty subset of $X \times X$ such that $\chi = \chi^{-1}$ and the set $\chi^y = \{x \in X : (x,y) \in \chi\}$ is a linear subspace of X, for all $y \in X$. A function $\|.,.\| : \chi \longrightarrow [0, \infty)$ is called a generalized symmetric 2-norm on χ if it satisfies the following conditions:

(S_1) $\|x,y\| = \|y,x\|$, for all $(x,y) \in \chi$.
(S_2) $\|x, \alpha y\| = |\alpha| \|x,y\|$, for all $(x,y) \in \chi$ and every scalar α.
(S_3) $\|x+y, z\| \leq \|x,z\| + \|y,z\|$, for all $(x,y), (x,z) \in \chi$.

Then, $(\chi, \|.,.\|)$ is called a generalized symmetric 2-normed set. In particular, if $\chi = X \times X$, the function $\|.,.\|$ is called a generalized symmetric 2-norm on χ and $(X, \|.,.\|)$ is called a generalized symmetric 2-normed space.

Definition 1.3. ([5]) Let $(X \times Y, \|.,.\|)$ be a generalized 2-normed space.

(a) The family β of all sets defined by $\bigcap_{i=1}^{n}\{x \in X : \|x, y_i\| < \varepsilon\}$, where $n \in \mathbb{N}$, $y_1, ..., y_n \in Y$ and $\varepsilon > 0$, forms a complete system of neighborhoods of zero for a locally convex topology in Y.

(b) The family β of all sets defined by $\bigcap_{i=1}^{n}\{y \in Y : \|x_i, y\| < \varepsilon\}$, where $n \in \mathbb{N}$, $x_1, ..., x_n \in X$ and $\varepsilon > 0$, forms a complete system of neighborhoods of zero for a locally convex topology in X.

We will denote the above topologies by the symbols $\tau(X, Y)$ and $\tau(Y, X)$, respectively. In the case when $X = Y$, we will denote these topologies by $\tau_1(X) = \tau(X, Y)$ and $\tau_2(X) = \tau(Y, X)$.

Let us consider the linear spaces X and Y and let $D \subseteq X \times Y$ be a 2-normed set and Z be a normed space. A map $f : D \longrightarrow Z$ is called 2-linear if it satisfies the following conditions:
(i) $f(x_1+x_2, y_1+y_2) = f(x_1, y_1)+f(x_1, y_2)+f(x_2, y_1)+f(x_2, y_2)$, for all $x_1, x_2, y_1, y_2 \in X$ such that $x_1, x_2 \in D^{y_1} \bigcap D^{y_2}$,
(ii) $f(\delta x, \lambda y) = \delta\lambda f(x, y)$, for all scalars δ, λ and all $(x, y) \in D$.

A 2-linear map f is said to be bounded if there exists a non-negative real number M such that $\|f(x, y)\| \leq M\|x, y\|$ for all $(x, y) \in D$. Also, the norm of a 2-linear map f is defined by

$$\|f\| = \inf\{M \geq 0 : \|f(x, y)\| \leq M\|x, y\| \; for \; all \; (x, y) \in D\}.$$

We denote by $< b >$ the subspace of linear space X genereted by the element $b \in X$. For a generalized 2-normed space $(X \times Y, \|.,.\|)$, a subspace W of X and $b \in Y$, we denote by $W_b^{\sharp}$ the Banach space of all K-valued bounded 2-linear maps on $W\times < b >$.

Let $(X \times Y, \|.,.\|)$ be a generalized 2-normed space, W be a subspace of X and $b \in Y$.

(i) $w_0 \in W$ is called b-best approximation of $x \in X$ in W, if

$$\|x - w_0, b\| = \inf\{\|x - w, b\| : \; w \in W\}.$$

The set of all b-best approximations of x in W is denoted by $P_W^b(x)$.

(ii) W is called b-proximinal if for every $x \in X\backslash(\overline{W}\backslash W)$, there exists $w_0 \in W$ such that $\|x - w_0, b\| = \inf\{\|x - w, b\| : w \in W\}$, where $\overline{W}$ denotes the closure of W in the seminormed space (X, p_b).

Note that, W is b-proximinal if and only if $P_W^b(x) \neq \emptyset$ for all $x \in X\backslash\overline{W}$.

The following basic lemma is important in the proof of main results.

Proposition 1.1 ([3]; Theorem 3.6)**.** . *Let $(X, \|.,.\|)$ be a 2-normed space, W be a subspace of X and $b \in X$. If $x_0 \in X$ is such that*

$$\delta = \inf\{\|x_0 - w, b\| : w \in W\} > 0,$$

then there exists a bounded 2-linear map $F : X\times < b >\longrightarrow K$ such that $F|_{W\times<b>} = 0$, $F(x_0, b) = 1$ and $\|F\| = \frac{1}{\delta}$.

By review of [3], we find that the following similar Lemma holds for generalized 2-normed spaces.

Lemma 1.2. *Let $(X \times Y, \|.,.\|)$ be a generalized 2-normed space, W be a subspace of X and $b \in Y$. If $x_0 \in X$ is such that*

$$\delta = \inf\{\|x_0 - w, b\| : w \in W\} > 0,$$

then there exists a bounded 2-linear map $F : X \times < b > \longrightarrow K$ such that $F|_{W\times<b>} = 0$, $F(x_0, b) = 1$ and $\|F\| = \frac{1}{\delta}$.

Lemma 1.3. *Let $(X \times Y, \|.,.\|)$ be a generalized 2-normed space, W be a subspace of X, $b \in Y$ and $x \in X\backslash\overline{W}$, where $\overline{W}$ denotes the closure of W in the seminormed space (X, p_b). Then, $M \subseteq P^b_W(x)$ if and only if there exists $f \in X^\sharp_b$ such that $f|_{W\times<b>} = 0$, $\|f\| = 1$ and $f(x_0 - m, b) = \|x_0 - m, b\|$ for all $m \in M$.*

Proof. First suppose that there exists $f \in X^\sharp_b$ such that $f|_{W\times<b>} = 0$, $\|f\| = 1$ and $f(x_0 - m, b) = \|x_0 - m, b\|$ for all $m \in M$. Then,

$$\|x_0 - m, b\| = f(x_0 - m, b) = f(x_0, b) = f(x_0 - w, b)$$

$$\leq \|f\| \, \|x_0 - w, b\| = \|x_0 - w, b\|,$$

for all $m \in M$ and all $w \in W$. Hence, $m \in P^b_W(x_0)$ for all $m \in M$. Conversely, fix $m_0 \in M$. Then,

$$\delta = \|x_0 - m_0, b\| = \inf\{\|x_0 - w, b\| : w \in W\} > 0.$$

By Lemma 1.2, there exists $g \in X^\sharp_b$ such that $g|_{W\times<b>} = 0$, $g(x_0, b) = 1$ and $\|g\| = \frac{1}{\delta}$. Now for $f = \delta g$ we have, $f|_{W\times<b>} = 0$, $f(x_0 - m_0, b) = \|x_0 - m_0, b\|$ and $\|f\| = 1$. Note that, $f(x_0 - m, b) = \|x_0 - m_0, b\| = \|x_0 - m, b\|$ for all $m \in M$. ∎

2. 1-TYPE PSEUDO-CHEBYSHEV SUBSPACES

Definition 2.1. Let $(X \times Y, \|.,.\|)$ be a generalized 2-normed space, W be a subspace of X and $b \in Y$.

(i) W is called b-pseudo Chebyshev if for every $x \in X\backslash\overline{W}$, where $\overline{W}$ denotes the closure of W in the seminormed space (X, p_b), $P^b_W(x)$ is non-empty and finite dimensional.

(ii) W is called 1-type pseudo-Chebyshev if W is b-pseudo Chebyshev for every $0 \neq b \in Y$.

Example 2.1. *Let $X = \mathbb{R}^3$, $W = \{(x, y, 0) : x, y \in \mathbb{R}\}$ and*

$$\|(x_1, x_2, x_3), (y_1, y_2, y_3)\| =$$

$$max\{|x_1y_2 - x_2y_1| + |x_1y_3 - x_3y_1|, |x_1y_2 - x_32y_1| + |x_2y_3 - x_3y_2|\}$$

for all $(x_1, x_2, x_3), (y_1, y_2, y_3) \in X$. Then, $\|.,.\|$ is a 2-norm on X and W is 1-type pseudo-Chebyshev subspace.

Example 2.2. *Let W be a pseudo-Chebyshev subspace of a normed space $(X, \|.\|_1)$ and let $(Y, \|.\|_2)$ be an arbitrary normed space. Then, $\|x, y\| = \|x\|_1\|y\|_2$ is a generalized 2-norm on $X \times Y$ and W is 1-type pseudo-Chebyshev subspace.*

Proposition 2.1. *Let $(X \times Y, \|.,.\|)$ be a generalized 2-normed space, W be a subspace of X and $b \in Y$. Then, W is b-pseudo Chebyshev subspace of X if and only if there do not exist $f \in X^\sharp_b$, $x_0 \in X\backslash\overline{W}$, where $\overline{W}$ denotes the closure of W in the seminormed space (X, p_b), and infinitely many linearly independent elements $w_1, w_2, ...$ in W such that $f|_{W\times<b>} = 0$, $\|f\| = 1$ and $f(x_0 - w_n, b) = \|x_0 - w_n, b\|$, for all $n \geq 1$.*

Proof. Suppose that W is not b-pseudo Chebyshev subspace. Then, there exists $x \in X \backslash \overline{W}$, such that $P_W^b(x)$ is not finite dimensional. Fix $w_0 \in P_W^b(x)$. Then, there exist infinitely many elements $w_1, w_2, ...$ in $P_W^b(x)$ such that $w_0 - w_1, w_0 - w_2, ...$ are infinitely many linearly independent elements of W. Put $x_0 = x - w_0$ and $g_n = w_n - w_0$ for all $n \geq 1$ and note that, $g_1, g_2, ...$ are infinitely many linearly independent elements of $P_W^b(x_0)$. By Lemma 1.3, there exists $f \in X_b^\sharp$ such that $f|_{W \times <b>} = 0$, $\|f\| = 1$ and $f(x_0 - g_n, b) = \|x_0 - g_n, b\|$ for all $n \geq 1$. This is a contradiction. ∎

Corollary 2.2. *Let $(X \times Y, \|., .\|)$ be a generalized 2-normed space and let W be a subspace of X. Then, W is 1-type pseudo-Chebyshev subspace if and only if there do not exist $0 \neq b_0 \in Y$, $x_0 \in X \backslash \overline{W}$, $f_{b_0} \in X_{b_0}^\sharp$, where $\overline{W}$ denotes the closure of W in the seminormed space (X, p_{b_0}), and infinitely many linearly independent elements $w_1, w_2, ...$ in W such that $\|f_{b_0}\| = 1$, $f_{b_0}|_{W \times <b_0>} = 0$ and $f_{b_0}(x_0 - w_n, b_0) = \|x_0 - w_n, b_0\|$ for all $n \geq 1$.*

3. (*b*,ε)-PSEUDO CHEBYSHEV SUBSPACES

Definition 3.1. Let $(X \times Y, \|., .\|)$ be a generalized 2-normed space, W be a subspace of X, $0 \neq b \in Y$ and $\varepsilon > 0$ be given.

(i) $w_0 \in W$ is called (b,ε)-best approximation of $x \in X$ in W, if

$$\|x - w_0, b\| \leq \inf\{\|x - w, b\| : w \in W\} + \varepsilon.$$

The set of all b-best approximations of x in W is denoted by $P_{W,\varepsilon}^b(x)$.

(ii) W is called (b,ε)-pseudo Chebyshev if $P_{W,\varepsilon}^b(x)$ is finite dimensional for every $x \in X$.

Theorem 3.1. *Let $(X \times Y, \|., .\|)$ be a generalized 2-normed space, W be a subspace of X, $w_0 \in W$, $0 \neq b \in Y$ and $\varepsilon > 0$ be given. Then, $w_0 \in P_{W,\varepsilon}^b(x)$ if and only if there exist $f \in X_b^\sharp$ such that $f|_{W \times <b>} = 0$, $\|f\| = 1$ and $f(x - w_0, b) \geq \|x - w_0, b\| - \varepsilon$.*

Proof. First suppose that there exist $f \in X_b^\sharp$ such that $f|_{W \times <b>} = 0$, $\|f\| = 1$ and $f(x - w_0, b) \geq \|x - w_0, b\| - \varepsilon$. Then, $\|x - w_0, b\| \leq f(x - w_0, b) + \varepsilon = f(x - w, b) + \varepsilon \leq \|x - w, b\|\|f\| + \varepsilon = \|x - w, b\| + \varepsilon$ for all $w \in W$. Hence, $w_0 \in P_{W,\varepsilon}^b(x)$. Conversely, Let $w_0 \in P_{W,\varepsilon}^b(x)$. If $x \in \overline{W}$, where $\overline{W}$ denotes the closure of W in the seminormed space (X, p_b), choose $w_0 \in W$ such that $\|x - w_0, b\| < \varepsilon$. Then, every $f \in X_b^\sharp$ with $f|_{W \times <b>} = 0$ and $\|f\| = 1$, satisfies $f(x - w_0, b) \geq \|x - w_0, b\| - \varepsilon$. If $x \in X \backslash \overline{W}$, $\delta = \inf\{\|x_0 - w, b\| : w \in W\} > 0$. Then by Lemma 1.2, there exists $g \in X_b^\sharp$ such that $g|_{W \times <b>} = 0$, $g(x, b) = 1$ and $\|g\| = \frac{1}{\delta}$. Put $f = \delta g$. Then, $f \in X_b^\sharp$, $f|_{W \times <b>} = 0$, $\|f\| = 1$ and $f(x - w_0, b) + \varepsilon = \delta + \varepsilon \geq \|x - w_0, b\|$. ∎

Lemma 3.2. *Let $(X \times Y, \|., .\|)$ be a generalized 2-normed space, W be a subspace of X, $\varepsilon > 0$ be given and $0 \neq b \in Y$. Then, $M \subseteq P_{W,\varepsilon}^b(x)$ if and only if there exists $f \in X_b^\sharp$ such that $f|_{W \times <b>} = 0$, $\|f\| = 1$ and $f(x_0 - m, b) \geq \|x_0 - m, b\| - \varepsilon$ for all $m \in M$.*

Proof. Let $M \subseteq P_{W,\varepsilon}^b(x)$ and choose $w_0 \in P_{W,\varepsilon}^b(x)$ with $\|x - w_0, b\| = \lambda + \varepsilon$, where $\lambda = \inf\{\|x - w, b\| : w \in W\}$. By Theorem 3.1, there exist $f \in X_b^\sharp$ such that $f|_{W \times <b>} = 0$, $\|f\| = 1$ and $f(x - w_0, b) \geq \|x - w_0, b\| - \varepsilon$. Then, $f(x - m, b) = f(x - w_0, b) \geq \|x - w_0, b\| - \varepsilon = \lambda \geq \|x - m, b\| - \varepsilon$, for all $m \in M$. ∎

Theorem 3.3. *$(X \times Y, \|., .\|)$ be a generalized 2-normed space, W be a subspace of X, $0 \neq b \in Y$ and $\varepsilon > 0$ be given. Then, W is (b,ε)-pseudo Chebyshev subspace if and only if there do not exist $f \in X_b^\sharp$, $x \in X$ and and infinitely many linearly independent elements $w_1, w_2, ...$ in W such that $\|x, b\| \leq 1$, $f|_{W \times <b>} = 0$, $\|f\| = 1$ and $f(x - w_n, b) \geq \|x - w_n, b\| - \varepsilon$, for all $n \geq 1$.*

Proof. First assume that there exist $f \in X_b^\sharp$, $x \in X$ and infinitely many linearly independent elements $w_1, w_2, \ldots$ in W such that $\|x, b\| \leq 1$, $f|_{W\times<b>} = 0$, $\|f\| = 1$ and $f(x-w_n, b) \geq \|x-w_n, b\| - \varepsilon$, for all $n \geq 1$. Then, $w_n \in P^b_{W,\varepsilon}(x)$ for all $n \geq 1$. It follows that $\dim P^b_{W,\varepsilon}(x_0) = \infty$ and hence W is not (b,ε)-pseudo Chebyshev subspace. Now, suppose that W is not (b,ε)-pseudo Chebyshev subspace. Since $P^b_{W,\varepsilon}(\lambda x) = \lambda P^b_{W,\varepsilon/\lambda}(x)$ and $P^b_{W,\varepsilon_1}(x) \subseteq P^b_{W,\varepsilon_2}(x)$ for all $0 < \varepsilon_1 \leq \varepsilon_2$, $x \in X$ and $\lambda > 0$, there exist $x_0 \in X$ with $\|x_0, b\| \leq 1$ such that $\dim P^b_{W,\varepsilon}(x_0) = \infty$. Hence, $P_{W,\varepsilon}(x_0)$ contains infinitely many linearly independent elements $g_1, g_2, \ldots$. By Lemma 3.2, there exists $f \in X_b^\sharp$ such that $\|f\| = 1$, $f|_{W\times<b>} = 0$ and $f(x_0 - g_n, b) \geq \|x_0 - g_n, b\| - \varepsilon$ for all $n \geq 1$. ∎

Definition 3.2. Let $(X \times Y, \|.,.\|)$ be a generalized 2-normed space, $0 \neq b \in Y$, $\varepsilon > 0$ be given and $f \in X_b^\sharp$. Define

$$M^b_{f,\varepsilon} = \{x \in X : f(x, b) \geq \|x, b\| - \varepsilon, \|x, b\| \leq 1 + \varepsilon\}.$$

Theorem 3.4. *Let $(X \times Y, \|.,.\|)$ be a generalized 2-normed space, W be a subspace of X, $0 \neq b \in Y$ and $\varepsilon > 0$ be given. If $M^b_{f,\varepsilon}$ is finite dimensional for all $f \in X_b^\sharp$ with $\|f\| = 1$ and $f|_{W\times<b>} = 0$, then W is (b,ε)-pseudo Chebyshev subspace.*

Proof. Assume that W is not (b,ε)-pseudo Chebyshev subspace. Then by Theorem 3.3, there exist $f \in X_b^\sharp$, $x_0 \in X$ with $\|x_0, b\| \leq 1$ and infinitely many linearly independent elements $w_1, w_2, \ldots$ in W such that $\|f\| = 1$, $f|_{W\times<b>} = 0$, and $f(x_0 - w_n, b) \geq \|x_0 - w_n, b\| - \varepsilon$ for all $n \geq 1$. Since $\|x_0 - w_n, b\| \leq f(x_0 - w_n, b) + \varepsilon = f(x_0, b) + \varepsilon \leq 1 + \varepsilon$, $x_0 - w_n \in M^b_{f,\varepsilon}$ for all $n \geq 1$. This is a contradiction. ∎

Definition 3.3. Let $(X \times Y, \|.,.\|)$ be a generalized 2-normed space, $0 \neq b \in Y$, $\varepsilon > 0$ be given and let M be a subspace of $X_b^\sharp$. For each $x \in X$, put

$$D^{M,b}_{x,\varepsilon} = \{y \in X : f(y, b) = f(x, b) \ for\ all\ f \in M \ \& \ \|y, b\| \leq \|x, b\|_M + \varepsilon\},$$

where $\|x, b\|_M = \sup\{|f(x, b)| : \|f\| \leq 1, f \in M\}$.

It is clear that $D^{M,b}_{x,\varepsilon}$ is a non-empty, closed and convex subset of (X, p_b), for all $x \in X$.

We say that M has the property (b,ε) $- F^*$ if $D^{M,b}_{x,\varepsilon}$ is finite dimensional for all $x \in X$.

Theorem 3.5. *Let $(X \times Y, \|.,.\|)$ be a generalized 2-normed space, W be a subspace of X, $\varepsilon > 0$ be given, $0 \neq b \in Y$ and let $M_0 = \{f \in X_b^\sharp : f|_{W\times<b>} = 0\}$. Then, W is (b,ε)-pseudo Chebyshev subspace if and only if M_0 has the property (b,ε) $- F^*$.*

Proof. If $\dim D^{M_0,b}_{x,\varepsilon} = \infty$ for some $x \in X$, then there exist infinitely many linearly independent elements $y_1, y_2, \ldots$ in $D^{M_0,b}_{x,\varepsilon}$. Hence, $y_1 - y_n \in W$ for all $n \geq 1$ and

$$\|y_1 - (y_1 - y_n), b\| = \|y_n, b\| \leq \|x, b\|_{M_0} + \varepsilon = \|y_1 - (y_1 - y_n), b\|_{M_0} + \varepsilon$$

for all $n \geq 1$. Therefore, $y_1 - y_n \in P^b_{W,\varepsilon}(y_1)$ for all $n \geq 1$. It follows that W is not (b,ε)-pseudo Chebyshev subspace. Now, suppose that $\dim P^b_{W,\varepsilon}(x_0) = \infty$ for some $x_0 \in X$. Then, there exist infinitely many linearly independent elements $g_1, g_2, \ldots$ in $P^b_{W,\varepsilon}(x_0)$. It is easy to see that, $\|x_0 - g_n, b\| \leq \|x_0 - g_n, b\|_{M_0} + \varepsilon = \|x_0, b\|_{M_0} + \varepsilon$ for all $n \geq 1$. It follows that $x_0 - g_n \in D^{M_0,b}_{x_0,\varepsilon}$ for all $n \geq 1$, which is a contradiction. ∎

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