The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 4, Issue 1, Article 7, pp. 1-9, 2007

KOMATU INTEGRAL TRANSFORMS OF ANALYTIC FUNCTIONS SUBORDINATE TO CONVEX FUNCTIONS

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Received 7 November, 2005; accepted 12 September, 2006; published 20 February, 2007.

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ABSTRACT. In this paper, we consider the class $\mathcal{A}$ of the functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \Delta := \{z \in \mathbb{C} : |z| < 1\}),$$

which are analytic in an open disk $\Delta := \{z \in \mathbb{C} : |z| < 1\})$ and study certain subclass of the class $\mathcal{A}$, for which

$$I_a^{\sigma} f(z) = \frac{(1+a)^{\sigma}}{z^a \Gamma(\sigma)} \int_0^z \left[log \frac{z}{t}\right]^{\sigma - 1} t^{a-1} f(t) dt$$

has some property. Certain inclusion and the closure properties like convolution with convex univalent function etc. are studied.

Key words and phrases: Analytic functions, Starlike functions, Convex functions, Close-to-convex functions, Quasi-convex functions, Integral operator, Subordination and Convolution.

2000 *Mathematics Subject Classification.* Primary 30C45.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Let $H(\Delta)$ be the class of all analytic functions f defined in an open disk of the form $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ denote the subclass of $H(\Delta)$ consisting of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \Delta := \{z \in \mathbb{C} : |z| < 1\}. \tag{1.1}$$

Also let $\mathcal{S}, \mathcal{S}^*$ and $\mathcal{C}$ denote the subclasses of $\mathcal{A}$ consisting of functions which are univalent, starlike and convex in Δ respectively. A function f in $\mathcal{A}$ is said to be a close-to-convex univalent function if

$$Re\left\{\frac{f'(z)}{g'(z)}\right\} > 0, z \in \Delta,$$

for some $g \in \mathcal{C}$. This class is denoted by $\mathcal{K}$. A function f in $\mathcal{A}$ is said to be a quasi- convex univalent function if there exists a function $g \in \mathcal{C}$ such that

$$Re\left\{\frac{(zf'(z))'}{g'(z)}\right\} > 0, z \in \Delta.$$

Denote the class of all such functions by $\mathcal{K}^*$. Let h be a convex univalent function with $h(0) = 1$ and $Re\{h(z)\} > 0, z \in \Delta$. We denote several subclass of $\mathcal{A}$ respectively by

$$\begin{aligned}
\mathcal{S}^*(h) &= \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec h(z)\right\};\\
\mathcal{C}(h) &= \left\{f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec h(z)\right\};\\
\mathcal{K}(h) &= \left\{f \in \mathcal{A} : \frac{f'(z)}{g'(z)} \prec h(z)\right\};
\end{aligned}$$

where $\prec$ denote the subordination. For details regarding this and other terms mentioned above, we refer to Goodman [6] and references therein. Let $h_i(z), i = 1, 2$ be two convex univalent function with $h_i(0) = 1, i = 1, 2$ and $Re\{h_i(z)\} > 0, z \in \Delta, i = 1, 2$. We denote by,

$$\mathcal{K}(h_1, h_2) = \left\{f \in \mathcal{A} : \frac{f'(z)}{g'(z)} \prec h_1(z), z \in \Delta\right\};$$

$$\mathcal{K}^*(h_1, h_2) = \left\{f \in \mathcal{A} : 1 + \frac{(zf'(z))'}{g'(z)} \prec h_1(z), z \in \Delta\right\};$$

for some $g \in \mathcal{C}(h_2)$. Clearly $f \in \mathcal{K}^*(h_1, h_2)$ if and only if $zf' \in \mathcal{K}(h_1, h_2)$.
There are various operators in the literature, related to the study of function theory. One such operator, called Komatu operator which generalizes various operators extensively (for example see, Balasubramanian, Ponnusamy and Prabhakaran [1]) is given by

$$I_a^\sigma f(z) = \frac{(1+a)^\sigma}{z^a \Gamma(\sigma)} \int_0^z \left[log \frac{z}{t}\right]^{\sigma-1} t^{a-1} f(t) dt;$$

$$I_a^\sigma f(z) = \frac{(1+a)^\sigma}{\Gamma(\sigma)} \int_0^1 \left[\log \frac{1}{t}\right]^{\sigma-1} t^{a-1} f(tz) dt; \tag{1.2}$$

$a > 0, \sigma > 0, f \in \mathcal{A}$. It can be observed easily that,

$$I_a^\sigma f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+a}{n+a}\right)^\sigma a_n z^n;$$

that is ,

$$I_a^\sigma f(z) = f(z) * \psi_{\sigma,a}(z), \tag{1.3}$$

where,

$$\psi_{\sigma,a}(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+a}{n+a}\right)^\sigma z^n.$$

In particular for $a = 1$, this operator have been discussed by Jung, Kim and Srivastava [5] and various results were obtained which are generalized in this paper. More precisely, the properties of

$$I_1^\sigma f(z) = \frac{(1+a)^\sigma}{z\Gamma(\sigma)} \int_0^z \left[log \frac{z}{t}\right]^{\sigma-1} f(t)dt, \sigma > 0, f \in \mathcal{A}, \tag{1.4}$$

subordinate to certain convex univalent functions were discussed. This operator $I_1^\sigma f$ is closely related to multiplier transformations studied earlier by Flett [13]. It follows from(1.4) that one can define the operator$I_1^\sigma f$ for any real number σ. Certain properties of this operator have been studied by Jung, Kim and Srivastava [5], Uralegadi and Somanatha [18], Li [8] and Liu [9]. We define the following subclasses of $\mathcal{A}$ of functions, using the operator$I_a^\sigma f$.

$$\begin{aligned}
\mathcal{S}^*_{\sigma,a}(h) &= \left\{f \in \mathcal{A} : I_a^\sigma f \in \mathcal{S}^*(h)\right\};\\
\mathcal{C}_{\sigma,a}(h) &= \left\{f \in \mathcal{A} : I_a^\sigma f \in \mathcal{C}(h)\right\};\\
\mathcal{K}_{\sigma,a}(h) &= \left\{f \in \mathcal{A} : I_a^\sigma f \in \mathcal{K}(h)\right\};\\
\mathcal{K}_{\sigma,a}(h_1,h_2) &= \left\{f \in \mathcal{A} : I_a^\sigma f \in \mathcal{K}(h_1,h_2)\right\};\\
\mathcal{K}^*_{\sigma,a}(h_1,h_2) &= \left\{f \in \mathcal{A} : I_a^\sigma f \in \mathcal{K}^*(h_1,h_2)\right\}.
\end{aligned}$$

Noor [14] introduced and studied some classes of functions defined through Ruschweyh derivatives.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}$ if it satisfies the inequality

$$Re\left\{f'(z)\right\} > 0, z \in \Delta. \tag{1.5}$$

The class $\mathcal{R}$ was studied systematically by MacGregor [12]. Let $\mathcal{H}^p, (0 < p \leq \infty)$ denote the Hardy space of analytic functions $f \in \Delta$, and define the integral means

$$M_p(r,f) = \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta, 0 < p < \infty \\ \\ \max_{|z|=r} |f(z)|, p = \infty \end{cases} \tag{1.6}$$

Then, by definition, an analytic function $f \in \Delta$ to the Hardy space $\mathcal{H}^p, 0 < p \leq \infty$ if

$$\lim_{r\to 1-} \left\{M_p(r,f)\right\} < \infty, 0 < p \leq \infty. \tag{1.7}$$

For $1 \leq p \leq \infty$, $\mathcal{H}^p$ is a Banach space with the norm defined by Duren [4].

$$||f||_p = \lim_{r\to 1-} \left\{M_p(r,f)\right\}, 1 \leq p \leq \infty. \tag{1.8}$$

Furthermore, $\mathcal{H}^\infty$ is the class of bounded analytic functions in Δ, while $\mathcal{H}^2$ is the class of the power series $\sum_{n=2}^{\infty} a_n z^n$ with $\sum_{n=2}^{\infty} |a_n|^2 \leq \infty$. In the present paper, we prove certain inclusion relations for the subclasses $\mathcal{S}^*_{\sigma,a}(h), \mathcal{C}_{\sigma,a}(h), \mathcal{K}_{\sigma,a}(h), \mathcal{K}_{\sigma,a}(h_1,h_2), \mathcal{K}^*{}_{\sigma,a}(h_1,h_2)$ and discuss certain integral operators defined on those classes. In addition to , we aim at proving a number of inclusion theorems involving the Hardy space $\mathcal{H}^p$, the class $\mathcal{R}$ of function in $\mathcal{A}$ satisfying the inequality (1.5), and the following integral operator $I_a^\sigma f$. In avoiding repetition we say once and for all in this paper, unless otherwise specified that $h(z)$ denote a convex univalent function on Δ with $h(0) = 1$ and $Re\{h(z)\} > 0, z \in \Delta$.

2. Main Results and Preliminary Lemmas

Theorem 2.1. *If $f \in \mathcal{S}^*_{\sigma,a}(h)$, then $f \in \mathcal{S}^*_{\sigma+1,a}(h)$, for every real number $\sigma > 0$, that is,*

$$\mathcal{S}^*_{\sigma,a}(h) \subseteq \mathcal{S}^*_{\sigma+1,a}(h).$$

Theorem 2.2. *If $f \in \mathcal{C}_{\sigma,a}(h)$, then $f \in \mathcal{C}_{\sigma+1,a}(h)$, for every real number $\sigma > 0$, that is,*

$$\mathcal{C}_{\sigma,a}(h) \subseteq \mathcal{C}_{\sigma+1,a}(h).$$

Theorem 2.3. *If $f \in \mathcal{K}_{\sigma,a}(h_1,h_2)$, then $\mathcal{K}_{\sigma+1,a}(h_1,h_2)$, for every real number $\sigma > 0$, that is,*

$$\mathcal{K}_{\sigma,a}(h_1,h_2) \subseteq \mathcal{K}_{\sigma+1,a}(h_1,h_2).$$

Theorem 2.4. *If $f \in \mathcal{K}^*_{\sigma,a}(h_1,h_2)$, then $\mathcal{K}^*_{\sigma+1,a}(h_1,h_2)$, for every real number $\sigma > 0$, that is,*

$$\mathcal{K}^*_{\sigma,a}(h_1,h_2) \subseteq \mathcal{K}^*_{\sigma+1,a}(h_1,h_2).$$

We note that, for the case $a = 1$, all the theorems stated above were obtained by Liu [10]. In order to prove our main results, we need the following Lemmas.

Lemma 2.5. *If $f \in \mathcal{S}^*(h)$, then $(zI_a^\sigma f)' = I_a^\sigma(zf')$*

Proof. Since $f \in \mathcal{A}$, we have

$$z(I_a^\sigma f)' \;=\; z(f * \psi_{\sigma,a})' = zf' * \psi_{\sigma,a} = I_a^\sigma(zf').$$

∎

Lemma 2.6. *If $f(z)$ be the function defined by* (1.1) *in the class $\mathcal{A}$. Then*

$$I_a^\sigma f(z) \;=\; \frac{(1+a)^\sigma}{z^a\Gamma(\sigma)} \int_0^z \left[log\frac{z}{t}\right]^{\sigma-1} t^{a-1} f(t)dt, \sigma > 0,$$

satisfying the equations,

$$z(I_a^{\sigma+1} f)' = (a+1)I_a^\sigma(f) - aI_a^{\sigma+1}(f). \tag{2.1}$$

Lemma 2.7.

$$I_a^\sigma f(z) \;=\; f(z) * \psi_{\sigma,a}(z)$$

is a convex function for every integer $\sigma > 0, a > 0$.
Where

$$\psi_{\sigma,a}(z) \;=\; z + \sum_{n=2}^{\infty} \left(\frac{1+a}{n+a}\right)^\sigma z^n.$$

Proof. From the equation $I_a^\sigma f(z) = f(z) * \psi_{\sigma,a}(z)$. If $\sigma > 0, a > 0$ is an integer, then

$$\psi_{\sigma,a}(z) \quad = \quad \psi_{1,a}(z) * \psi_{1,a}(z) * ... * \psi_{1,a}(z),$$

where $\psi_{1,a}(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+a}{n+a}\right) z^n$. Now $\psi_{1,a}(z)$ is the Libra transformation [7] of the function $\frac{z}{1-z}$, which is a convex univalent function. As convolution of two convex univalent functions is a convex univalent function; for example see, Ruscheweyh [16], we have $\psi_{\sigma,a}(z)$ is a convex univalent function. Clearly, this is true for every integer $\sigma > 0$. ∎

Lemma 2.8. [3]*; Let $\beta, \gamma \in C$ and h be convex function in Δ with $h(0) = 1$ and $Re\{\beta h(z) + \gamma\} > 0, z \in \Delta$ and let $p \in H(\Delta)$ with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z). \tag{2.2}$$

A modification of Lemma 2.8 is given by ,

Lemma 2.9. [15]*; Let $\beta, \gamma \in C$ and h be convex function in Δ with $h(0) = 1$ and $Re\{\beta h(z) + \gamma\} > 0, z \in \Delta$ and let $q \in H(\Delta)$ with $q(0) = 1$ and $q(z) \prec h(z), z \in \Delta$, if $p \in H(\Delta)$ with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\beta q(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z). \tag{2.3}$$

3. PROOF OF THEOREMS 2.1, 2.2, 2.3, 2.4

Proof of the Theorem 2.1. Let $f \in S^*_{\sigma,a}(h)$ and taking $p(z) = \dfrac{z(I_a^{\sigma+1} f)'}{I_a^{\sigma+1} f}$, therefore

$$\frac{p(z)}{z(I_a^{\sigma+1} f)'} = \frac{1}{I_a^{\sigma+1} f} \Rightarrow \frac{p(z) I_a^\sigma f}{z(I_a^{\sigma+1} f)'} = \frac{I_a^\sigma f}{I_a^{\sigma+1} f}.$$

Using (2.1), we have

$$\frac{I_a^\sigma f}{I_a^{\sigma+1} f} = \frac{1}{a+1}\left(\frac{z(I_a^{\sigma+1} f)'}{I_a^{\sigma+1} f} + a\right)$$

that is,

$$\frac{I_a^\sigma f}{I_a^{\sigma+1} f} = \frac{1}{a+1}\left(p(z) + a\right).$$

Now, taking the logarithmic derivative we get,

$$\begin{aligned} \frac{z(I_a^\sigma f)'}{I_a^\sigma f} &= \frac{z(I_a^{\sigma+1} f)'}{I_a^{\sigma+1} f} + \frac{zp'(z)}{p(z)+a} \\ \frac{z(I_a^\sigma f)'}{I_a^\sigma f} &= p(z) + \frac{zp'(z)}{p(z)+a} \prec h(z), \end{aligned}$$

an application of Lemma 2.8 with $\beta = 1$ and $\gamma = a$ gives $f \in S^*_{\sigma+1,a}(h)$. ∎

Remark 3.1. If $h(z) = \frac{1+(1-2\gamma)z}{1-z}$ and $a = 1$, then the Theorem 2.1 reduces to of Theorem 1 of Liu [9].

Proof of the Theorem 2.2. We have, $f \in C_{\sigma,a}(h) \Rightarrow I_a^\sigma f \in C(h) \Rightarrow z(I_a^\sigma f)' \in S^*(h)$. Therefore By Lemma 2.6, $I_a^\sigma (zf') \in S^*(h) \Rightarrow zf' \in S^*_{\sigma,a}(h)$ which by Theorem 2.1 , gives $(zf') \in S^*_{\sigma+1,a}(h)$ and hence $I_a^{\sigma+1}(zf') \in S^*(h) \Rightarrow z(I_a^{\sigma+1} f)' \in S^*(h)$, by Lemma 2.6 , gives

$$I_a^{\sigma+1} f \in C(h) \Rightarrow f \in C_{\sigma+1,a}(h).$$

∎

Proof of the Theorem 2.3. Let $f \in K(h_1, h_2)$, then there is a function $k(z) \in S^*(h_2)$ such that $\frac{z(I_a^\sigma f)'}{k(z)} \prec h_1(z)$. Taking the function $g(z)$ which satisfies $I_a^\sigma g = k(z)$, we have $g(z) \in S_\sigma^*(h_2)$ and $\frac{z(I_a^\sigma f)'}{I_a^\sigma g} \prec h_1(z), \ \ z \in \Delta$. Consider,

$$\begin{aligned} \frac{z(I_a^\sigma f)'}{I_a^\sigma g} &= \frac{I_a^\sigma(zf')}{I_a^\sigma g}, \\ &= \frac{1}{a+1}\left\{\frac{z(I_a^{\sigma+1}zf')' + aI_a^{\sigma+1}(zf')}{z(I_a^{\sigma+1}g)' + aI_a^{\sigma+1}g}\right\} \text{ by (2.1)} \\ &= \frac{1}{a+1}\left\{\frac{\frac{z(I_a^{\sigma+1}zf')'}{I_a^{\sigma+1}g} + a\frac{I_a^{\sigma+1}(zf')}{I_a^{\sigma+1}g}}{\frac{z(I_a^{\sigma+1}g)'}{I_a^{\sigma+1}g} + a}\right\}. \end{aligned}$$

Now, taking $p(z) = \frac{z(I_a^{\sigma+1}f)'}{I_a^{\sigma+1}g}$ and $q(z) = \frac{z(I_a^{\sigma+1}g)'}{I_a^{\sigma+1}g}$ subject to $q(0) = 1$ and $Re\{q(z)\} > 0, z \in U$, we get,

$$\frac{z(I_a^\sigma f)'}{I_a^\sigma g} = \frac{1}{a+1}\left\{\frac{\frac{z(I_a^{\sigma+1}zf')'}{I_a^{\sigma+1}g} + ap(z)}{q(z)+a}\right\}. \tag{3.1}$$

Differentiating $z(I_a^{\sigma+1}f)' = p(z)(I_a^{\sigma+1}g)$ on both sides, we have,

$$z(I_a^{\sigma+1}(zf'))' = z(I_a^{\sigma+1}g)p'(z) + z(I_a^{\sigma+1}g)'p(z)$$

which gives,

$$\frac{z(I_a^{\sigma+1}(zf'))'}{I_a^{\sigma+1}g} = zp'(z) + \frac{z(I_a^{\sigma+1}g)'}{I_a^{\sigma+1}g}p(z) = p(z)q(z) + zp'(z). \tag{3.2}$$

From (3.1) and (3.2), we get

$$\begin{aligned} \frac{z(I_a^\sigma f)'}{I_a^\sigma g} &= \frac{1}{a+1}\left\{\frac{(a+q(z))p(z) + zp'(z)}{a+q(z)}\right\} \\ &= \frac{1}{a+1}\left\{p(z) + \frac{zp'(z)}{a+q(z)}\right\} \prec \ h(z) \end{aligned}$$

an application of Lemma 2.9 gives $p(z) \prec h(z) \ \Rightarrow f \in K_{\sigma+1,a}(h_1, h_2)$. ∎

Proof of the Theorem 2.4. Since

$$\begin{aligned} f \in K_{\sigma,a}^*(h_1,h_2) &\Rightarrow \ I^\sigma f \in K^*(h_1,h_2) \\ &\Rightarrow \ z(I_a^\sigma f)' \in K(h_1,h_2) \\ &\Rightarrow \ I_a^\sigma(zf') \in K(h_1,h_2), \end{aligned}$$

which gives

$$\begin{aligned} zf' \in K_{\sigma,a}(h_1,h_2) &\Rightarrow \ zf' \in K_{\sigma+1,a}(h_1,h_2) \\ &\Rightarrow \ I_a^{\sigma+1}(zf') \in \ K^*(h_1,h_2), \end{aligned}$$

by Theorem 2.2

$$\begin{aligned} &\Rightarrow \quad z(I_a^{\sigma+1}f)' \in K^*(h_1, h_2) \\ &\Rightarrow \quad I_a^{\sigma+1}f \in K^*(h_1, h_2). \end{aligned}$$

Hence $f \in K^*_{\sigma+1,a}(h_1, h_2)$. ∎

4. A PARTICULAR INTEGRAL OPERATOR

From the integral operator,

$$I_a^\sigma f(z) = \frac{(1+a)^\sigma}{z^a \Gamma(\sigma)} \int_0^z \left[log\frac{z}{t}\right]^{\sigma-1} t^{a-1} f(t)dt, \sigma > 0, a > 0, f \in \mathcal{A}, z \in \Delta,$$

now taking $\sigma = 1$, we have

$$I_a^1 f(z) = \frac{1+a}{z^a} \int_0^z t^{a-1} f(t)dt, a > 0, f \in \mathcal{A}, z \in \Delta. \tag{4.1}$$

It is the Bernardi-Libra-Livingston integral operator $L_a(f) = I_a^1 f(z) = f(z) * \psi_{1,a}(z)$, where

$$\psi_{1,a}(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+a}{n+a}\right) z^n, Re(z) > 0,$$

which is a Bernardi transformation of the convex univalent function $\frac{z}{1-z}$ and hence a convex univalent function. The operator $L_a(f)$, for $a = 1$, $L_1(f)$ was investigated by Libra [7], where $a \in \mathcal{N} = 1, 2, ...$ was studied by Bernardi [2].

Theorem 4.1. *If f is in class $\mathcal{R}$, then $I_a^\sigma \in \mathcal{H}^\infty$ at least for $\sigma > 1$.*

Remark 4.1. Theorem 4.1 for particular values of σ and a are available in the literature. For example, Jung, Kim and Srivastava[5] obtained independent results for $\sigma = 1$ and $a = 1$.

We state the theorem without proof, as they can be easily obtained in a way similar to the results stated in Section 2.

Theorem 4.2. *For every real number $\sigma > 0$, we have*

(i) *If $f \in \mathcal{S}^*_{\sigma,a}(h)$, then $L_a(f) \in \mathcal{S}^*_{\sigma,a}(h)$.*
(ii) *If $f \in \mathcal{C}_{\sigma,a}(h)$, then $L_a(f) \in \mathcal{C}_{\sigma,a}(h)$.*
(iii) *If $f \in \mathcal{K}_{\sigma,a}(h_1, h_2)$, then $L_a(f) \in \mathcal{K}_{\sigma,a}(h_1, h_2)$.*
(iv) *If $f \in \mathcal{K}^*_{\sigma,a}(h_1, h_2)$, then $L_a(f) \in \mathcal{K}^*_{\sigma,a}(h_1, h_2)$.*

For proving the Theorem 4.1, we required the following Lemmas.

Lemma 4.3. [4]*; If f is analytic in Δ, and if $Ref(z) > 0, z \in \Delta$, then $f \in \mathcal{H}^p$, for all $p < 1$.*

Lemma 4.4. [4]*; If f is in the class $\mathcal{R}$, then $f \in \mathcal{H}^p$, for all $0 < p < \infty$.*

Lemma 4.5. [17]*; If $f \in \mathcal{S}^*(h)$, then $L_a(f) \in \mathcal{S}^*(h)$.*

Proof of the Theorem 4.1. Making use of (2.1) of Lemma 2.6, we obtain

$$(I_a^{\sigma+1}f)' = \frac{1}{z}\left\{(a+1)I_a^\sigma(f) - aI_a^{\sigma+1}(f)\right\}, \qquad \sigma > 1, a > 0, \tag{4.2}$$

which, in view of the elementary inequality

$$\max(A^p, B^p) \le (A+B)^p \le 2^p(A^p + B^p), \qquad 0 < p < \infty, \quad A \ge 0, \quad B \ge 0,$$

readily yields

$$|(I_a^\sigma f)'|^p \leq \left(\frac{2}{r}\right)^p \left\{ a^p |I_a^\sigma f|^p + (a+1)^p |I_a^{\sigma-1} f|^p \right\}, \tag{4.3}$$

where $|z| = r$, $0 < p < \infty$, $\sigma > 1$, $a > 0$. For $p = 1$, a substantially improved inequality would follow directly from (4.2), and we have

$$|(I_a^\sigma f)'| \leq \frac{1}{r} \left\{ a|I_a^\sigma f| + (a+1)|I_a^{\sigma-1} f| \right\}, \tag{4.4}$$

which may be compared with a special case of (4.3) when $p = 1$. Now, from the equation

$$\begin{aligned} I_a^\sigma f(z) &= \frac{(1+a)^\sigma}{z^a \Gamma(\sigma)} \int_0^z \left[log \frac{z}{t} \right]^{\sigma-1} t^{a-1} f(t) dt \\ &= \frac{(1+a)^\sigma}{\Gamma(\sigma)} \int_0^1 \left[log \frac{1}{t} \right]^{\sigma-1} t^{a-1} f(tz) dt, \qquad \sigma > 0, \quad a > 0, \quad f \in \mathcal{A}. \end{aligned}$$

we set

$$Re(I_a^\sigma f)' = \frac{(1+a)^\sigma}{\Gamma(\sigma)} \int_0^1 \left[log \frac{1}{t} \right]^{\sigma-1} t^a Re[f'(tz)] dt, \qquad \sigma > 0, \quad a > 0. \tag{4.5}$$

Since $f \in \mathcal{R}$, it follows from (4.5) that $I_a^\sigma f \in \mathcal{R}, \sigma > 0, a > 0$. Thus, by Lemma 4.4, we have

$$I_a^\sigma f \in \mathcal{H}^p, \qquad 0 < p < \infty, \qquad \sigma > 0, \quad a > 0. \tag{4.6}$$

In view of the definitions (1.6) and (1.8), the inequality (4.3) yields

$$M_1(r, (I_a^\sigma f)') \leq \frac{1}{r} \left\{ a M_1(r, I_a^\sigma f) + (a+1) M_1(r, I_a^{\sigma-1} f) \right\}, \quad |z| = r, \quad \sigma > 1, \quad a > 0$$

and

$$\|((I_a^\sigma f)'\|_1 \leq |a| \|I_a^\sigma f\|_1 + |a+1| \|I_a^{\sigma-1} f\|_1, \qquad \sigma > 1, \quad a > 0. \tag{4.7}$$

Since $\sigma > 1$, it follows readily from the inclusion relation (4.6) that

$$I_a^\sigma f \in \mathcal{H}^1 \qquad \text{and} \qquad I_a^{\sigma-1} f \in \mathcal{H}^1, \tag{4.8}$$

and hence (4.7) gives $(I_a^\sigma f)' \in \mathcal{H}^1$, $\sigma > 1$, $a > 0$. Therefore, by appealing to a known result Duren [4] [3, p. 42, Theorem 3.11], we conclude from (4.8) that $I_a^\sigma f, (\sigma > 1, a > 0)$, is continuous in

$$\bar{\Delta} = \Delta \cup \partial\Delta = \{z \in \Delta : |z| \leq 1\}.$$

Finally, since $\bar{\Delta}$ is compact, $I_a^\sigma f$ is bounded in $\bar{\Delta}$. Thus $I_a^\sigma f$ is bounded analytic function in Δ, which leads us to the assertion Theorem 4.1. This evidently completes the proof of the Theorem 4.1. ∎

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