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SALAGEAN-TYPE HARMONIC UNIVALENT FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

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ABSTRACT. A necessary and sufficient coefficient are given for functions in a class of complex-valued harmonic univalent functions of the form $f = h + \overline{g}$ using Salagean operator where h and g are analytic in the unit disk $U = \{z : |z| < 1\}$. Furthermore, distortion theorems, extreme points, convolution condition, and convex combinations for this family of harmonic functions are obtained.

Key words and phrases: Harmonic, Univalent, Starlike with respect to symmetric points, Convex with respect to symmetric points.

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1. INTRODUCTION

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain C if both u and v are real harmonic in C. In any simply connected domain $D \subset C$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation-preserving in D is that $|g'(z)| < |h'(z)|$ in D, see [1].

Denote by H the class of functions $f = h + \overline{g}$ which are harmonic univalent and orientation-preserving in the open unit disk $U = \{z : |z| < 1\}$ so that $f = h + \overline{g}$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Therefore, for $f = h + \overline{g} \in H$, we can express h and g by the following power series expansion

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ g(z) = \sum_{k=1}^{\infty} b_k\, z^k, \quad |b_1| < 1. \tag{1.1}$$

Observe that H reduces to S, the class of normalized univalent analytic functions, if the co-analytic part of f is zero.

For $f = h + \overline{g}$ given by (1.1), Jahangiri [2] defined the modified Salagean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)}, \qquad (n \in N_0 = N \cup \{0\}), \tag{1.2}$$

where $D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$ and $D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k$.

Definition 1.1. For $0 \le \alpha < 1$, we let $SH_s(n, \alpha)$ denote the class of harmonic functions f of the form (1.1) such that

$$\text{Re}\left(\frac{2D^{n+1} f(z)}{D^n f(z) - D^n f(-z)}\right) > \alpha, \quad z \in U. \tag{1.3}$$

We let the subclass $\overline{SH}_s(n, \alpha)$ consist of harmonic functions $f_n = h + \overline{g}_n$ in $SH_s(n, \alpha)$ so that h and g_n are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k,\ g_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \quad a_k \ge 0,\ b_k \ge 0. \tag{1.4}$$

If the co-analytic part of $f = h + \overline{g}$ is zero, then $SH_s(0, \alpha)$ turns out to be the class $S_s^*(\alpha)$ of starlike functions with respect to symmetric points which was introduced by K. Sakaguchi [3]. Also, $SH_s(1, \alpha)$ turns out to be the class $K_s(\alpha)$ of convex functions with respect to symmetric points which was introduced by Das and Singh [4].

In this paper, we have obtained the coefficient conditions for the classes $SH_s(n, \alpha)$ and $\overline{SH}_s(n, \alpha)$. Further a representation theorem, inclusion properties and distortion bound for the class $\overline{SH}_s(n, \alpha)$ are established. Finally, properties of the integral operator $\frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt$, $c > -1$ are also studied.

2. MAIN RESULTS

First, we introduce a sufficient coefficient bound for functions in the class $SH_s(n, \alpha)$.

Theorem 2.1. *Let $f = h + \overline{g}$ be given by (1.1). If*

$$\begin{aligned}\sum_{k=1}^{\infty} \big((2k-1)^n (2k-1-\alpha)|a_{2k-1}| + (2k)^{n+1}|a_{2k}| \\ + (2k-1)^n (2k-1+\alpha)|b_{2k-1}| + (2k)^{n+1}|b_{2k}|\big) \le 1-\alpha\end{aligned} \tag{2.1}$$

where $a_1 = 1$ *and* $0 \le \alpha < 1$*, then* f *is orientation-preserving, harmonic univalent in* U*, and* $f \in SH_s(n, \alpha)$*.*

Proof. If the inequality (2.1) holds for the coefficients of $f = h + \overline{g}$, then by (1.2), f is orientation-preserving and harmonic univalent in U. It remains to show that

$$\operatorname{Re}\left\{\frac{2D^{n+1}f(z)}{D^nf(z) - D^nf(-z)}\right\} \ge \alpha.$$

According to (1.2) and (1.3), we have

$$\begin{aligned}\operatorname{Re}&\left\{\frac{2D^{n+1}f(z)}{D^nf(z) - D^nf(-z)}\right\}\\ &= \operatorname{Re}\left\{\frac{2D^{n+1}h(z) - 2(-1)^nD^{n+1}g(z)}{D^nh(z) + (-1)^n\overline{D^ng(z)} - D^nh(-z) - (-1)^n\overline{D^ng(-z)}}\right\} \ge \alpha,\\ &\qquad\qquad 0 \le \alpha < 1.\end{aligned}$$

Using the fact that $\operatorname{Re}\omega \ge \alpha$ if and only if $|1 - \alpha + \omega| \ge |1 + \alpha - \omega|$, it suffices to show that

$$\left|1 - \alpha + \frac{2D^{n+1}f(z)}{D^nf(z) - D^n(-z)}\right| \ge \left|\frac{2D^{n+1}f(z)}{D^nf(z) - D^n(-z)} - (1+\alpha)\right|$$

which is equivalent to

$$\begin{aligned}&\left|2D^{n+1}f(z) + (1-\alpha)(D^nf(z) - D^nf(-z))\right|\\ &\qquad - \left|2D^{n+1}f(z) - (1+\alpha)(D^nf(z) - D^nf(-z))\right| \ge 0.\end{aligned} \tag{2.2}$$

Substituting for D^nf and $D^{n+1}f$ in (2.2) yields

$$\begin{aligned}&\Bigg|2z + 2\sum_{k=2}^{\infty} k^{n+1}a_kz^k - 2(-1)^n\sum_{k=1}^{\infty} k^{n+1}\overline{b}_k\overline{z}^k\\ &\qquad + (1-\alpha)\Bigg[z + \sum_{k=2}^{\infty} k^na_kz^k + (-1)^n\sum_{k=1}^{\infty} k^n\overline{b}_k\overline{z}^k + z\\ &\qquad - \sum_{k=2}^{\infty} k^n(-1)^ka_kz^k - (-1)^n\sum_{k=1}^{\infty} k^n(-1)^k\overline{b}_k\overline{z}^k\Bigg]\Bigg|\\ &\qquad - \Bigg|2z + 2\sum_{k=2}^{\infty} k^{n+1}a_kz^k - 2(-1)^n\sum_{k=1}^{\infty} k^{n+1}\overline{b}_k\overline{z}_k\\ &\qquad - (1+\alpha)\Bigg[z + \sum_{k=2}^{\infty} k^na_kz^k + (-1)^n\sum_{k=1}^{\infty} k^n\overline{b}_k\overline{z}^k\\ &\qquad + z - \sum_{k=2}^{\infty}(-1)^kk^na_kz^k - (-1)^n\sum_{k=1}^{\infty}(-1)^kk^n\overline{b}_k\overline{z}^k\Bigg]\Bigg|\end{aligned}$$

$$
\begin{aligned}
&= \Bigg| 2(2-\alpha)z + \sum_{k=2}^{\infty} k^n\big(2k+(1-\alpha)-(-1)^k(1-\alpha)\big)a_k z^k \\
&-(-1)^n \sum_{k=1}^{\infty} k^n\big(2k-(1-\alpha)+(-1)^k(1-\alpha)\big)\overline{b}_k \overline{z}^k \Bigg| \\
&- \Bigg| -2\alpha z + \sum_{k=2}^{\infty} k^n\big(2k-(1+\alpha)+(-1)^k(1+\alpha)\big)a_k z^k \\
&-(-1)^n \sum_{k=1}^{\infty} k^n\big(2k+(1+\alpha)-(-1)^k(1+\alpha)\big)\overline{b}_k \overline{z}^k \Bigg| \\
&= \Bigg| 2(2-\alpha)z + 2\sum_{k=2}^{\infty}\big\{(2k-2)^{n+1}a_{2k-2}z^{2k-2} \\
&+ (2k-1)^n(2k-\alpha)a_{2k-1}z^{2k-1}\big\} - 2(-1)^n \sum_{k=1}^{\infty}\big\{(2k)^{n+1}\overline{b}_{2k}\overline{z}^{2k} \\
&+ (2k-1)^n(2k+\alpha-2)\overline{b}_{2k-1}\overline{z}^{2k-1}\big\}\big| \\
&- \big| -2\alpha z + 2\sum_{k=2}^{\infty}\big\{(2k-2)^{n+1}a_{2k-2}z^{2k-2} \\
&+ (2k-1)^n(2k-\alpha-2)a_{2k-1}z^{2k-1}\big\} - 2(-1)^n \sum_{k=1}^{\infty}\big\{(2k)^{n+1}\overline{b}_{2k}\overline{z}^{2k} \\
&+ (2k-1)^n(2k+\alpha)\overline{b}_{2k-1}\overline{z}^{2k-1}\big\}\big| \\
&\geq 2(2-\alpha)|z| - 2\sum_{k=2}^{\infty}(2k-2)^{n+1}|a_{2k-2}|\;|z|^{2k-1} \\
&- 2\sum_{k=2}^{\infty}(2k-1)^n(2k-\alpha)|a_{2k-1}|\;|z|^{2k-1} - 2\sum_{k=1}^{\infty}(2k)^{n+1}|b_{2k}|\;|z|^{2k} \\
&- 2\sum_{k=1}^{\infty}(2k-1)^n(2k+\alpha-2)|b_{2k-1}|\;|z|^{2k-1} - 2\alpha|z| \\
&- 2\sum_{k=2}^{\infty}(2k-2)^{n+1}|a_{2k-2}|\;|z|^{2k-2} - 2\sum_{k=2}^{\infty}(2k-1)^n(2k-\alpha-2) \\
&|a_{2k-1}|\;|z|^{2k-1} - 2\sum_{k=1}^{\infty}(2k)^{n+1}|b_{2k}|\;|z|^{2k} - 2\sum_{k=1}^{\infty}(2k-1)^n(2k+\alpha)|b_{2k-1}\;|z|^{2k-1} \\
&= 4(1-\alpha)|z| - 4\sum_{k=2}^{\infty}(2k-2)^{n+1}|a_{2k-2}|\;|z|^{2k-2} \\
&- 2\sum_{k=2}^{\infty}(2k-1)^n(2k-\alpha+2k-\alpha-2)|a_{2k-1}|\;|z|^{2k-1} \\
&- 4\sum_{k=1}^{\infty}(2k)^{n+1}|b_{2k}|\;|z|^{2k} - 2\sum_{k=1}^{\infty}(2k-1)^n(2k+\alpha-2+2k+\alpha)|b_{2k-1}|\;|z^{2k-1}| \\
&= 4(1-\alpha)|z| - 4\sum_{k=2}^{\infty}(2k-2)^{n+1}|a_{2k-2}|\;|z|^{2k-2}
\end{aligned}
$$

$$
\begin{aligned}
&- 4\sum_{k=2}^{\infty}(2k-1)^n(2k-1-\alpha)|a_{2k-1}|\,|z|^{2k-1}\\
&- 4\sum_{k=1}^{\infty}(2k)^{n+1}|b_{2k}|\,|z|^{2k} - 4\sum_{k=1}^{\infty}(2k-1)^n(2k-1+\alpha)|b_{2k-1}|\,|z|^{2k-1}\\
&= 4(1-\alpha)|z|\left[1-\sum_{k=2}^{\infty}\left\{\frac{(2k-2)^{n+1}}{1-\alpha}|a_{2k-2}|\,|z|^{2k-3}\right.\right.\\
&+\left.\frac{(2k-1)^n(2k-1-\alpha)}{1-\alpha}|a_{2k-1}|\,|z|^{2k-2}\right\}\\
&-\sum_{k=1}^{\infty}\left\{\frac{(2k-1)^n(2k-1+\alpha)}{1-\alpha}|b_{2k-1}\,|z|^{2k-2}\right.\\
&\left.\left.+\frac{(2k)^{n+1}}{1-\alpha}|b_{2k}|\,|z|^{2k-1}\right\}\right]\\
&\geq 4(1-\alpha)|z|\left[1-\sum_{k=2}^{\infty}\left\{\frac{(2k-2)^{n+1}}{1-\alpha}|a_{2k-2}|+\frac{(2k-1)^n(2k-\alpha-1)}{1-\alpha}|a_{2k-1}|\right\}\right.\\
&\left.-\sum_{k=1}^{\infty}\left\{\frac{(2k-1)^n(2k-1+\alpha)}{1-\alpha}|b_{2k-1}|+\frac{(2k)^{n+1}}{1-\alpha}|b_{2k}|\right\}\right].
\end{aligned}
$$

This last expression is non-negative by (2.1), and so the proof is complete. ∎

The harmonic function

$$
\begin{aligned}
f(z) \;&=\; z+\sum_{k=2}^{\infty}\left\{\frac{1-\alpha}{(2k-2)^{n+1}}x_{2k-2}z^{2k-2}+\frac{1-\alpha}{(2k-1)^n(2k-1-\alpha)}x_{2k-1}z^{2k-1}\right\}\\
&\quad+\sum_{k=1}^{\infty}\left\{\frac{1-\alpha}{(2k)^{n+1}}\overline{y}_{2k}\overline{z}^{2k}+\frac{1-\alpha}{(2k-1)^n(2k-1+\alpha)}\overline{y}_{2k-1}\overline{z}^{2k-1}\right\}, \qquad (2.3)
\end{aligned}
$$

where $\sum_{k=2}^{\infty}\Big(|x_{2k-2}|+|x_{2k-1}|\Big)+\sum_{k=1}^{\infty}\Big(|y_{2k-1}|+|y_{2k}|\Big) = 1$ shows that the coefficient bound given by (2.1) is sharp.

The functions of the form (2.3) are in $SH_s(n,\alpha)$ because

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\left(\frac{(2k-1)^n(2k-1-\alpha)}{1-\alpha}|a_{2k-1}|+\frac{(2k)^{n+1}}{1-\alpha}|a_{2k}|\right.\\
&\qquad\left.+\frac{(2k-1)^n(2k-1+\alpha)}{1-\alpha}|b_{2k-1}|+\frac{(2k)^{n+1}}{1-\alpha}|b_{2k}|\right)\\
&\qquad= 1+\sum_{k=2}^{\infty}\left(\frac{(2k-2)^{n+1}}{1-\alpha}|a_{2k-2}|+\frac{(2k-1)^n(2k-1-\alpha)}{1-\alpha}|a_{2k-1}|\right)\\
&\qquad+\sum_{k=1}^{\infty}\left(\frac{(2k-1)^n(2k-1+\alpha)}{1-\alpha}|b_{2k-1}|+\frac{(2k)^{n+1}}{1-\alpha}|b_{2k}|\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 1 + \sum_{k=2}^{\infty}\Big(|x_{2k-2}| + |x_{2k-1}|\Big) + \sum_{k=1}^{\infty}\Big(|y_{2k-1}| + |y_{2k}|\Big) \\
&= 2.
\end{aligned}
$$

In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f_n = h + \overline{g}_n$ where h and g_n are of the form (1.4).

Theorem 2.2. *Let $f_n = h + \overline{g}_n$ be given by* (1.4)*. Then $f_n \in \overline{SH}_s(n,\alpha)$ if and only if*

$$
\begin{aligned}
\sum_{k=1}^{\infty}\big((2k-1)^n(2k-1-\alpha)a_{2k-1} + (2k)^{n+1}a_{2k} \\
+(2k-1)^n(2k-1+\alpha)b_{2k-1} + (2k)^{n+1}b_{2k}\big) \le 1-\alpha. \qquad (2.4)
\end{aligned}
$$

Proof. Since $\overline{SH}_s(n,\alpha) \subset SH_s(n,\alpha)$, we only need to prove the (only if) part of the theorem. To this end, for functions f_n of the form (1.4), we notice that the condition

$$
\operatorname{Re}\left(\frac{2D^{n+1}f_n(z)}{D^n f_n(z) - D^n f_n(-z)}\right) \ge \alpha
$$

is equivalent to

$$
\operatorname{Re}\left\{\frac{2(1-\alpha)z - \sum_{k=2}^{\infty} k^n\big(2k-\alpha+(-1)^k\alpha\big)a_k z^k - (-1)^{2n}\sum_{k=1}^{\infty} k^n\big(2k+\alpha-(-1)^k\alpha\big)b_k\overline{z}_k}{2z - \sum_{k=2}^{\infty} k^n\big(1-(-1)^k\big)a_k z^k + (-1)^{2n}\sum_{k=1}^{\infty} k^n(1-(-1)^k)b_k\overline{z}^k}\right\} \ge 0
$$

which implies

$$
\begin{aligned}
\operatorname{Re}\Bigg\{&\frac{(1-\alpha)z - \sum_{k=2}^{\infty}\{(2k-2)^{n+1}a_{2k-2}z^{2k-2} + (2k-1)^n(2k-1-\alpha)a_{2k-1}z^{2k-1}\}}{z - \sum_{k=2}^{\infty}(2k-1)^n a_{2k-1}z^{2k-1} + (-1)^{2n}\sum_{k=1}^{\infty}(2k-1)^n b_{2k-1}\overline{z}^{2k-1}} \\
&- \frac{(-1)^{2n}\sum_{k=1}^{\infty}\{(2k)^{n+1}b_{2k}\overline{z}^{2k} + (2k-1)^n(2k-1+\alpha)b_{2k-1}\overline{z}^{2k-1}\}}{z - \sum_{k=2}^{\infty}(2k-1)^n a_{2k-1}z^{2k-1} + (-1)^{2n}\sum_{k=1}^{\infty}(2k-1)^n b_{2k-1}\overline{z}^{2k-1}}\Bigg\} \ge 0. \qquad (2.5)
\end{aligned}
$$

The above required condition (2.5) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0 \le z = r < 1$, we must have

$$
\begin{aligned}
&\frac{1-\alpha - \sum_{k=2}^{\infty}\big((2k-2)^{n+1}a_{2k-2}r^{2k-3} + (2k-1)^n(2k-1-\alpha)a_{2k-1}r^{2k-2}\big)}{1 - \sum_{k=2}^{\infty}(2k-1)^n a_{2k-1}r^{2k-2} + \sum_{k=1}^{\infty}(2k-1)^n b_{2k-1}r^{2k-2}} \\
&\quad - \frac{\sum_{k=1}^{\infty}\big((2k)^{n+1}b_{2k}r^{2k-1} + (2k-1)^n(2k-1+\alpha)b_{2k-1}r^{2k-2}\big)}{1 - \sum_{k=2}^{\infty}(2k-1)^n a_{2k-1}r^{2k-2} + \sum_{k=1}^{\infty}(2k-1)^n b_{2k-1}r^{2k-2}} \ge 0. \qquad (2.6)
\end{aligned}
$$

If the condition (2.4) does not hold, then the number in (2.6) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0,1)$, for which the quotient in (2.6) is negative. This contradicts the required condition for $f_n \in \overline{SH}_s(n,\alpha)$ and so the proof is complete. ∎

The following theorem gives the distortion bounds for functions in $\overline{SH}_s(n,\alpha)$ which yields a covering result for this class.

Theorem 2.3. *Let $f_n \in \overline{SH}_s(n,\alpha)$. Then for $|z| = r < 1$,*

$$
\begin{aligned}
(1-b_1)r - \frac{1}{2^n}\left(\frac{(1-\alpha)}{2} - \frac{(1+\alpha)}{2}b_1\right)r^2 &\le |f_n(z)| \\
&\le (1+b_1)r + \frac{1}{2^n}\left(\frac{(1-\alpha)}{2} - \frac{(1+\alpha)}{2}b_1\right)r^2.
\end{aligned}
$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f_n \in \overline{SH}_s(n,\alpha)$. Taking the absolute value of f_n, we obtain

$$\begin{aligned} |f_n(z)| &\leq (1+b_1)r + \sum_{k=2}^{\infty}(a_k+b_k)r^k \\ &= (1+b_1)r + \sum_{k=2}^{\infty}(a_k+b_k)r^2 \\ &\leq (1+b_1)r + \frac{1-\alpha}{2^{n+1}} \sum_{k=2}^{\infty}\left(\frac{2^{n+1}}{1-\alpha}a_k + \frac{2^{n+1}}{1-\alpha}b_k\right) r^2 \\ &\leq (1+b_1)r + \frac{1-\alpha}{2^{n+1}}\left(1-\frac{1+\alpha}{1-\alpha}b_1\right) r^2 \\ &= (1+b_1)r + \frac{1}{2^{n+1}}\big((1-\alpha)-(1+\alpha)b_1\big)r^2. \end{aligned}$$

∎

The following covering result follows from the left hand inequality in Theorem 2.3.

Corollary 2.4. *Let f_n of the form* (1.4) *be so that $f_n \in \overline{SH}_s(n,\alpha)$. Then*

$$\left\{\omega : |w| < \frac{2^{n+1}-1+\alpha}{2^{n+1}} - \frac{2^{n+1}-1+\alpha}{2^{n+1}}b_1\right\} \subset f_n(U).$$

For our next theorem, we need to define the convolution of two harmonic functions of the form $f_n(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} b_k \overline{z}^k$ and $F_n(z) = z - \sum_{k-2}^{\infty} A_k z^k + (-1)^n \sum_{k=1}^{\infty} B_k \overline{z}^k$ as

$$\begin{aligned} (f_n * F_n)(z) &= f_n(z) * F_n(z) \\ &= z - \sum_{k=2}^{\infty} a_k A_k z^k + (-1)^k \sum_{k=1}^{\infty} b_k B_k \overline{z}^k \end{aligned} \tag{2.7}$$

to prove that the class $\overline{SH}_s(n,\alpha)$ is closed under convolution.

Theorem 2.5. *For $0 \leq \beta \leq \alpha < 1$, let $f_n \in \overline{SH}_s(n,\alpha)$ and $F_n \in \overline{SH}_s(n,\beta)$. Then the convolution $f_n * F_n \in \overline{SH}_s(n,\alpha) \subset \overline{SH}_s(n,\beta)$.*

Proof. For f_n and F_n as in Theorem 2.5. Then the convolution $f_n * F_n$ is given by (2.7). We wish to show that the coefficients of $f_n * F_n$ satisfy the required condition given in Theorem 2.2. For $F_n \in \overline{SH}_s(n,\alpha)$, we note that $|A_k| \leq 1$ and $|B_k| \leq 1$. Now, for $f_n * F_n$ we obtain

$$\begin{aligned} &\sum_{k=2}^{\infty}\left(\frac{(2k-2)^{n+1}}{1-\beta}a_{2k-2}A_{2k-2} + \frac{(2k-1)^n(2k-1-\beta)}{1-\beta}a_{2k-1}A_{2k-1}\right) \\ &\quad + \sum_{k=1}^{\infty}\left(\frac{(2k)^{n+1}}{1-\beta}b_{2k}B_{2k} + \frac{(2k-1)^n(2k-1+\beta)}{1-\beta}b_{2k-1}B_{2k-1}\right) \\ &\quad \leq \sum_{k=2}^{\infty}\left(\frac{(2k-2)^{n+1}}{1-\beta}a_{2k-2} + \frac{(2k-1)^n(2k-1-\beta)}{1-\beta}a_{2k-1}\right) \end{aligned}$$

$$\begin{aligned}
&+\sum_{k=1}^{\infty}\left(\frac{(2k)^{n+1}}{1-\beta}b_{2k}+\frac{(2k-1)^n(2k-1+\beta)}{1-\beta}b_{2k-1}\right)\\
&\leq\sum_{k=2}^{\infty}\left(\frac{(2k-2)^{n+1}}{1-\alpha}a_{2k-2}+\frac{(2k-1)^n(2k-1-\alpha)}{1-\alpha}a_{2k-1}\right)\\
&+\sum_{k=2}^{\infty}\left(\frac{(2k)^{n+1}}{1-\alpha}b_{2k}+\frac{(2k-1)^n(2k-1+\alpha)}{1-\alpha}a_{2k-1}\right)\\
&\leq 1
\end{aligned}$$

since $0\leq\beta\leq\alpha<1$ and $f_n\in\overline{SH}_s(n,\alpha)$. Therefore $f_n*F_n\in\overline{SH}_s(n,\alpha)\subset\overline{SH}_s(n,\beta)$. ∎

Now, we discuss the convex combinations of the class $\overline{SH}_s(n,\alpha)$.

Theorem 2.6. *The class $\overline{SH}_s(n,\alpha)$ is closed under convex combinations.*

Proof. For $i=1,2,\ldots,$ suppose that $f_{n_i}\in\overline{SH}_s(n,\alpha)$, where $f_{n_i}(z)=z-\sum_{k=2}^{\infty}a_{i_k}z^k+(-1)^n\sum_{k=1}^{\infty}b_{i_k}\overline{z}^k$. Then, by Theorem 2.2,

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left(\frac{(2k-2)^{n+1}}{1-\alpha}a_{i_{2k-2}}+\frac{(2k-1)^n(2k-1-\alpha)}{1-\alpha}a_{i_{2k-1}}\right)\\
&\quad+\sum_{k=2}^{\infty}\left(\frac{(2k)^{n+1}}{1-\alpha}b_{i_{2k}}+\frac{(2k-1)^n(2k-1+\alpha)}{1-\alpha}b_{i_{2k-1}}\right)\\
&\quad\leq 1.
\end{aligned}\tag{2.8}$$

For $\sum_{i=1}^{\infty}t_i=1,\ 0\leq t_i\leq 1$, the convex combination of f_{n_i} may be written as

$$\sum_{i=1}^{\infty}t_if_{n_i}(z)=z-\sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty}t_ia_{i_k}\right)z^k+(-1)^n\sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty}t_ib_{i_k}\right)\overline{z}^k.$$

Then, by (2.8)

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left\{\frac{(2k-2)^{n+1}}{1-\alpha}\left(\sum_{i=1}^{\infty}t_ia_{i_{2k-2}}\right)+\frac{(2k-1)^n(2k-1-\alpha)}{1-\alpha}\left(\sum_{i=1}^{\infty}t_ia_{i_{2k-1}}\right)\right\}\\
&\quad+\sum_{k=1}^{\infty}\left\{\frac{(2k)^{n+1}}{1-\alpha}\left(\sum_{i=1}^{\infty}t_ib_{i_{2k}}\right)+\frac{(2k-1)^n(2k-1+\alpha)}{1-\alpha}\left(\sum_{i=1}^{\infty}t_ib_{i_{2k-1}}\right)\right\}\\
&\quad=\sum_{i=1}^{\infty}t_i\left(\sum_{k=2}^{\infty}\left\{\frac{(2k-2)^{n+1}}{1-\alpha}a_{i_{2k-2}}+\frac{(2k-1)^n(2k-1-\alpha)}{1-\alpha}a_{i_{2k-1}}\right\}\right.\\
&\quad+\left.\sum_{k=1}^{\infty}\left\{\frac{(2k)^{n+1}}{1-\alpha}b_{i_{2k}}+\frac{(2k-1)^n(2k-1+\alpha)}{1-\alpha}b_{i_{2k-1}}\right\}\right)\\
&\quad\leq\sum_{i=1}^{\infty}t_i=1,
\end{aligned}$$

and therefore $\sum_{i=1}^{\infty} t_i f_{n_i} \in \overline{SH}_s(n,\alpha)$. ∎

Next, we determine the extreme points of the closed convex hulls of $\overline{SH}_s(n,\alpha)$ denoted by clco $\overline{SH}_s(n,\alpha)$.

Theorem 2.7. *Let f_n be given by (1.4). Then $f_n \in \overline{SH}_s(n,\alpha)$ if and only if*

$$f_n(z) = \sum_{k=1}^{\infty}\Big[\big(X_{2k-1}h_{2k-1}(z) + X_{2k}h_{2k}(z)\big) + \big(Y_{2k-1}g_{n_{2k-1}}(z) + Y_{2k}g_{n_{2k}}(z)\big)\Big] \tag{2.9}$$

where

$$h_1(z) = z, h_{2k-1}(z) = z - \frac{1-\alpha}{(2k-1)^n(2k-1-\alpha)}z^{2k-1}, \quad k = 2,3,\dots$$
$$h_{2k-2}(z) = z - \frac{1-\alpha}{(2k-2)^{n+1}}z^{2k-2}, \quad k = 2,3,\dots$$
$$g_{n_{2k-1}} = z + (-1)^n\frac{1-\alpha}{(2k-1)^n(2k-1+\alpha)}\overline{z}^{2k-1}, \quad k = 1,2,\dots$$
$$g_{n_{2k}} = z + (-1)^n\frac{1-\alpha}{(2k)^{n+1}}\overline{z}^{2k}, \quad k = 1,2,\dots$$
$$\sum_{k=1}^{\infty}\big[(X_{2k-1} + X_{2k}) + (Y_{2k-1} + Y_{2k})\big] = 1, \quad X_k \ge 0,\ Y_k \ge 0,$$

In particular, the extreme points of $\overline{SH}_s(n,\alpha)$ are $\{h_{2k-1}\}, \{h_{2k-2}\}, \{g_{n_{2k-1}}\}$ and $\{g_{n_{2k}}\}$.

Proof. For functions f_n of the form (2.9), we have

$$\begin{aligned} f_n &= \sum_{k=1}^{\infty}\big((X_{2k-1} + X_{2k})z + (Y_{2k-1} + Y_{2k})z\big) \\ &- \sum_{k=2}^{\infty}\left(\frac{1-\alpha}{(2k-2)^{n+1}}X_{2k-2}z^{2k-2} + \frac{1-\alpha}{(2k-1)^n(2k-1-\alpha)}X_{2k-1}z^{2k-1}\right) \\ &+ (-1)^n\sum_{k=1}^{\infty}\left(\frac{1-\alpha}{(2k)^{n+1}}Y_{2k}\overline{z}^{2k} + \frac{1-\alpha}{(2k-1)^n(2k-1+\alpha)}Y_{2k-1}\overline{z}^{2k-1}\right). \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=2}^{\infty}\left(\frac{(2k-2)^{n+1}}{1-\alpha}a_{2k-2} + \frac{(2k-1)^n(2k-1-\alpha)}{1-\alpha}a_{2k-1}\right) \\ &\quad + \sum_{k=1}^{\infty}\left(\frac{(2k)^{n+1}}{1-\alpha}b_{2k} + \frac{(2k-1)^n(2k-1+\alpha)}{1-\alpha}b_{2k-1}\right) \\ &\quad = \sum_{k=2}^{\infty}(X_{2k-2} + X_{2k-1}) + \sum_{k=1}^{\infty}(Y_{2k-1} + Y_{2k}) \\ &\quad = 1 - X_1 \le 1 \end{aligned}$$

and so $f_n \in \text{clco}\,\overline{SH}_s(n,\alpha)$.

Conversely, suppose that $f_n \in \text{clco}\,\overline{SH}_s(n,\alpha)$. Setting

$$X_{2k-1} = \frac{(2k-1)^n(2k-1-\alpha)}{1-\alpha}a_{2k-1}, X_{2k-2} = \frac{(2k-2)^{n+1}}{1-\alpha}a_{2k-2}, \quad k = 2, 3\ldots$$
$$Y_{2k-1} = \frac{(2k-1)^n(2k-1+\alpha)}{1-\alpha}b_{2k-1}, Y_{2k} = \frac{(2k)^{n+1}}{1-\alpha}b_{2k}, \quad k = 1, 2\ldots$$

where $\sum_{k=1}^{\infty}\left[(X_{2k-1}+X_{2k})+(Y_{2k-1}+Y_{2k})\right] = 1$, we obtain

$$f_n(z) = \sum_{k=1}^{\infty}\Big[\big(X_{2k-1}h_{2k-1}(z)+X_{2k}h_{2k}(z)\big)+\big(Y_{2k-1}g_{n_{2k-1}}(z)+Y_{2k}g_{n_{2k}}(z)\big)\Big]$$

as required. ∎

Finally, we study properties of an integral operator.

Theorem 2.8. *Let $f_n \in \overline{SH}_s(n,\alpha)$ and let c be a real number such that $c > -1$. Then the function $F_n(z)$ defined by*

$$F_n(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f_n(t)dt \tag{2.10}$$

belongs to the class $\overline{SH}_s(n,\alpha)$.

Proof. From the representation of $F_n(z)$, it follows that

$$\begin{aligned}
F_n(z) &= \frac{c+1}{z^c}\int_0^z t^{c-1}\left\{h(t)+\overline{g}(t)\right\}dt \\
&= \frac{c+1}{z^c}\left(\int_0^z t^{c-1}h(t)dt + \overline{\int_0^z t^{c-1}g(t)dt}\right) \\
&= \frac{c+1}{z^c}\left(\int_0^z t^{c-1}\left(t-\sum_{k=2}^{\infty}a_kt^k\right)dt\right. \\
&\quad \left.+(-1)^n\overline{\int_0^z t^{c-1}\left(\sum_{k=1}^{\infty}b_kt^k\right)dt}\right) \\
&= z-\sum_{k=2}^{\infty}A_kz^k+(-1)^n\sum_{k=1}^{\infty}B_k\overline{z}^k,
\end{aligned}$$

where $A_k = \frac{c+1}{c+k}a_k, \quad B_k = \frac{c+1}{c+k}b_k$. Therefore,

$$\begin{aligned}
&\sum_{k=2}^{\infty}\big((2k-2)^{n+1}A_{2k-2}+(2k-1)^n(2k-1-\alpha)A_{2k-1}\big) \\
&\qquad +\sum_{k=1}^{\infty}\big((2k)^{n+1}B_{2k}+(2k-1)^n(2k-1+\alpha)B_{2k-1}\big)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=2}^{\infty}\left\{(2k-2)^{n+1}\left(\frac{c+1}{c+2k-2}\right)a_{2k-2} + (2k-1)^n(2k-1-\alpha)\left(\frac{c+1}{c+2k-1}\right)a_{2k-1}\right\}\\
&+ \sum_{k=1}^{\infty}\left\{(2k)^{n+1}\left(\frac{c+1}{c+2k}\right)b_{2k} + (2k-1)^n(2k-1+\alpha)\left(\frac{c+1}{c+2k-1}\right)b_{2k-1}\right\}\\
&\le \sum_{k=2}^{\infty}\big\{(2k-2)^{n+1}a_{2k-2} + (2k-1)^n(2k-1-\alpha)a_{2k-1}\big\}\\
&+ \sum_{k=1}^{\infty}\big\{(2k)^{n+1}b_{2k} + (2k-1)^n(2k-1+\alpha)b_{2k-1}\big\}\\
&\le 1-\alpha.
\end{aligned}$$

Since $f_n(z) \in \overline{SH}_s(n,\alpha)$, therefore by Theorem 2.2 $F_n(z) \in \overline{SH}_s(n,\alpha)$. ∎

Theorem 2.9. *Let c be a real number such that $c > -1$. If $F_n(z) \in \overline{SH}_s(n,\alpha)$, then the function $f_n(z)$ defined by (2.10) is univalent in $|z| < R^*$, where $R^* = \min(r_1, r_2, r_3, r_4)$*

$$r_1 = \inf_k \left[\frac{(2k-2)^n(c+1)}{(1-\alpha)(c+2k-2)}\right]^{1/2k-3},$$

$$r_2 = \inf_k \left[\frac{(2k-1)^{n-1}(2k-1-\alpha)(c+1)}{(1-\alpha)(c+k-1)}\right]^{1/2k-2}, \quad k \ge 2$$

and

$$r_3 = \inf_k \left[\frac{(2k-1)^{n-1}(2k-1+\alpha)(c+1)}{(1-\alpha)(c-2k+1)}\right]^{1/2k-2},$$

and

$$r_4 = \inf_k \left[\frac{(2k)^n(c+1)}{(1-\alpha)(c-2k)}\right]^{1/2k-1}, \quad k \ge 1\,.$$

The result is sharp.

Proof. Let $F_n(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} b_k \overline{z}^k, \quad (a_k \ge 0, b_k \ge 0)$. It follows from (2.10) that

$$\begin{aligned}
f(z) &= \frac{z^{1-c}(z^c F_n(z))'}{c+1}, \qquad (c > -1)\\
&= \frac{z^{1-c}\big(cz^{c-1}F_n(z) + z^c F_n'(z)\big)}{c+1}\\
&= \frac{cF_n(z) + zF_n'(z)}{c+1}\\
&= \frac{1}{c+1}\left\{cF_n(z) + z\left(1 - \sum_{k=2}^{\infty} a_k z^{k-1} - (-1)^n \sum_{k=1} \overline{k b_k z^{k-1}}\right)\right\}\\
&= \frac{1}{c+1}\left\{cz - \sum_{k=2}^{\infty} c a_k z^k + (-1)^n \sum_{k=1}^{\infty} c b_k \overline{z}^k\right.
\end{aligned}$$

$$+ z - \sum_{k=2}^{\infty} k a_k z^k - (-1)^n \sum_{k=1}^{\infty} k b_k \overline{z}^k \Bigg\}$$

$$\begin{aligned} f_n &= z - \sum_{k=2}^{\infty} \frac{c+k}{c+1} a_k z^k + (-1)^n \sum_{k=1}^{\infty} \frac{c-k}{c+1} b_k \overline{z}^k \\ &= z - \sum_{k=2}^{\infty} \left(\frac{c+2k-2}{c+1} a_{2k-2} z^{2k-2} + \frac{c+k-1}{c+1} a_{2k-1} z^{2k-1} \right) \\ &+ (-1)^n \sum_{k=1}^{\infty} \left(\frac{c-2k+1}{c+1} b_{2k-1} \overline{z}^{2k-1} \frac{c-2k}{c+1} b_{2k} \overline{z}^{2k} \right) \\ &= H + \overline{G}_n. \end{aligned}$$

In order to obtain the required result it suffices to show that $|f_n'(z) - 1| < 1$ in $|z| < R^*$.

Now $|f_n'z) - 1| < 1$ if $|H' - \overline{G}_n' - 1| < 1$,

$$\Rightarrow \left| \sum_{k=2}^{\infty} \left(\frac{c+2k-2}{c+1} (2k-2) a_{2k-2} z^{2k-3} + \frac{c+k-1}{c+1} (2k-1) a_{2k-1} z^{2k-2} \right) \right.$$
$$- (-1)^n \sum_{k=1}^{\infty} \left(\frac{c-2k+1}{c+1} (2k-1) b_{2k-1} \overline{z}^{2k-2} + \frac{c-2k}{c+1} (2k) b_{2k} \overline{z}^{2k-1} \right)$$

$$\begin{aligned} &\leq \sum_{k=2}^{\infty} \left| \frac{c+2k-2}{c+1} (2k-2) a_{2k-2} z^{2k-3} + \frac{c+k-1}{c+1} (2k-1) a_{2k-1} z^{2k-1} \right| \\ &+ \sum_{k=1}^{\infty} \left| \frac{c-2k+1}{c+1} (2k-1) b_{2k-1} \overline{z}^{2k-2} + \frac{c-2k}{c+1} (2k) b_{2k} \overline{z}^{2k-1} \right| < 1. \end{aligned} \tag{2.11}$$

According to Theorem 2.2, we have

$$\begin{aligned} &\sum_{k=2}^{\infty} \left(\frac{(2k-2)^{n+1}}{1-\alpha} a_{2k-2} + \frac{(2k-1)^n (2k-1-\alpha)}{1-\alpha} a_{2k-1} \right) \\ &\qquad + \sum_{k=1}^{\infty} \left(\frac{(2k-1)^n (2k-1+\alpha)}{1-\alpha} b_{2k-1} + \frac{(2k)^{n+1}}{1-\alpha} b_{2k} \right) \leq 1. \end{aligned} \tag{2.12}$$

Hence (2.11) will be true if

$$\begin{aligned} &\frac{c+2k-2}{c+1} (2k-2) a_{2k-2} |z|^{2k-3} + \frac{c+k-1}{c+1} (2k-1) a_{2k-1} |z|^{2k-2} \\ &\qquad + \frac{c-2k+1}{c+1} (2k-1) b_{2k-1} |z|^{2k-2} + \frac{c-2k}{c+1} (2k) b_{2k} |z|^{2k-1} \\ &\qquad < \frac{(2k-2)^{n+1}}{1-\alpha} a_{2k-2} + \frac{(2k-1)^n (2k-1-\alpha)}{1-\alpha} a_{2k-1} \\ &\qquad + \frac{(2k-1)^n (2k-1+\alpha)}{1-\alpha} b_{2k-1} + \frac{(2k)^{n+1}}{1-\alpha} b_{2k} \end{aligned}$$

or if $|z| < R^* = \min(r_1, r_2, r_3, r_4)$ where

$$r_1 = \inf_k \left[\frac{(2k-2)^n(c+1)}{(1-\alpha)(c+2k-2)} \right]^{1/2k-3}$$

$$r_2 = \inf_k \left[\frac{(2k-1)^n(2k-1-\alpha)(c+1)}{(1-\alpha)(c+k-1)} \right]^{1/2k-2}, \quad k \geq 2$$

and

$$r_3 = \inf_k \left[\frac{(2k-1)^{n-1}(2k-1+\alpha)(c+1)}{(1-\alpha)(c-2k+1)} \right]^{1/2k-2}$$

$$r_4 = \inf_k \left[\frac{(2k)^n(c+1)}{(1-\alpha)(c-2k)} \right]^{1/2k-1}, \qquad k \geq 1.$$

Therefore, $f(z)$ is univalent in $|z| < R^*$. ∎

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