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A WALLIS TYPE INEQUALITY AND A DOUBLE INEQUALITY FOR PROBABILITY INTEGRAL

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ABSTRACT. In this short note, a Wallis type inequality with the best upper and lower bounds is established. As an application, a double inequality for the probability integral is found.

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1. INTRODUCTION

A double factorial $n!!$ is defined by

$$(2m)!! = \Pi_{i=1}^{m}(2i) \quad \text{and} \quad (2m-1)!! = \Pi_{i=1}^{m}(2i-1)$$

for given positive integer m. Let

$$P_n = \frac{(2n-1)!!}{(2n)!!}, \tag{1.1}$$

then

$$\frac{\sqrt{2}}{\sqrt{(2n+1)\pi}} < P_n < \frac{2}{\sqrt{(4n+1)\pi}} \tag{1.2}$$

for $n > 1$, which is called Wallis' inequality in [12, p. 96].

In [4, 5, 6, 7, 8, 9], the best lower and upper bounds for P_n were obtained:

$$\frac{1}{\sqrt{\pi(n+4/\pi-1)}} \le P_n < \frac{1}{\sqrt{\pi(n+1/4)}}. \tag{1.3}$$

For more information and recent developments on the Wallis' inequality, please refer to [11, 12, 15] and the references therein.

The first result of this short note is the following Wallis type inequality.

Theorem 1.1. *Let*

$$P_{n+1/2} = \frac{(2n)!!}{(2n+1)!!} \tag{1.4}$$

for $n \in \mathbb{N}$. *Then*

$$\frac{\sqrt{\pi}}{2\sqrt{n+9\pi/16-1}} \le P_{n+1/2} < \frac{\sqrt{\pi}}{2\sqrt{n+3/4}}. \tag{1.5}$$

The constants $9\pi/16 - 1$ *and* $3/4$ *in* (1.5) *are the best possible.*

The second result of this short note, as an application of inequalities (1.3) and (1.5), is the following double inequality for the probability integral.

Theorem 1.2. *For all natural number* n*,*

$$\frac{\sqrt{\pi}}{\sqrt{1+(9\pi/16-1)/n}} \le \int_{-\sqrt{n}}^{\sqrt{n}} e^{-x^2}\,\mathrm{d}x < \frac{\sqrt{\pi}}{\sqrt{1-3/4n}}. \tag{1.6}$$

In particular, taking $n \to \infty$ *in* (1.6) *leads to*

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}\,. \tag{1.7}$$

2. LEMMAS

The following lemmas are necessary for proving our main results.

Lemma 2.1. *For* $x > 0$*,*

$$\psi\left(x+\frac{1}{2}\right) - \psi(x) > \frac{2x+1}{x(4x+1)}. \tag{2.1}$$

As $x \to \infty$*,*

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\left(\frac{1}{x^2}\right). \tag{2.2}$$

Inequality (2.1) is given in [2, 3, 13] and the asymptotic expansion (2.2) can be found in [1, p. 257], [10] and [14, p. 378].

Lemma 2.2. *For* $x > -1$,

$$\frac{\Gamma(x+3/2)}{\Gamma(x+1)} < \frac{2(x+1)}{\sqrt{4x+5}}. \tag{2.3}$$

Proof. For positive real number x, let

$$f(x) = \ln[2(x+1)] + \ln\Gamma(x+1) - \frac{1}{2}\ln(4x+5) - \ln\Gamma\left(x+\frac{3}{2}\right).$$

Differentiation of $f(x)$ gives us

$$f'(x) = \frac{1}{x+1} - \frac{2}{4x+5} - \psi\left(x+\frac{3}{2}\right) + \psi(x+1).$$

Replacing x by $x+1$ in (2.1) yields

$$\psi\left(x+\frac{3}{2}\right) - \psi(x+1) > \frac{2x+3}{(x+1)(4x+5)} \tag{2.4}$$

for $x > -1$. Utilizing (2.4), we obtain

$$f'(x) < \frac{1}{x+1} - \frac{2}{4x+5} - \frac{2x+3}{(x+1)(4x+5)} = 0.$$

Therefore, $f(x)$ is strictly decreasing in $(0,\infty)$, and $f(x) > \lim_{x\to\infty} f(x) = 0$, which leads to inequality (2.3). ∎

Lemma 2.3. *The sequence*

$$Q_n \triangleq \left[\frac{\Gamma(n+3/2)}{\Gamma(n+1)}\right]^2 - n \tag{2.5}$$

is strictly decreasing.

Proof. Using the well known formulas

$$\Gamma(n+1) = n!, \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} \quad \text{and} \quad 2^n n! = (2n)!! \tag{2.6}$$

reveals that

$$\begin{aligned} Q_{n+1} - Q_n &= \left[\frac{\Gamma(n+5/2)}{\Gamma(n+2)}\right]^2 - (n+1) - \left[\frac{\Gamma(n+3/2)}{\Gamma(n+1)}\right]^2 + n \\ &= \left[\frac{\frac{(2n+3)!!}{2^{n+2}}\sqrt{\pi}}{(n+1)!}\right]^2 - \left[\frac{\frac{(2n+1)!!}{2^{n+1}}\sqrt{\pi}}{n!}\right]^2 - 1 \\ &= \left[\frac{(2n+1)!!}{(2n+2)!!}\right]^2 \frac{(4n+5)\pi}{4} - 1. \end{aligned} \tag{2.7}$$

Replacing n by $n+1$ in (1.3) yields

$$\frac{1}{\sqrt{\pi(n+4/\pi)}} \le \frac{(2n+1)!!}{(2n+2)!!} < \frac{1}{\sqrt{\pi(n+5/4)}},$$

that is,

$$\frac{1}{\pi(n+4/\pi)} \le \left[\frac{(2n+1)!!}{(2n+2)!!}\right]^2 < \frac{1}{\pi(n+5/4)}.$$

Hence,

$$Q_{n+1} - Q_n < \frac{1}{\pi(n+5/4)} \cdot \frac{(4n+5)\pi}{4} - 1 = 0.$$

As a result, the sequence Q_n is strictly decreasing. ∎

Lemma 2.4. *For natural number* n*,*

$$\int_{-\sqrt{n}}^{\sqrt{n}} \left(1 - \frac{x^2}{n}\right)^n \mathrm{d}\,x = 2\sqrt{n} \int_0^{\pi/2} \sin^{2n+1} x \,\mathrm{d}\,x = 2\sqrt{n}\, P_{n+1/2} \tag{2.8}$$

and

$$\int_{-\sqrt{n}}^{\sqrt{n}} \left(1 + \frac{x^2}{n}\right)^{-n} \mathrm{d}\,x = \sqrt{n} \int_{\pi/4}^{3\pi/4} \sin^{2n-2} x \,\mathrm{d}\,x < \pi\sqrt{n}\, P_{n-1}. \tag{2.9}$$

Proof. Letting $x = \sqrt{n}\,\cos t$ in the left side of (2.8) yields

$$\int_{-\sqrt{n}}^{\sqrt{n}} \left(1 - \frac{x^2}{n}\right)^n \mathrm{d}\,x = -\sqrt{n} \int_{\pi}^{0} \sin^{2n+1} t \,\mathrm{d}\,t = 2\sqrt{n} \int_0^{\pi/2} \sin^{2n+1} t \,\mathrm{d}\,t.$$

Using Wallis sine formula gives

$$\int_0^{\pi/2} \sin^{2n+1} x \,\mathrm{d}\,x = \frac{(2n)!!}{(2n+1)!!} = P_{n+1/2}.$$

Letting $x = \sqrt{n}\,\tan t$ in the left side of (2.9) shows

$$\begin{aligned}\int_{-\sqrt{n}}^{\sqrt{n}} \left(1 + \frac{x^2}{n}\right)^{-n} \mathrm{d}\,x &= \sqrt{n} \int_{-\pi/4}^{\pi/4} \cos^{2n-2} t \,\mathrm{d}\,t \\ &< 2\sqrt{n} \int_0^{\pi/2} \sin^{2n-2} t \,\mathrm{d}\,t = \pi\sqrt{n}\, P_{n-1}.\end{aligned}$$

The proof of Lemma 2.4 is complete. ∎

Lemma 2.5. *For* $|x| < \sqrt{n}$*,*

$$\left(1 - \frac{x^2}{n}\right)^n \le e^{-x^2} \le \left(1 + \frac{x^2}{n}\right)^{-n}. \tag{2.10}$$

Proof. It was given in [12, p. 289] that for $x < 1$,

$$1 + x \le e^x \le \frac{1}{1-x}. \tag{2.11}$$

Replacing x by $-x^2/n > -1$ and taking n-times power on the both sides of inequality (2.11) leads to (2.10). The proof is complete. ∎

3. Proofs of Theorems

Proof of Theorem 1.1. From Lemma 2.3, it follows that

$$\lim_{n\to\infty} Q_n < Q_n \le Q_1 = \left[\frac{\Gamma(5/2)}{\Gamma(2)}\right]^2 - 1 = \frac{9}{16}\pi - 1.$$

Using the asymptotic formula (2.2), we conclude from

$$Q_n = n\left[n^{-1/2}\frac{\Gamma(n+3/2)}{\Gamma(n+1)} - 1\right]\left[n^{-1/2}\frac{\Gamma(n+3/2)}{\Gamma(n+1)} + 1\right]$$

that

$$\lim_{n\to\infty} Q_n = \lim_{n\to\infty} n\left[\frac{(1/2)(3/2)}{2n} + O\left(\frac{1}{n^2}\right)\right] \times \quad \left[2 + \frac{(1/2)(3/2)}{2n} + O\left(\frac{1}{n^2}\right)\right] = \frac{3}{4}.$$

Thus, Theorem 1.1 follows. ∎

Proof of Theorem 1.2. By formula (2.8) and inequality (2.10), we obtain

$$\int_{-\sqrt{n}}^{\sqrt{n}} e^{-x^2}\,\mathrm{d}x \ge \int_{-\sqrt{n}}^{\sqrt{n}} \left(1-\frac{x^2}{n}\right)^n \mathrm{d}x = 2\sqrt{n}\,P_{n+1/2}$$

and

$$\int_{-\sqrt{n}}^{\sqrt{n}} e^{-x^2}\,\mathrm{d}x \le \int_{-\sqrt{n}}^{\sqrt{n}} \left(1+\frac{x^2}{n}\right)^{-n} \mathrm{d}x < \pi\sqrt{n}\,P_{n-1}.$$

By virtue of the right hand side of (1.3) and the left hand side of (1.5), inequality (1.6) is proved. The proof of Theorem 1.2 is complete. ∎

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