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ON THE GENERALIZED INVERSE $A^{(2)}_{T,S}$ OVER INTEGRAL DOMAINS

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ABSTRACT. In this paper, we study further the generalized inverse $A^{(2)}_{T,S}$ of a matrix A over an integral domain. We give firstly some necessary and sufficient conditions for the existence of the generalized inverse $A^{(2)}_{T,S}$, an explicit expression for the elements of the generalized inverse $A^{(2)}_{T,S}$ and an explicit expression for the generalized inverse $A^{(2)}_{T,S}$, which reduces to the $\{1\}$ inverse. Secondly, we verify that the group inverse, the Drazin inverse, the Moore-Penrose inverse and the weighted Moore-Penrose inverse are identical with the generalized inverse $A^{(2)}_{R(G),N(G)}$ for an appropriate matrix G, respectively, and then we unify the conditions for the existence and the expression for the elements of the weighted Moore-Penrose inverse, the Moore-Penrose inverse, the Drazin inverse and the group inverse over an integral domain. Thirdly, as a simple application, we give the relation between some rank equation and the existence of the generalized inverse $A^{(2)}_{T,S}$, and a method to compute the generalized inverse $A^{(2)}_{T,S}$. Finally, we give an example of evaluating the elements of $A^{(2)}_{T,S}$ without calculating $A^{(2)}_{T,S}$.

Key words and phrases: Generalized inverse $A^{(2)}_{T,S}$, Rank equation, Integral domain.

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1. INTRODUCTION

Over complex number fields, there are a great number of the results about the generalized inverse $A^{(2)}_{T,S}$ of a matrix A, for example, [2, 11, 4, 5, 12]. These results reveal that the known generalized inverses, such as Moore-Penrose inverses, Drazin inverses and group inverses and so on, are all the generalized inverse $A^{(2)}_{T,S}$, that some algorithms for the known generalized inverses are unified by the generalized inverse $A^{(2)}_{T,S}$, and that some characteristics of the generalized inverse $A^{(2)}_{T,S}$ are common characteristics of the known generalized inverses. Over integral domains, there are many results for the existence and computing formulas about the Moore-Penrose inverse, the Drazin inverse and the group inverse (see [3, 7, 8, 9, 13]). In [14], we gave an explicit expression for the generalized inverse $A^{(2)}_{T,S}$ of a matrix A over an integral domain, which reduces to the group inverse, and proved that over an integral domain the Moore-Penrose inverse, the Drazin inverse and the group inverse are all the generalized inverse $A^{(2)}_{T,S}$.

In this paper, we study further the generalized inverse $A^{(2)}_{T,S}$ of a matrix A over an integral domain. We give firstly some necessary and sufficient conditions for the existence of the generalized inverse $A^{(2)}_{T,S}$, an explicit expression for the elements of the generalized inverse $A^{(2)}_{T,S}$ and another explicit expression for the generalized inverse $A^{(2)}_{T,S}$, which reduces to the $\{1\}$ inverse. Secondly, we verify that the group inverse, the Drazin inverse and the Moore-Penrose inverse, and the weighted Moore-Penrose inverse are identical with the generalized inverse $A^{(2)}_{R(G),N(G)}$ for an appropriate matrix G, respectively, and then we unify the conditions for the existence and the expression for the elements of the weighted Moore-Penrose inverse, the Moore-Penrose inverse, the Drazin inverse and the group inverse over an integral domain. Thirdly, as a simple application, we give the relation between some rank equation and the existence of the generalized inverse with prescribed image and kernel, and, consequently, obtain a method to compute the generalized inverse $A^{(2)}_{T,S}$. Finally, we give an example of evaluating the elements of $A^{(2)}_{T,S}$ without calculating $A^{(2)}_{T,S}$ by using Theorem 2.3, in which we compute the generalized inverse $A^{(2)}_{T,S}$ by using Equation (3.25) in Corollary 3.18.

Throughout this paper, R denotes an integral domain unless otherwise mentioned, and $R^{m\times n}$ denotes the set of $m \times n$ matrices over R. Especially, the notation $R^m = R^{m\times 1}$. By 'a module' we mean 'a right R-module'. If S is a R-submodule of R-module M, then we write $S \subset M$.

We write R with an involution $a \to \overline{a}$ for the meaning of R with a function $a \to \overline{a}$ such that $\overline{a+b} = \overline{a} + \overline{b}$, $\overline{ab} = \overline{b}\,\overline{a}$, $\overline{0} = 0$ and $\overline{1} = 1$. For an $m \times n$ matrix $A = (a_{ij})$ over R, we write A^* for the matrix $\overline{A}^T = (\overline{a}_{ji})$.

Let $A \in R^{m\times n}$. We write the image of A by $R(A) = \{Ax \mid x \in R^n\}$ and the kernel of A by $N(A) = \{x \in R^n \mid Ax = 0\}$. It is obvious that $R(A) \subset R^m$ and $N(A) \subset R^n$. We denote the maximal order of a nonvanishing minor of A by $\rho(A)$, called the determinantal rank of A over R. Obviously, $\rho(AB) \le \min\{\rho(A), \rho(B)\}$ and

$$\rho(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}) = \rho(A) + \rho(B) \tag{1.1}$$

When R is a complex number field, $\rho(A) = \mathrm{rank}(A)$.

For integers $m \ge 1$ and $1 \le k \le m$, let $Q_{k,m}$ stand for the set $\{\alpha : \alpha = \{\alpha_1, \alpha_2, \dots, \alpha_k\}, 1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_k \le m\}$. If $A = (a_{ij})$ is an $m\times n$ matrix, $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_k\} \in Q_{k,m}$ and $\beta = \{\beta_1, \beta_2, \dots, \beta_l\} \in Q_{l,n}$, then $A_{\alpha,\beta}$, called the (α, β)-submatrix of A, stands for the $k \times l$ matrix whose (i, j)th element is the (α_i, β_j) element of A, especially if $|\beta| = n$ (or $|\alpha| = m$), $A_{\alpha *}$ (or$A_{*\beta}$) $= A_{\alpha\beta}$.

If $A = (a_{ij})$ is an $n \times n$ matrix, for fixed i and j, $\frac{\partial |A|}{\partial a_{ij}}$ stands for the coefficient of a_{ij} in the expansion of the determinant of A. It is evident from the Laplace Expansion Theorem that $|A| = \sum\limits_{j} a_{ij} \frac{\partial |A|}{\partial a_{ij}}$.

Moreover If $A = (a_{ij})$ is an $m \times n$ matrix and $A_{\alpha,\beta}$ is a submatrix of A, where $\alpha \in Q_{k,m}$ and $\beta \in Q_{k,n}$, then we define $\frac{\partial |A_{\alpha,\beta}|}{\partial a_{ij}}$ as above if $i \in \alpha$ and $j \in \beta$ and $\frac{\partial |A_{\alpha,\beta}|}{\partial a_{ij}} = 0$ if $i \notin \alpha$ or $j \notin \beta$.

Let $C_k(A)$ denote the kth compound matrix of an $m \times n$ matrix A over R, where $k \leq \min\{m, n\}$. It is easy to see that $C_k(AB) = C_k(A)C_k(B)$ and $\rho(C_{\rho(A)}(A)) = 1$.

Let A be an $m \times n$ matrix over R with an involution $a \longrightarrow \overline{a}$, M and N be invertible matrices of orders m and n over R, respectively, and consider following equations

$$\begin{array}{llcllrcl} (1) & AXA & = & A, & (2) & XAX & = & A, \\ (3) & (AX)^* & = & AX, & (4) & (XA)^* & = & XA, \\ (3M) & (MAX)^* & = & MAX, & (4N) & (NXA)^* & = & NXA. \end{array}$$

X is called a {1} inverse (or g-inverse) of A if X is satisfies (1), and denoted by $X = A^{(1)}$ and A is said to be regular. X is called a $\{2\}$-inverse of A if it satisfies (2), and denoted by $A^{(2)}$. X is called the Moore-Penrose inverse of A if it satisfies (1), (2), (3) and (4), and denoted by $A^{\dagger}$. X is called the weighted Moore-Penrose inverse of A (or generalized Moore-Penrose inverse of A with respect to M, N) if it satisfies (1), (2), $(3M)$ and $(4N)$, and denoted by $A^{\dagger}_{MN}$. (About the weighted Moore-Penrose inverse, one can see [6]).

An $n \times n$ matrix A over R is said to have Drazin inverse if for some positive integer k there exists a matrix X such that

$$(1^k) \quad A^kXA = A^k, \quad (2) \quad XAX = X, \quad (5) \quad AX = XA.$$

If X exists, then it is unique, and called the Drazin inverse of A and denoted by A_d. If k is the smallest positive integer such that X and A satisfy $(1^k), (2)$ and (5), then it is called the Drazin index and denoted by k=Ind(A). By using [9, Proposition 6.22], it is clear to get that Ind(A) is the smallest positive integer k satisfying $\rho(A^k) = \rho(A^{k+1})$. If $k = 1$, then X is called the group inverse of A and denoted by A_g.

Lemma 1.1. ([14, Theorem 1]) *Let A be an $m \times n$ matrix over a commutative ring R with identity, $T \subset R^n$ and $S \subset R^m$. Then the following are equivalent:*

(i) *There exists a unique $X \in R^{n \times m}$ such that*

$$XAX = X, \quad R(X) = T, \quad N(X) = S. \tag{1.2}$$

(ii) *$AT \oplus S = R^m$ and $N(A) \cap T = \{0\}$.*

An $n \times m$ matrix X over a commutative ring R is called the generalized inverse which is $\{2\}$ inverse of matrix A with prescribed image T and kernel S over a commutative ring R if it satisfies the equivalent conditions of Lemma 1.1, and denoted by $A^{(2)}_{T,S}$.

2. MAIN RESULTS

We begin with the following lemma.

Lemma 2.1. [15, Theorem 2] *Let A be a matrix over R. If $A^{(2)}_{T,S}$ exists and there exists a matrix G over R satisfying $R(G) = T$ and $N(G) = S$, then there exists a matrix W over R such that*

$$GAGW = G. \tag{2.1}$$

Proof. Suppose $A^{(2)}_{T,S}$ exists with $R(G) = T$ and $N(G) = S$ for a matrix G. Then $AR(G) \oplus N(G) = R^m$ and so there exists an epimorphism $R^m \to N(G) \to 0$. By [1, Theorem 8.1], $N(G)$ has a finite spanning set whose elements constitute a matrix, denoted by L. Thus $GL = 0$, and the columns of (AG, L) generate R^m, that is, there exists a matrix $\begin{pmatrix} W \\ W_1 \end{pmatrix}$ such that

$$AGW + LW_1 = I_m.$$

If we multiply with G the left side, then we have

$$GAGW = G.$$

∎

Next theorem is our main result.

Theorem 2.2. *Let A be an $m \times n$ matrix over an integral domain R, $T \subset R^n$ and $S \subset R^m$. If there exists an $n \times m$ matrix G with $r = \rho(G)$ over R such that $R(G) = T$ and $N(G) = S$, then the following statements are equivalent:*

(i) $A^{(2)}_{R(G),N(G)}$ *exists.*

(ii) $u = \sum\limits_{\gamma \in Q_{r,m}} |(AG)_{\gamma,\gamma}|$ *is a unit in R.*

(iii) $\rho(G) = \rho(GAG)$ *and GAG is regular.*

In case u is a unit,

$$A^{(2)}_{R(G),N(G)} = G(AG)_g = (GA)_g G \tag{2.2}$$

$$= G(GAG)^{(1)}G, \tag{2.3}$$

and the elements of $A^{(2)}_{R(G),N(G)}$ are

$$w_{ij} = u^{-1} \sum_{\alpha \in Q_{r,m}, \beta \in Q_{r,n}} |G_{\beta,\alpha}| \frac{\partial |A_{\alpha,\beta}|}{\partial a_{ji}}, \ i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m. \tag{2.4}$$

Proof. "(i)⟹(iii)" Suppose $A^{(2)}_{R(G),N(G)}$ exists. By Equation (2.1), we have

$$\rho(G) = \rho(GAGW) \leq \rho(GAG) \leq \rho(G),$$

and then $\rho(G) = \rho(GAG)$. Hence there exists a matrix M over the field of quotients of R such that $G = MGAG$. From this and Equation (2.1) we have

$$\begin{aligned} GAG &= (MGAG)AG &&= M(GAGW)(AG)^2 &&= GW(AG)^2 \\ &= (GAGW)W(AG)^2 &&= (GAG)(W^2A)(GAG). \end{aligned}$$

Hence GAG is regular.

"(iii)⟹(ii)" Since $\rho(G) = \rho(GAG)$, there exist matrices M and N over the field of quotients of R such that $G = MGAG = GAGN$. Let K be a {1} inverse of GAG. Then

$$AG = AGAGN = A(GAGKGAG)N = (AG)^2KG.$$

Hence $\rho(G) = \rho(GAG) \leq \rho(AG) \leq \rho((AG)^2) \leq \rho(G)$, and therefore $\rho(G) = \rho(AG) = \rho((AG)^2)$. Since

$$\begin{aligned} (AG)^2 &= A(GAG)K(GAG) &&= (AG)^2K(MGAG)(AG) \\ &= (AG)^2KM(GAGKGAG)(AG) &&= (AG)^2(KGKG)(AG)^2, \end{aligned}$$

we have that $(AG)^2$ is regular.

By [7, Theorem 5], $\sum\limits_{\gamma \in Q_{r,m}} |(AG)_{\gamma,\gamma}|$ is a unit in R.

"(ii)$\Longrightarrow$(i)"

$$Tr(C_r(AG)) = \sum_{\gamma \in Q_{r,m}} |(AG)_{\gamma,\gamma}| = u$$

is a unit in R, $C_r(AG)$ is a nonzero matrix. This implies that $\rho(AG) \geq r = \rho(G)$. However $\rho(AG) \leq \rho(G)$. Thus we obtain that $\rho(AG) = \rho(G) = r$.

Because of the invertibility of u, AG has the group inverse $(AG)_g$ by [7, Theorem 5].

Similarly, GA has the group inverse $(GA)_g$ because

$$Tr(C_r(GA)) = Tr(C_r(G)C_r(A)) = Tr(C_r(A)C_r(G)) = Tr(C_r(AG)).$$

Therefore, we have that

$$\begin{aligned} G(AG)_g &= G(AG)((AG)_g)^2 = ((GA)_g)^2(GA)^3G((AG)_g)^2 \\ &= ((GA)_g)^2G(AG)^3((AG)_g)^2 = ((GA)_g)^2G(AG) \\ &= (GA)_gG. \end{aligned} \tag{2.5}$$

Since $\rho(AG) = \rho(G)$, there exists a matrix M over the field of quotients of R such that $G = MAG$. So we have

$$G(AG)_gAG = MAG(AG)_gAG = MAG = G. \tag{2.6}$$

From Equations (2.5) and (2.6), we have that

$$\begin{aligned} N(G) &\subset N((GA)_gG) = N(G(AG)_g) \subset N(AG(AG)_g) \\ &= N((AG)_gAG) \subset N(G(AG)_gAG) = N(G) \end{aligned}$$

and

$$R(G) = R(G(AG)_gAG) \subset R(G(AG)_g) \subset R(G).$$

Thus, we have

$$N(G(AG)_g) = N(AG(AG)_g) = N(G) \tag{2.7}$$

and

$$R(G(AG)_g) = R(G). \tag{2.8}$$

Let $x \in N(A) \cap R(G)$. Then there exists a y such that $x = Gy$. So $AGy = 0$. By Equation (2.6),

$$x = Gy = G(AG)_gAGy = 0.$$

Thus,

$$N(A) \cap R(G) = \{0\}. \tag{2.9}$$

From the idempotent of $AG(AG)_g$ and Equations (2.7) and (2.8), by [1, Lemma 5.6] we have

$$\begin{aligned} R^m &= R(AG(AG)_g) \oplus N(AG(AG)_g) = AR(G(AG)_g) \oplus N(G) \\ &= AR(G) \oplus N(G). \end{aligned} \tag{2.10}$$

By Equations (2.9) and (2.10) and Lemma 1.1, $A^{(2)}_{R(G),N(G)}$ exists. Therefore, we reach (i).

Since $(G(AG)_g)A(G(AG)_g) = G(AG)_g$, we get Equation (2.2) by Equations (2.7), (2.8) and (2.5), and Lemma 1.1.

Now we shall verify Equation (2.3). Since $\rho(G) = \rho(GAG)$, there exist matrices M and N over the field of quotients of R such that $G = MGAG = GAGN$. From this and the regularity of GAG, we have

$$G = MGAG = M(GAG)(GAG)^{(1)}(GAG) = G(GAG)^{(1)}GAG \tag{2.11}$$

and

$$G = GAGN = (GAG)(GAG)^{(1)}(GAG)N = GAG(GAG)^{(1)}G. \tag{2.12}$$

From the above equations, we have that

$$R(G(GAG)^{(1)}G) \subset R(G) = R(G(GAG)^{(1)}GAG) \subset R(G(GAG)^{(1)}G)$$

and

$$N(G) \subset N(G(GAG)^{(1)}G) \subset N(GAG(GAG)^{(1)}G) = N(G).$$

Thus,

$$R(G(GAG)^{(1)}G) = R(G) \tag{2.13}$$

and

$$N(G(GAG)^{(1)}G) = N(G). \tag{2.14}$$

From the idempotent of $(GAG)^{(1)}GAG$, by [1, Lemma 5.6] we have

$$R((GAG)^{(1)}GAG) \oplus N((GAG)^{(1)}GAG) = R^m.$$

By Equation (2.12), we have that

$$R((GAG)^{(1)}G) = R((GAG)^{(1)}GAG(GAG)^{(1)}G) \subset R((GAG)^{(1)}GAG)$$

Thus, by [14, Lemma 1(1)], we have that

$$\begin{aligned}(G(GAG)^{(1)}G)A(G(GAG)^{(1)}G) &= GP_{R((GAG)^{(1)}GAG),N((GAG)^{(1)}GAG)}((GAG)^{(1)}G)\\ &= G(GAG)^{(1)}G,\end{aligned}$$

and therefore we obtain Equation (2.3) by Equations (2.13) and (2.14) and Lemma 1.1.

Finally, we shall prove Equation (2.4). Set $G = (g_{ij})$. By [7, Theorem 8(ii)], the element of $(AG)_g$ is

$$y_{kj} = \sum_{\alpha,\gamma\in Q_{r,m}} u^{-2}|(AG)_{\gamma,\alpha}|\frac{\partial|(AG)_{\alpha,\gamma}|}{\partial x_{jk}},$$

where x_{jk} is the element of AG. Thus, the element w_{ij} of $A^{(2)}_{R(G),N(G)} = G(AG)_g$ is

$$w_{ij} = \sum_{k=1}^{m} g_{ik}y_{kj} = \sum_{k=1}^{m} g_{ik} \sum_{\alpha,\gamma\in Q_{r,m}} u^{-2}|(AG)_{\gamma,\alpha}|\frac{\partial|(AG)_{\alpha,\gamma}|}{\partial x_{jk}}.$$

Note that for all $\alpha \in Q_{r,m}$ for which $j \in \alpha$ and all $\gamma \in Q_{r,m}$,

$$\sum_{k=1}^{m} g_{ik}\frac{\partial|(AG)_{\alpha,\gamma}|}{\partial x_{jk}} = |U_{\alpha,\gamma}|$$

where U is the matrix obtained from AG by replacing the jth row of AG with the ith row of G. We write as B the matrix obtained from A by replacing the jth row of A with the row $(0, 0, \dots, 0, 1, \dots, 0)$ where the ith entry is 1 and all otherwise are zero. Thus, we have $U = BG$. Hence, by using the Cauchy-Binet Formula, we have

$$|U_{\alpha,\gamma}| = \sum_{\beta\in Q_{r,n}} |B_{\alpha,\beta}|\,|G_{\beta,\gamma}| = \sum_{\beta\in Q_{r,n}} \frac{\partial|A_{\alpha,\beta}|}{\partial a_{ji}}|G_{\beta,\gamma}|.$$

Thus

$$\begin{aligned} w_{ij} &= u^{-2} \sum_{\alpha,\beta \in Q_{r,m}} |(AG)_{\gamma,\alpha}| \sum_{\beta \in Q_{r,n}} \frac{\partial |A_{\alpha,\beta}|}{\partial a_{ji}} |G_{\beta,\gamma}| \\ &= u^{-2} \sum_{\alpha,\gamma \in Q_{r,m}} \sum_{\beta,\delta \in Q_{r,n}} |A_{\gamma,\delta}|\, |G_{\delta,\alpha}|\, |G_{\beta,\gamma}| \frac{\partial |A_{\alpha,\beta}|}{\partial a_{ji}}. \end{aligned}$$

Since $\rho(C_r(G)) = 1$, for all $\alpha, \gamma \in Q_{r,m}$ and $\beta, \delta \in Q_{r,n}$ we have

$$|G_{\delta,\alpha}|\, |G_{\beta,\gamma}| = |G_{\delta,\gamma}|\, |G_{\beta,\alpha}|$$

Thus, we get that

$$\begin{aligned} w_{ij} &= u^{-2} \sum_{\alpha,\gamma \in Q_{r,m}} \sum_{\beta,\delta \in Q_{r,n}} |A_{\gamma,\delta}|\, |G_{\delta,\gamma}|\, |G_{\beta,\alpha}| \frac{\partial |A_{\alpha,\beta}|}{\partial a_{ji}} \\ &= u^{-2} \sum_{\alpha,\gamma \in Q_{r,m}} \sum_{\beta \in Q_{r,n}} |(AG)_{\gamma,\gamma}|\, |G_{\beta,\alpha}| \frac{\partial |A_{\alpha,\beta}|}{\partial a_{ji}} \\ &= u^{-1} \sum_{\alpha \in Q_{r,m}, \beta \in Q_{r,n}} |G_{\beta,\alpha}| \frac{\partial |A_{\alpha,\beta}|}{\partial a_{ji}}. \end{aligned}$$

∎

If R is a complex number field, then we have the following result because every matrix is regular over a complex number field.

Theorem 2.3. *Let $A \in \mathbb{C}^{m\times n}$, $T \subset \mathbb{C}^n$ and $S \subset \mathbb{C}^m$. If there exits a matrix $G \in \mathbb{C}_r^{n\times m}$ such that $R(G) = T$ and $N(G) = S$, then the following statements are equivalent:*

(i) $A^{(2)}_{R(G),N(G)}$ *exists.*
(ii) $u = \sum_{\gamma \in Q_{r,m}} |(AG)_{\gamma,\gamma}| \neq 0$.
(iii) $\mathrm{rank}(G) = \mathrm{rank}(GAG)$.

In case $u \neq 0$,

$$A^{(2)}_{R(G),N(G)} = G(AG)_g = (GA)_g G \tag{2.15}$$

$$= G(GAG)^{(1)}G, \tag{2.16}$$

and the elements of $A^{(2)}_{R(G),N(G)}$ are

$$w_{ij} = u^{-1} \sum_{\alpha \in Q_{r,m}, \beta \in Q_{r,n}} |G_{\beta,\alpha}| \frac{\partial |A_{\alpha,\beta}|}{\partial a_{ji}},\ i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m. \tag{2.17}$$

Remark 1: It is easy to judge whether $A^{(2)}_{R(G),N(G)}$ exists by using the condition $\mathrm{rank}(G) = \mathrm{rank}(GAG)$. And if $A^{(2)}_{T,S}$ exists, we can evaluate its elements without calculating itself by using Equation (2.17). ∎

Next theorem shows that the group inverse, the Drazin inverse, the Moore-Penrose inverse and the weighted Moore-Penrose inverse are identical with the generalized inverse $A^{(2)}_{R(G),N(G)}$ for an appropriate matrix G, respectively.

Theorem 2.4. (i) *Let A be an $m \times n$ matrix over R with an involution $a \to \overline{a}$, M and N invertible matrices of orders m and n over R, respectively. Set $A^{\#} = N^{-1}A^*M^*$. Then $A^{(2)}_{R(A^{\#}),N(A^{\#})}$ exists and A^*MA and $AN^{-1}A^*$ are symmetric if and only if $A^{\dagger}_{MN}$ exists. Moreover, $A^{(2)}_{R(A^{\#}),N(A^{\#})} = A^{\dagger}_{MN}$.*

(ii) *Let A be an $m \times n$ matrix over R with an involution $a \to \overline{a}$. Then $A^{(2)}_{R(A^*),N(A^*)}$ exists if and only if $A^{\dagger}$ exists. Moreover, $A^{(2)}_{R(A^*),N(A^*)} = A^{\dagger}$.*

(iii) *Let A be an $n \times n$ over R and k a positive integer. Then $A^{(2)}_{R(A^k),N(A^k)}$ exists if and only if A_d exists. Moreover, $A^{(2)}_{R(A^k),N(A^k)} = A_d$.*

(iv) *Let A be an $n \times n$ over R. Then $A^{(2)}_{R(A),N(A)}$ exists if and only if A_g exists. Moreover, $A^{(2)}_{R(A),N(A)} = A_g$.*

Proof. (i) "$\Longleftarrow$" Since $AA^{\dagger}_{MN}$ is idempotent, by [1, Lemma 5.6], we have that

$$R(AA^{\dagger}_{M,N}) \oplus N(AA^{\dagger}_{M,N}) = R^m.$$

Since

$$\begin{array}{rcllcl} R(A^{\dagger}_{MN}) & = & R(N^{-1}NA^{\dagger}_{MN}AA^{\dagger}_{MN}) & & = & R(N^{-1}A^*A^{\dagger *}_{MN}N^*A^{\dagger}_{MN}) \\ & = & R(N^{-1}A^*M^*M^{*-1}A^{\dagger *}_{MN}N^*A^{\dagger}_{MN}) & & \subset & R(N^{-1}A^*M^*) \\ & = & R(N^{-1}A^*A^{\dagger *}_{MN}A^*M^*) & & = & R(A^{\dagger}_{MN}AN^{*-1}A^*M^*) \\ & \subset & R(A^{\dagger}_{MN}), & & & \\ N(AA^{\dagger}_{MN}) & \subset & N(N^{-1}A^*MAA^{\dagger}_{MN}) & & = & N(N^{-1}A^*A^{\dagger *}_{MN}A^*M^*) \\ & = & N(N^{-1}A^*M^*) & & \subset & N(M^{-1}A^{\dagger *}_{MN}NN^{-1}A^*M^*) \\ & = & N(AA^{\dagger}_{MN}), & & & \end{array}$$

we have

$$R(A^{\dagger}_{MN}) = R(A^{\#}), \quad N(A^{\dagger}_{MN}) = N(A^{\#})$$

and then

$$AR(A^{\#}) \oplus N(A^{\#}) = R^M.$$

Let $x \in R^m$ such that $AA^{\#}x = 0$. Then $A^{\#}x = A^{\#}AA^{\#}x = 0$ and so $N(A) \cap R(A^{\#}) = \{0\}$. By Lemma 1.1, $A^{(2)}_{R(A^{\#}),N(A^{\#})}$ exists and $A^{\dagger}_{MN} = A^{(2)}_{R(A^{\#}),N(A^{\#})}$.

By [6, Theorem 8], A^*MA and $AN^{-1}A^*$ are symmetric.

"$\Longrightarrow$" If $A^{(2)}_{R(A^{\#}),N(A^*)}$ exists, then we can write $X = A^{(2)}_{R(A^{\#}),N(A^{\#})} = A^{\#}(AA^{\#})_g = (A^{\#}A)_g A^{\#}$ by Theorem 2.2. Replacing G with $A^{\#}$ in Equation (2.1), we have

$$A^{\#}AA^{\#}W = A^{\#}$$

and then

$$A = M^{*-1}W^*A^{\#^*}A^*A^{\#^*}N^* = M^{*-1}W^*A^{\#^*}A^*MA = M^{*-1}W^*A^{\#^*}A^*M^*A.$$

Thus

$$\begin{array}{rcllcl} AXA & = & A((A^{\#}A)_g A^{\#})A & & = & M^{*-1}W^*A^{\#^*}A^*M^*A(A^{\#}A)_g A^{\#}A \\ & = & M^{*-1}W^*A^{\#^*}NA^{\#}A(A^{\#}A)_g A^{\#}A & & = & M^{*-1}W^*A^{\#^*}NA^{\#}A \\ & = & M^{*-1}W^*A^{\#^*}A^*M^*A & & = & A, \\ XAX & = & X, & & & \end{array}$$

$$\begin{aligned}
(MAX)^* &= X^*A^*M^* = (A^{\#}(AA^{\#})_g)^*A^*M^* \\
&= (A^{\#^*}A^*)_g A^{\#^*}A^*M^* = (A^{\#^*}A^*)_g MAN^{*-1}A^*M^* \\
&= (A^{\#^*}A^*)_g MAN^{-1}A^*M^* = (A^{\#^*}A^*)_g MAA^{\#} \\
&= (A^{\#^*}A^*)_g MAA^{\#}AA^{\#}(AA^{\#})_g \\
&= (A^{\#^*}A^*)_g MAN^{-1}A^*M^*AN^{-1}A^*M^*(AA^{\#})_g \\
&= (A^{\#^*}A^*)_g MAN^{*-1}A^*MAN^{*-1}A^*M^*(AA^{\#})_g \\
&= (A^{\#^*}A^*)_g (A^{\#^*}A^*)^2 M^*(AA^{\#})_g \\
&= A^{\#^*}A^*M^*(AA^{\#})_g = MAN^{*-1}A^*M^*(AA^{\#})_g \\
&= MAN^{-1}A^*M^*(AA^{\#})_g = MAA^{\#}(AA^{\#})_g \\
&= MAX,
\end{aligned}$$

$$\begin{aligned}
(NXA)^* &= A^*X^*N^* = A^*((A^{\#}A)_g A^{\#})^*N^* \\
&= A^*A^{\#^*}(A^*A^{\#^*})_g N^* = NN^{-1}A^*MAN^{*-1}(A^*A^{\#^*})_g N^* \\
&= NN^{-1}A^*M^*AN^{*-1}(A^*A^{\#^*})_g N^* = N(A^{\#}A)N^{*-1}(A^*A^{\#^*})_g N^* \\
&= N(A^{\#}A)_g (A^{\#}A)^2 N^{*-1}(A^*A^{\#^*})_g N^* \\
&= N(A^{\#}A)_g N^{-1}A^*M^*AN^{-1}A^*M^*AN^{*-1}(A^*A^{\#^*})_g N^* \\
&= N(A^{\#}A)_g N^{-1}A^*MAN^{*-1}A^*MAN^{*-1}(A^*A^{\#^*})_g N^* \\
&= N(A^{\#}A)_g N^{-1}A^*A^{\#^*}A^*A^{\#^*}(A^*A^{\#^*})_g N^* \\
&= N(A^{\#}A)_g N^{-1}A^*A^{\#^*}N^* = N(A^{\#}A)_g N^{-1}A^*MAN^{*-1}N^* \\
&= N(A^{\#}A)_g N^{-1}A^*M^*A = N(A^{\#}A)_g A^{\#}A \\
&= NXA.
\end{aligned}$$

Hence, $A^{\dagger}_{MN}$ exists.

(ii) Take $M = I_m$ and $N = I_n$.

(iii) "$\Longleftarrow$" By the proof in [14, Theorems 4].

"$\Longrightarrow$" If $A^{(2)}_{R(A^k),N(A^k)}$ exists. Let $X = A^{(2)}_{R(A^k),N(A^k)}$ = $A^k(A^{k+1})_g$ = $(A^{k+1})_g A^k$ by by Theorem 2.2. Replacing G with A^k in Equation (2.1), we have $A^k = A^{2k+1}W$ and then

$$\begin{aligned}
AX &= A^{k+1}(A^{k+1})_g &&= (A^{k+1})_g A^{k+1} &&= XA, \\
A^kXA &= A^k((A^{k+1})_g A^k)A &&= A^k(A^{k+1})_g AA^{2k+1}W &&= A^kA^{k+1}W \\
&= A^k, \\
XAX &= X.
\end{aligned}$$

Hence A_d exists.

(iv) Take $k = 1$ in (iii). ∎

In order to see that the conditions for the existence and expressions for the elements of the known generalized inverses are unified by Theorem 2.2, we need the next result.

Lemma 2.5. (i) *Let A be an $m \times n$ matrix over R with an involution $a \to \overline{a}$, M and N invertible matrices of orders m and n over R, respectively. Set $A^{\#} = N^{-1}A^*M^*$. Then $AA^{\#}$ and $A^{\#}A$ are regular and $\rho(A) = \rho(A^{\#}A) = \rho(AA^{\#})$ if and only if $A^{\#}AA^{\#}$ is regular and $\rho(A^{\#}) = \rho(A^{\#}AA^{\#})$.*

(ii) *Let A be an $m \times n$ matrix over R with an involution $a \to \overline{a}$. Then AA^* and A^*A are regular and $\rho(A) = \rho(A^*A) = \rho(AA^*)$ if and only if A^*AA^* is regular and $\rho(A^*) = \rho(A^*AA^*)$.*

(iii) *Let A be an $n \times n$ over R and k a positive integer. Then A^{2k+1} is regular and $\rho(A^k) = \rho(A^{2k+1})$ if and only if A^{k+1} is regular and $\rho(A^k) = \rho(A^{k+1})$.*
(iv) *Let A be an $n \times n$ over R. Then A^3 is regular and $\rho(A) = \rho(A^3)$ if and only if A^2 is regular and $\rho(A) = \rho(A^2)$.*

Proof. (i)"$\Longrightarrow$" From the hypothesis, there exist matrices P and Q over the field of quotients of R such that $A = PA^{\#}A = AA^{\#}Q$. So we have

$$\begin{array}{lllllll} A(A^{\#}A)^{(1)}A^{\#}A & = & PA^{\#}A(A^{\#}A)^{(1)}A^{\#}A & = & PA^{\#}A & = & A, \\ AA^{\#}(AA^{\#})^{(1)}A & = & AA^{\#}(AA^{\#})^{(1)}AA^{\#}Q & = & AA^{\#}Q & = & A. \end{array}$$

From these, we have

$$A = AA^{\#}(AA^{\#})^{(1)}A(A^{\#}A)^{(1)}A^{\#}A \text{ and } A = A(A^{\#}A)^{(1)}A^{\#}AA^{\#}(AA^{\#})^{(1)}A.$$

Thus,

$$A^{\#}AA^{\#} = (A^{\#}AA^{\#})((AA^{\#})^{(1)}A(A^{\#}A)^{(1)})(A^{\#}AA^{\#}).$$

and

$$\rho(A) \leq \rho(A^{\#}AA^{\#}) \leq \rho(A).$$

Hence, we obtain that $A^{\#}AA^{\#}$ is regular and $\rho(A) = \rho(A^{\#}AA^{\#})$.

It is easy to see that $\rho(A) = \rho(A^{\#})$. Consequently, we have $\rho(A^{\#}) = \rho(A^{\#}AA^{\#})$.

"$\Longleftarrow$"Since

$$\rho(A) = \rho(A^{\#}) = \rho(A^{\#}AA^{\#}) \leq \rho(A^{\#}A) \leq \rho(A),$$

we have $\rho(A) = \rho(A^{\#}A)$ and then there exists a matrix P over the field of quotients of R such that $A = PA^{\#}A$. Hence

$$AA^{\#} = PA^{\#}AA^{\#} = PA^{\#}AA^{\#}(A^{\#}AA^{\#})^{(1)}A^{\#}AA^{\#} = AA^{\#}((A^{\#}AA^{\#})^{(1)}A^{\#})AA^{\#}.$$

Thus, $AA^{\#}$ is regular.

Similarly, we have that $A^{\#}A$ is regular and $\rho(A) = \rho(AA^{\#})$.

(ii)Take $M = I_m$ and $N = I_n$ in (i).

(iii)"$\Longrightarrow$"Since

$$\rho(A^k) = \rho(A^{2k+1}) \leq \rho(A^{k+1}) \leq \rho(A^k),$$

we have $\rho(A^k) = \rho(A^{k+1})$. Since $\rho(A^k) = \rho(A^{2k+1})$, there exists a matrix N over the field of quotients of R such that $A^k = A^{2k+1}N$. Then

$$A^{k+1} = A^{2k+2}N = A^{2k+1}(A^{2k+1})^{(1)}A^{2k+1}AN = A^{k+1}(A^k(A^{2k+1})^{(1)})A^{k+1}.$$

Hence, A^{k+1} is regular.

"$\Longleftarrow$"Since $\rho(A^k) = \rho(A^{k+1})$, there exists a matrix N over the field of quotients of R such that $A^k = A^{k+1}N$. Since

$$A^k = A^{k+1}N = A^{k+1}(A^{k+1})^{(1)}A^{k+1}N = A^{k+1}(A^{k+1})^{(1)}A^k,$$

there exists a matrix Y over R such that $A^k = A^{2k+1}Y$. Thus, we have $\rho(A^k) = \rho(A^{2k+1})$ and

$$A^{2k+1} = A^{k+1}A^k = A^{k+1}(A^{k+1})^{(1)}A^{k+1}A^k = A^{2k+1}(YA(A^{k+1})^{(1)})A^{2k+1}.$$

That is, A^{2k+1} is regular.

(iv)Take $k = 1$ in (iii). ∎

From Theorems 2.2 and 2.4 and Lemma 2.5, it is easy to get the following corollaries.

Corollary 2.6. *Let A be an $m \times n$ matrix with $r = \rho(A)$ over R with an involution $a \to \overline{a}$, M and N invertible matrices of orders m and n over R, respectively. Set $A^{\#} = N^{-1}A^*M^*$. Then the following are equivalent:*

(i) *$A^{\dagger}_{MN}$ exists.*

(ii) $u = \sum_{\gamma \in Q_{r,m}} |(AA^{\#})_{\gamma,\gamma}|$ *is a unit in* R*, and* $AN^{-1}A^*$ *and* A^*MA *are symmetric.*

(iii) $\rho(A) = \rho(A^{\#}A) = \rho(AA^{\#})$ *and* $A^{\#}A$ *and* $AA^{\#}$ *are both regular.*

In case u *is a unit,*

$$A^{\dagger}_{MN} = A^{\#}(AA^{\#})_g = (A^{\#}A)_g A^{\#} \tag{2.18}$$

$$= A^{\#}(A^{\#}AA^{\#})^{(1)}A^{\#}, \tag{2.19}$$

and the element of $A^{\dagger}_{MN}$ *is*

$$w_{ij} = u^{-1} \sum_{\alpha \in Q_{r,m}, \beta \in Q_{r,n}} |A^{\#}_{\beta,\alpha}| \frac{\partial |A_{\alpha,\beta}|}{\partial a_{ji}}. \tag{2.20}$$

Remark 2: Some of the results in Corollary 2.6 is identical with some of the results in [6, Theorems 8]. ∎

Corollary 2.7. *Let* A *be an* $m \times n$ *matrix with* $r = \rho(A)$ *over* R *with an involution* $a \to \overline{a}$*. Then the following are equivalent:*

(i) $A^{\dagger}$ *exists.*

(ii) $u = \sum_{\gamma \in Q_{r,m}} |(AA^*)_{\gamma,\gamma}|$ *is a unit in* R*.*

(iii) $\rho(A) = \rho(A^*A) = \rho(AA^*)$ *and* A^*A *and* AA^* *are both regular.*

In case u *is a unit,*

$$A^{\dagger} = A^*(AA^*)_g = (A^*A)_g A^* \tag{2.21}$$

$$= A^*(A^*AA^*)^{(1)}A^*, \tag{2.22}$$

and the element of $A^{\dagger}$ *is*

$$w_{ij} = u^{-1} \sum_{\alpha \in Q_{r,m}, \beta \in Q_{r,n}} |A^*_{\beta,\alpha}| \frac{\partial |A_{\alpha,\beta}|}{\partial a_{ji}}. \tag{2.23}$$

Remark 3: Some of the results in Corollary 2.7 is identical with some of the results in [9, Theorems 6.7 and 6.8]. In addition, Equation (2.22) extends the formula of Zlobec [16]. ∎

Corollary 2.8. *Let* A *be an* $n \times n$ *matrix over* R*. Then the following are equivalent:*

(i) A_d *exists.*

(ii) *For some positive integral* k*,* $u = \sum_{\gamma \in Q_{r,m}} |(A^{k+1})_{\gamma,\gamma}|$ *is a unit in* R*, where* $r = \rho(A^k)$*.*

(iii) *For some positive integral* k*,* $\rho(A^k) = \rho(A^{k+1})$ *and* A^{k+1} *is regular.*

In case u *is a unit and for* $k \geq \mathrm{Ind}(A)$*,*

$$A_d = A^k(A^{k+1})_g = (A^{k+1})_g A^k \tag{2.24}$$

$$= A^k(A^{2k+1})^{(1)}A^k, \tag{2.25}$$

and the element of A_d *is*

$$w_{ij} = u^{-1} \sum_{\alpha \in Q_{r,m}, \beta \in Q_{r,n}} |(A^k)_{\beta,\alpha}| \frac{\partial |A_{\alpha,\beta}|}{\partial a_{ji}}, \tag{2.26}$$

where $r = \rho(A^k)$*.*

Remark 4: Some of the results in Corollary 2.8 is identical with some of results in [9, Theorems 6.24] and [7, Theorem 9], and Equation (2.26) extends Equation (3.1) in [10, Theorem 3.1]. ∎

Remark 5: The formula in [7, Theorem 8(ii)] can rewritten the following form, which is identical with Equation (2.26) when $k=1$,

$$w_{ij} = u^{-1} \sum_{\alpha\in Q_{r,m},\beta\in Q_{r,n}} |A_{\beta,\alpha}| \frac{\partial |A_{\alpha,\beta}|}{\partial a_{ji}}, \tag{2.27}$$

where $r=\rho(A)$. The reason is that

$$\begin{aligned}\sum_{\gamma\in Q_{r,n}} |(A^2)_{\gamma,\gamma}| &= \sum_{\gamma,\delta\in Q_{r,n}} |A_{\gamma,\delta}|\,|A_{\delta,\gamma}| \\ &= \sum_{\gamma,\delta\in Q_{r,n}} |A_{\gamma,\gamma}|\,|A_{\delta,\delta}| \quad (\text{since } \rho(C_r(A))=1) \\ &= (\sum_{\gamma\in Q_{r,m}} |A_{\gamma,\gamma}|)^2.\end{aligned}$$

∎

From Remarks 2 ~ 5, Theorem 2.2 unifies the conditions for the existence and the expression for the elements of the weighted Moore-Penrose inverse, the Moore-Penrose inverse, the Drazin inverse and the group inverse over an integral domain.

Corollary 2.9. *Let A be an $n\times n$ matrix over R. If the group inverse A_g of A exists, then*

$$A_g = A(A^3)^{(1)}A. \tag{2.28}$$

3. RANK EQUATIONS

As a simple application we consider following theorem which characters the relation between some rank equation and the existence of the generalized inverse with prescribed image and kernel. In the proof we use the results in Theorem 2.2.

Firstly we need the following lemma.

Lemma 3.1. [13, Lemma 1] *Let $\alpha_1,\alpha_2,\cdots,\alpha_w \in R^n$ $(w\le n)$ and $T=(\alpha_1,\,\alpha_2,\,\cdots,\,\alpha_w)$. Then $\alpha_1,\alpha_2,\cdots,\,\alpha_w$ are linearly dependent over R if and only if $\det(T_{\alpha,w})=0$, for every $\alpha\in Q_{w,n}$.*

Now we show the main theorem in the section.

Theorem 3.2. *Let A be an $m\times n$ matrix over an integer ring R, $T\subset R^n, S\subset R^m$, Suppose $G\in R^{n\times m}$ such that $R(G)=T$ and $N(G)=S$. Then the generalized inverse $A^{(2)}_{T,S}$ exists if and only if there exists a solution in the rank equation*

$$\rho(\begin{pmatrix} GAG & G \\ G & X\end{pmatrix}) = \rho(GAG), \quad R(X)\subset T \tag{3.1}$$

In this case, the solution is unique.

Proof. "$\Longrightarrow$"Suppose $A^{(2)}_{T,S}$ exists. We will show that $A^{(2)}_{T,S}$ is a solution of the rank equation (3.1).

Obviously, $R(A^{(2)}_{T,S})=R(G)=T$. Since $A^{(2)}_{T,S}$ exists, we have $A^{(2)}_{T,S}=G(GAG)^{(1)}G$ and $\rho(G)=\rho(GAG)$ by Theorem 2.2. Thus

$$\begin{aligned}\rho(A^{(2)}_{T,S}) &\le \rho(G) = \rho(GAG) = \rho(GAG(GAG)^{(1)}GAG) \\ &= \rho(GA(A^{(2)}_{T,S})AG) \le \rho(A^{(2)}_{T,S}),\end{aligned}$$

So, $\rho(A^{(2)}_{T,S})=\rho(GAG)$. It is easy to see that $A^{(2)}_{T,S}AG=G$ and $GAA^{(2)}_{T,S}=G$. Hence

$$\begin{pmatrix} I & -GA \\ 0 & I\end{pmatrix}\begin{pmatrix} GAG & G \\ G & A^{(2)}_{T,S}\end{pmatrix}\begin{pmatrix} I & 0 \\ -AG & I\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ G & A^{(2)}_{T,S}\end{pmatrix}\begin{pmatrix} I & 0 \\ -AG & I\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & A^{(2)}_{T,S}\end{pmatrix}$$

and then

$$\rho(\begin{pmatrix} GAG & G \\ G & A^{(2)}_{T,S} \end{pmatrix}) = \rho(\begin{pmatrix} 0 & 0 \\ 0 & A^{(2)}_{T,S} \end{pmatrix}) = \rho(A^{(2)}_{T,S}) = \rho(GAG).$$

That is, $A^{(2)}_{T,S}$ satisfies Equation (3.1).

"$\Longleftarrow$"Suppose that

$$\operatorname{rank}\begin{pmatrix} GAG & G \\ G & X \end{pmatrix} = \operatorname{rank}(GAG)$$

has a solution. Denote $r = \rho(GAG)$, $GAG = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$, $G = \{\beta_1, \beta_2, \dots, \beta_m\}$. If $r = 0$, then $G = 0$ by Equation (3.1). So by Theorem 2.2, $A^{(2)}_{T,S}$ exists and $A^{(2)}_{T,S} = 0$.

Now suppose $r \neq 0$. By Lemma 3.1, there exist r columns of GAG, which are linearly independent. Also, there exist r rows of GAG, which are linearly independent, because $|A^T| = |A|$.

Without loss of generality, let the first r columns of GAG, say $\alpha_1, \alpha_2, \dots, \alpha_r$, be linearly independent. By Equation (3.1), we obtain that $\alpha_1, \alpha_2, \dots, \alpha_r$ and β_j $(j = 1, 2, \dots, m)$ are linearly dependent. That is, there exist $a_{1,j}, a_{2,j}, \dots, a_{r,j}, b_j$ $(j = 1, 2, \dots, m)$, not all 0, such that

$$\sum_{i=1}^{r} a_{i,j}\alpha_i = b_j\beta_j, \quad j = 1, \dots, m.$$

Clearly, $b_j \neq 0, j = 1, \dots, m$. Written the above equation, we get

$$(\alpha_1, \alpha_2, \dots, \alpha_r, \alpha_{r+1}, \dots, \alpha_m)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rm} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = (\beta_1, \beta_2, \dots, \beta_m)\begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_m \end{pmatrix}$$

and write down

$$(GAG)P = GD. \tag{3.2}$$

Similarly we take into account the row. Then, there exist matrices Q and D' such that

$$Q(GAG) = D'G. \tag{3.3}$$

where D', as well as D, is a diagonal matrix and $|D'| \neq 0$. Using Equations (3.2) and (3.3), we have

$$\begin{aligned}
&\begin{pmatrix} I & 0 \\ -Q & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & D' \end{pmatrix}\begin{pmatrix} GAG & G \\ G & X \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} I & -P \\ 0 & I \end{pmatrix} \\
= &\begin{pmatrix} I & 0 \\ -Q & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & D' \end{pmatrix}\begin{pmatrix} GAG & GD \\ G & XD \end{pmatrix}\begin{pmatrix} I & -P \\ 0 & I \end{pmatrix} \\
= &\begin{pmatrix} I & 0 \\ -Q & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & D' \end{pmatrix}\begin{pmatrix} GAG & 0 \\ G & XD - GP \end{pmatrix} \\
= &\begin{pmatrix} I & 0 \\ -Q & I \end{pmatrix}\begin{pmatrix} GAG & 0 \\ D'G & D'(XD - GP) \end{pmatrix} \\
= &\begin{pmatrix} GAG & 0 \\ 0 & D'(XD - GP) \end{pmatrix}.
\end{aligned}$$

Thus, using Equation (1.1), we obtain

$$\begin{aligned}\rho(GAG) &= \rho(\begin{pmatrix} GAG & G \\ G & X \end{pmatrix}) = \rho(\begin{pmatrix} GAG & 0 \\ 0 & D'(XD-GP) \end{pmatrix}) \\ &= \rho(GAG) + \rho(D'(XD-GP)).\end{aligned}$$

So $D'(XD-GP)=0$. Since there exist no nonzero divisors of zero in an integral domain,

$$XD = GP. \tag{3.4}$$

Analogously, we can get

$$D'X = QG. \tag{3.5}$$

Hence, by Equations (3.2) $\sim$ (3.5), we have

$$\begin{aligned} D'XAXD &= QGAGP = QGD = D'XD, \\ D'XAG &= QGAG = D'G, \\ GAXD &= GAGP = GD. \end{aligned}$$

Since R has no nonzero divisors of zero, we have

$$XAX = X, \tag{3.6}$$

$$XAG = G, \tag{3.7}$$

$$GAX = G. \tag{3.8}$$

Thus, by Equations (3.7) and (3.8),

$$R(G) = R(XAG) \subset R(X) \subset T = R(G),$$

$$N(X) \subset N(GAX) = N(G) \subset N(QG) = N(D'X) = N(X).$$

and then

$$R(G) = R(X), \tag{3.9}$$

$$N(G) = N(X). \tag{3.10}$$

Hence, by Theorem 2.2 and Equations (3.6), (3.9) and (3.10), we reach that $X = A^{(2)}_{T,S}$ exists uniquely. ∎

In Theorem 3.2, we take $G = A^{\#}$, $G = A^*$, $G = A^l, l \geq \mathrm{Ind}(A)$, and $G = A$, respectively. Then by using Theorem 2.4, we get the results of the existences of the weighted Moore-Penrose inverse $A^{\dagger}_{MN}$, the Moore-Penrose inverse $A^{\dagger}$, the Drazin inverse A_d and the group inverse A_g of the matrix A over R.

Corollary 3.3. *Let R be an integer ring with an involution $a \to \overline{a}$, $A \in R^{m\times n}$, M and N be invertible matrices of orders m and n over R. Set $A^{\#} = N^{-1}A^*M^*$. Then $A^{\dagger}_{MN}$ exists if and only if the rank equation*

$$\rho(\begin{pmatrix} A^{\#}AA^{\#} & A^{\#} \\ A^{\#} & X \end{pmatrix}) = \rho(A^{\#}AA^{\#}), \quad R(X) \subset R(A^{\#}) \tag{3.11}$$

has a solution $X \in R^{n\times m}$, and $AN^{-1}A^*$ and A^*MA are symmetric.

In this case, the solution is unique.

Proof. Take $G = A^{\#}$ in Theorem 3.2 and afterward use Theorem 2.4(i). ∎

Corollary 3.4. *Let R be an integer ring with an involution $a \to \overline{a}$, $A \in R^{m\times n}$. Then $A^{\dagger}$ exists if and only if the rank equation*

$$\rho(\begin{pmatrix} A^*AA^* & A^* \\ A^* & X \end{pmatrix}) = \rho(A^*AA^*), \quad R(X) \subset R(A^*) \tag{3.12}$$

has a solution $X \in R^{n\times m}$.

In this case, the solution is unique.

Corollary 3.5. *Let R be an integer ring, $A \in R^{n\times n}$. Then A_d exists if and only if the rank equation*

$$\rho(\begin{pmatrix} A^{2l+1} & A^l \\ A^l & X \end{pmatrix}) = \rho(A^{2l+1}), \quad R(X) \subset R(A^l) \tag{3.13}$$

has a solution $X \in R^{n\times n}$, where $l \geq \mathrm{Ind}(A)$.

In this case, the solution is unique.

Corollary 3.6. *Let R be an integer ring, $A \in R^{n\times n}$. Then A_g exists if and only if the rank equation*

$$\rho(\begin{pmatrix} A^3 & A \\ A & X \end{pmatrix}) = \rho(A^3), \quad R(X) \subset R(A) \tag{3.14}$$

has a solution $X \in R^{n\times n}$.

In this case, the solution is unique.

If R is the complex number field, we can omit the restricted condition $R(X) \subset T$ in Theorem 3.2. Thus, we have the following theorem and its corollaries.

Theorem 3.7. *Let $A \in \mathbb{C}^{m\times n}_r$, $T \subset \mathbb{C}^n, S \subset \mathbb{C}^m, \dim(T) = s \leq r, \dim(S) = m - s$. Let $G \in \mathbb{C}^{n\times m}$ such that $R(G) = T$ and $N(G) = S$. Then $A^{(2)}_{T,S}$ exists if and only if there exists a solution in the rank equation*

$$\mathrm{rank}(\begin{pmatrix} GAG & G \\ G & X \end{pmatrix}) = \mathrm{rank}(GAG). \tag{3.15}$$

In this case, the solution is unique.

Proof. Using Equation (3.4) in the proof of Theorem 3.2, we have that $X = GPD^{-1}$. and then $R(X) \subset R(G) = T$. It is to say that the rank equation in (3.1) implies $R(X) \subset T$. Therefore, by Theorem 3.2, the result is true. ∎

When $G = N^{-1}A^*M$, where M and N are Hermitian positive definite matrices of orders m and n respectively, $G = A^*$ or $G = A^l$, where $l \geq \mathrm{Ind}(G)$, in the above theorem, we have following corollaries respectively in view of the existence of $A^{\dagger}_{MN}$, $A^{\dagger}$ and A_d of a matrix A over $\mathbb{C}$.

Corollary 3.8. *Let $A \in \mathbb{C}^{m\times n}$, M and N are Hermitian positive definite matrices of orders m and n, respectively. Then $A^{\dagger}_{MN}$ is a unique solution of*

$$\mathrm{rank}(\begin{pmatrix} A^{\#}AA^{\#} & A^{\#} \\ A^{\#} & X \end{pmatrix}) = \mathrm{rank}(A^{\#}AA^{\#}), \tag{3.16}$$

where $A^{\#} = N^{-1}A^*M$.

Corollary 3.9. *Let $A \in \mathbb{C}^{m\times n}$. Then $A^{\dagger}$ is a unique solution of*

$$\mathrm{rank}(\begin{pmatrix} A^*AA^* & A^* \\ A^* & X \end{pmatrix}) = \mathrm{rank}(A^*AA^*). \tag{3.17}$$

Corollary 3.10. *Let $A \in \mathbb{C}^{n\times n}$. Then A_d is a unique solution of*

$$\operatorname{rank}(\begin{pmatrix} A^{2l+1} & A^l \\ A^l & X \end{pmatrix}) = \operatorname{rank}(A^{2l+1}), \tag{3.18}$$

for any integer $l \geq \operatorname{Ind}(A)$.

Taking $G = A$ in Theorem 3.2, we get the following result.

Corollary 3.11. *Let $A \in \mathbb{C}^{n\times n}$. Then A_g exist if and only if there exists a solution of*

$$\operatorname{rank}(\begin{pmatrix} A^3 & A \\ A & X \end{pmatrix}) = \operatorname{rank}(A^3). \tag{3.19}$$

In this case, the solution is unique.

Now we continue to study the properties of $A^{(2)}_{T,S}$of A over an integer domain. First, we show the following lemma, whose proof is similar to that over $\mathbb{C}$. Here we need Equation (1.1).

Lemma 3.12. *Over an integer domain R, suppose that the submatrix A of the matrix*

$$P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is invertible. Then $\rho(P) = \rho(A)$ if and only if $D = CA^{-1}B$.

The following theorem fines Equation (3.17).

Theorem 3.13. *Let R an integer domain, $A \in R^{m\times n}$, $T \subset R^n$ and $S \subset R^m$. $G \in R^{n\times m}$ with $R(G) = T$ and $N(G) = S$. Denote $r = \rho(G)$. If there exist $\alpha \in Q_{r,n}, \beta \in Q_{r,m}$ such that $(GAG)_{\alpha,\beta}$ is invertible in R, then the generalized inverse $A^{(2)}_{T,S}$ exists and*

$$A^{(2)}_{T,S} = G_{*\beta}((GAG)_{\alpha,\beta})^{-1}G_{\alpha *}. \tag{3.20}$$

Proof. Clearly, we have $\rho(GAG) \leq \rho(G)$. From the invertibility of $(GAG)_{\alpha,\beta}$, we get $\rho(GAG) \geq r = \rho(G)$. Thus, $\rho(GAG) = \rho(G)$.

From the invertibility of $(GAG)_{\alpha,\beta}$, we obtain

$$\frac{1}{|(GAG)_{\alpha,\beta}|}|(GAG)_{\alpha,\beta}| + \sum_{\gamma\in Q_{r,n},\delta\in Q_{r,m},(\gamma,\delta)\neq(\alpha,\beta)} 0 \cdot |(GAG)_{\gamma,\delta}| = 1.$$

By [8, Theorem 8], we know that GAG is regular. Therefore, by Theorem 3.2, $A^{(2)}_{T,S}$ exists.

Now we shall show Equation (3.20). Set

$$P = \begin{pmatrix} (GAG)_{\alpha,\beta} & G_{\alpha *} \\ G_{*\beta} & A^{(2)}_{T,S} \end{pmatrix}$$

It is evident that

$$\rho(P) \geq \rho((GAG)_{\alpha,\beta}) = \rho(G) = \rho(GAG).$$

Since P is a submatrix of the matrix $\begin{pmatrix} GAG & G \\ G & A^{(2)}_{T,S} \end{pmatrix}$,

$$\rho(P) \leq \rho(\begin{pmatrix} GAG & G \\ G & A^{(2)}_{T,S} \end{pmatrix}).$$

Thus, by Equation (3.1) in Theorem 3.2 we obtain

$$\rho(\begin{pmatrix} GAG & G \\ G & A^{(2)}_{T,S} \end{pmatrix}) = \rho(GAG).$$

because $A^{(2)}_{T,S}$ exists. Hence
$$\rho(P) = \rho((GAG)_{\alpha,\beta}).$$
According to Lemma 3.12, we have
$$A^{(2)}_{T,S} = G_{*\beta}((GAG)_{\alpha,\beta})^{-1}G_{\alpha *}.$$
∎

From the above theorem we have following results about known generalized inverses.

Corollary 3.14. *Let R be an integer ring with an involution $a \to \overline{a}$, $A \in R^{m\times n}$, Denote $r = \rho(A)$. Let M and N be invertible matrices over R of orders m and n, respectively, and $AN^{-1}A^*$ and A^*MA be symmetric. Denote $A^{\#} = N^{-1}A^*M^*$. If there exist $\alpha \in Q_{r,n}, \beta \in Q_{r,m}$ such that $(A^{\#}AA^{\#})_{\alpha,\beta}$ is invertible, then the weighted Moore-Penrose inverse $A^{\dagger}_{MN}$ exists, and*

$$A^{\dagger}_{MN} = A^{\#}_{*\beta}((A^{\#}AA^{\#})_{\alpha,\beta})^{-1}A^{\#}_{\alpha *}. \tag{3.21}$$

Corollary 3.15. *Let R be an integer ring with an involution $a \to \overline{a}$, $A \in R^{m\times n}$. Denote $r = \rho(A)$. If there exist $\alpha \in Q_{r,n}, \beta \in Q_{r,m}$ such that $(A^*AA^*)_{\alpha,\beta}$ is invertible, then the Moore-Penrose inverse $A^{\dagger}$ exists, and*

$$A^{\dagger} = A^*_{*\beta}((A^*AA^*)_{\alpha,\beta})^{-1}A^*_{\alpha *}. \tag{3.22}$$

Corollary 3.16. *Let R be an integer ring, $A \in R^{n\times n}$, Denote $r = \rho(A^l)$, where $l \geq \mathrm{Ind}(A)$. If there exist $\alpha, \beta \in Q_{r,n}$ such that $(A^{2l+1})_{\alpha,\beta}$ is invertible, then the Drazin inverse A_d exists, and*

$$A_d = A^l_{*\beta}((A^{2l+1})_{\alpha,\beta})^{-1}A^l_{\alpha *}. \tag{3.23}$$

Corollary 3.17. *Let R be an integer ring, $A \in R^{n\times n}$, Denote $r = \rho(A)$. If there exist $\alpha, \beta \in Q_{r,n}$ such that $(A^3)_{\alpha,\beta}$ is invertible, then the group inverse A_g exists, and*

$$A_g = A_{*\beta}((A^3)_{\alpha,\beta})^{-1}A_{\alpha *}. \tag{3.24}$$

Especially, for matrices over $\mathbb{C}$, we have

Corollary 3.18. *Let $A \in \mathbb{C}^{m\times n}$, $T \subset \mathbb{C}^n$ and $S \subset \mathbb{C}^m$. $G \in \mathbb{C}^{n\times m}$ with $R(G) = T$ and $N(G) = S$, Denote $r = \rho(G)$. If there exist $\alpha \in Q_{r,n}, \beta \in Q_{r,m}$ such that $(GAG)_{\alpha,\beta}$ is nonsingular, then the generalized $A^{(2)}_{T,S}$ exists, and*

$$A^{(2)}_{T,S} = G_{*\beta}((GAG)_{\alpha,\beta})^{-1}G_{\alpha *}. \tag{3.25}$$

4. EXAMPLE

Here we give an example of evaluating the elements of $A^{(2)}_{T,S}$ without calculating $A^{(2)}_{T,S}$ by using Theorem 2.3.

Example. Let
$$A = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \in \mathbb{R}^{4\times 4}_3.$$
Given $S = R((-1533/4072, 479/694, -1549/2743, 511/2036)^T) \subset \mathbb{R}^4$ and $T = R(V) \subset \mathbb{R}^4$ where
$$V = \begin{pmatrix} 2 & 3 & 5 \\ 3 & 4 & 6 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Taking

$$G = \begin{pmatrix} 2 & 3 & 5 & 6 \\ 3 & 4 & 6 & 7 \\ 3 & 1 & 1 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \in \mathbb{R}_3^{4\times 4},$$

we can show easily that $R(G) = T$, $N(G) = S$ and rank(G)=rank(GAG). Thus, by Theorem 2.3, $A^{(2)}_{T,S}$ exists.

Consider the element w_{21} of $A^{(2)}_{T,S}$. Thus, we have

$$\begin{aligned} \{(\alpha,\beta)|1\in\alpha, 2\in\beta\} \;=\; & \{(\{1,2,3\},\{1,2,3\}),(\{1,2,3\},\{1,2,4\}),(\{1,2,3\},\{2,3,4\}),\\ & (\{1,2,4\},\{1,2,3\}),(\{1,2,4\},\{1,2,4\}),(\{1,2,4\},\{2,3,4\}),\\ & (\{1,3,4\},\{1,2,3\}),(\{1,3,4\},\{1,2,4\}),(\{1,3,4\},\{2,3,4\})\} \end{aligned}$$

and make two lists as follows

α	β	$G_{\beta,\alpha}$	$\lvert G_{\beta,\alpha}\rvert$	$A_{\alpha,\beta}$	$\frac{\partial}{\partial a_{12}}\lvert A_{\alpha,\beta}\rvert$
{1,2,3}	{1,2,3}	$\begin{pmatrix} 2 & 3 & 5 \\ 3 & 4 & 6 \\ 3 & 1 & 1 \end{pmatrix}$	-4	$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$	$(-1)^{1+2}\times\begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} = 0$
{1,2,3}	{1,2,4}	$\begin{pmatrix} 2 & 3 & 5 \\ 3 & 4 & 6 \\ 1 & 1 & 1 \end{pmatrix}$	0	don't need	don't calculate
{1,2,3}	{2,3,4}	$\begin{pmatrix} 3 & 4 & 6 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	4	$\begin{pmatrix} 0 & 3 & 0 \\ -2 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$	$(-1)^{1+1}\times\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} = -2$
{1,2,4}	{1,2,3}	$\begin{pmatrix} 2 & 3 & 6 \\ 3 & 4 & 7 \\ 3 & 1 & 4 \end{pmatrix}$	-9	$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$(-1)^{1+2}\times\begin{vmatrix} 0 & 0 \\ 0 & -1 \end{vmatrix} = 0$
{1,2,4}	{1,2,4}	$\begin{pmatrix} 2 & 3 & 6 \\ 3 & 4 & 7 \\ 1 & 1 & 1 \end{pmatrix}$	0	don't need	don't calculate
{1,2,4}	{2,3,4}	$\begin{pmatrix} 3 & 4 & 7 \\ 3 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix}$	9	$\begin{pmatrix} 0 & 3 & 0 \\ -2 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$	$(-1)^{1+1}\times\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 1$
{1,3,4}	{1,2,3}	$\begin{pmatrix} 2 & 5 & 6 \\ 3 & 6 & 7 \\ 3 & 1 & 4 \end{pmatrix}$	-11	$\begin{pmatrix} 1 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 0 & -1 \end{pmatrix}$	$(-1)^{1+2}\times\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = 1$
{1,3,4}	{1,2,4}	$\begin{pmatrix} 2 & 5 & 6 \\ 3 & 6 & 7 \\ 1 & 1 & 1 \end{pmatrix}$	0	don't need	don't calculate
{1,3,4}	{2,3,4}	$\begin{pmatrix} 3 & 6 & 7 \\ 3 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix}$	11	$\begin{pmatrix} 0 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{pmatrix}$	$(-1)^{1+1}\times\begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} = 0$

and

γ	$(AG)_{\gamma,\gamma}$	$\lvert(AG)_{\gamma,\gamma}\rvert$
{1,2,3}	$\begin{pmatrix} 11 & 6 & 8 \\ -5 & -7 & -11 \\ 8 & 5 & 7 \end{pmatrix}$	-4
{1,2,4}	$\begin{pmatrix} 11 & 6 & 18 \\ -5 & -7 & -13 \\ -3 & -1 & -4 \end{pmatrix}$	-9
{1,3,4}	$\begin{pmatrix} 11 & 8 & 18 \\ 8 & 7 & 14 \\ -3 & -1 & -4 \end{pmatrix}$	0
{2,3,4}	$\begin{pmatrix} -7 & -11 & -13 \\ 5 & 7 & 14 \\ -1 & -1 & -4 \end{pmatrix}$	6

where

$$AG = \begin{pmatrix} 11 & 6 & 8 & 18 \\ -5 & -7 & -11 & -13 \\ 8 & 5 & 7 & 14 \\ -3 & -1 & -1 & -4 \end{pmatrix}.$$

Hence, $u = -4 - 9 + 0 + 6 = -7$. Thus, by using Equation (2.17), we have

$$w_{21} = -\frac{1}{7}(4 \times (-2) + 9 \times 1 + (-11) \times 1) = \frac{10}{7}.$$

In order to compare with the method for evaluating elements by the aid of $A^{(2)}_{T,S}$, we calculate a matrix X by using Equation (3.25) in Corollary 3.18 and show that X satisfies the conditions in [2, Theorem 2.13], that is, X is the generalized inverse $A^{(2)}_{T,S}$.

We compute

$$GAG = \begin{pmatrix} 29 & 10 & 12 & 43 \\ 40 & 13 & 15 & 58 \\ 24 & 12 & 16 & 39 \\ 11 & 3 & 3 & 15 \end{pmatrix}$$

and then take $\alpha = \{1, 3, 4\}, \beta = \{2, 3, 4\}$. Since

$$|(GAG)_{\alpha,\beta}| = \begin{vmatrix} 10 & 12 & 43 \\ 12 & 16 & 39 \\ 3 & 3 & 15 \end{vmatrix} = -42 \neq 0,$$

we have

$$\begin{aligned} X &= G_{*,\beta}(GAG)^{-1}_{\alpha,\beta}G_{\alpha,*} = \begin{pmatrix} 3 & 5 & 6 \\ 4 & 6 & 7 \\ 1 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 10 & 12 & 43 \\ 12 & 16 & 39 \\ 3 & 3 & 15 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 3 & 5 & 6 \\ 3 & 1 & 1 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1/7 & 0 & 6/7 & 15/7 \\ 10/7 & -1 & -17/7 & -4/7 \\ 4/7 & 0 & -4/7 & -3/7 \\ 9/7 & -1 & -23/7 & -19/7 \end{pmatrix}. \end{aligned}$$

Obviously, $\dim(T) = \dim(S^{\perp}) = 3 = \text{rank}(A)$. It is easy to show that $AT \oplus S = \mathbb{R}^4$ and X is $\{2\}$ inverse of A with $N(X) = S$ and $R(X) = T$. Therefore, by [2, Theorem 2.13], $A^{(2)}_{T,S} = (w_{ij})$ exists and $A^{(2)}_{T,S} = X$.

Thus, we have $w_{21} = \frac{10}{7}$. They are identical. ∎

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