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GENERALIZATIONS OF HERMITE-HADAMARD'S INEQUALITIES FOR LOG-CONVEX FUNCTIONS

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ABSTRACT. In this article, Hermite-Hadamard's inequalities are extended in terms of the weighted power mean and log-convex function. Several refinements, generalizations and related inequalities are obtained.

Key words and phrases: Hermite-Hadamard's inequalities, log-convex function, weighted power mean.

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1. INTRODUCTION

Let $f(x)$ be a convex function on the closed interval $[a,b]$, the well known Hermite-Hadamard's inequalities are expressed as

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The middle term of inequality (1) is called the arithmetic mean of the function $f(x)$ on the interval [a; b]; the right term in (1) is the arithmetic mean of numbers$f(a)$ and $f(b)$.

It is well known that Hermite-Hadamard's inequalities are an important cornerstone in mathematical analysis and optimization. There exists a very extensive literature on its refinements and generations, please refer to [1, 2, 3, 10, 11] and the references therein.

In 1976, the generations of Hermite-Hadamard's inequalities were obtained by Vasić and Lacković [15] and Lupaş [9]:

Theorem A. *Let p,q be given positive numbers and $a_1 < a < b < b_1$. Then the inequalities*

$$f\left(\frac{pa+qb}{p+q}\right) \leq \frac{1}{2y}\int_{M-y}^{M+y} f(x)\,\mathrm{d}x \leq \frac{pf(a)+qf(b)}{p+q} \tag{1.2}$$

hold for $M = \frac{pa+qb}{p+q}$, and all continuous convex functions $f : [a_1, b_1] \to \mathbb{R}$ iff

$$y \leq \frac{b-a}{p+q}\min\{p,q\}.$$

If $p = q = 1$ and $y = \frac{b-a}{2}$, (1.2) are the Hermite-Hadamard's inequalities.

In 1998, S. S. Dragomir and B. Mond [4] proved that

Theorem B. *Let $f : I \to [0,\infty)$ be a log-convex mapping on I, $a,b \in I$ with $a < b$. Then we have*

$$f[A(a,b)] \leq \frac{1}{b-a}\int_a^b G[f(x), f(a+b-x)]\,\mathrm{d}x \leq G[f(a), f(b)], \tag{1.3}$$

where A is arithmetic mean, G geometric mean and I an interval of real numbers. In what follows, I will be used to denote an interval of real numbers.

Log-convex function is defined in [12, p.7] and presented as :

Definition. *A function $f : I \to [0,\infty)$ is said to be* log-convex *or* multiplicatively convex *if $\log f$ is convex, or, equivalently, if for all $x,y \in I$ and $t \in [0,1]$ one has inequality:*

$$f(tx+(1-t)y) \leq [f(x)]^t [f(y)]^{1-t}. \tag{1.4}$$

Since $f = \exp(\log f)$, it follows that a log-convex function is convex, but the converse may not necessarily be true [12, p.7]. Following directly from (1.4), by the arithmetic-geometric mean inequality, we have

$$[f(t)]^t [f(y)]^{1-t} \leq tf(t) + (1-t)f(y) \tag{1.5}$$

for all $x,y \in I$ and $t \in [0,1]$.

Moreover, in 1999, weighted power mean was defined by Jeong Sheok Vme and Young Ho Kim [8, p. 49] as

$$M_p(r;a,b) = [ra^p + (1-r)b^p]^{1/p} \quad (0 \leq r \leq 1)$$

for $a,b > 0, p \neq 0$.

Let $p = 1$. Then

$$M_1(r;a,b) = ra + (1-r)b \triangleq A(r;a,b)$$

is weighted arithmetic mean, and let $p = 0$,

$$M_0(r;a,b) = \lim_{p\to 0} M_p(r;a,b) = a^r b^{1-r} \triangleq G(r;a,b)$$

is weighted geometric mean.

In 1998, Feng Qi [13] proved that the weighted power mean $M_p(r;a,b)$ is increasing in both p and r and in both a and b. For more information about weighted power mean please see [13, 14].

In this paper, motivated by S. S. Dragomir's results, we will use weighted arithmetic mean and weighted geometric mean to extend the Hermite-Hadamard's inequalities. Some new inequalities will be deduced.

Theorem 1. *Let $f : I \to [0,\infty)$ be a log-convex mapping on I, and $a,b \in I$ with $a < b$. Then we have two inequalities:*

$$f[A(r;a,b)] \le \frac{1}{2y}\int_{M-y}^{M+y} A[r; f(x), f(2A-x)]\,\mathrm{d}x \le A[r; f(a), f(b)] \tag{1.6}$$

for $M = \frac{pa+qb}{p+q}$ and $y \le \frac{b-a}{p+q}\min\{p,q\}$;

$$f[A(r;a,b)] \le \frac{1}{b-a}\int_a^b G[r; f(x), f(a+b-x)]\,\mathrm{d}x \le G[r; f(a), f(b)], \tag{1.7}$$

where $r = \frac{p}{p+q}$ and p,q are positive numbers.

Corollary 1. *If $a,b \in I$ with $0 \le a < b$ and f is nondecreasing on I, we get*

$$f[G(r;a,b)] \le \frac{1}{b-a}\int_a^b G[r; f(x), f(a+b-x)]\,\mathrm{d}x \le G[r; f(a), f(b)], \tag{1.8}$$

where $0 \le r \le 1$.

Corollary 2. *Let $g : I \to \mathbb{R}$ be a convex mapping on I and $a,b \in I$ with $0 \le a < b$. Then we have*

$$\begin{aligned} g[A(r;a,b)] &\le \ln\left[\frac{1}{b-a}\int_a^b \exp\big[A[r; g(x), g(a+b-x)]\big]\,\mathrm{d}x\right] \\ &\le A[r; g(a), g(b)], \end{aligned} \tag{1.9}$$

where $0 \le r \le 1$.

The following theorem for log-convex functions also holds.

Theorem 2. *Let $f : I \to (0,\infty)$ be a log-convex mapping on I, $0 \le r \le 1$ and $a,b \in I$ with $a < b$, then we have*

$$\begin{aligned} f[A(r;a,b)] &\le \exp\left[\frac{1}{2y}\int_{M-y}^{M+y} \ln f(x)\,\mathrm{d}x\right] \\ &\le \frac{1}{2y}\int_{M-y}^{M+y} G\big[f(x), f(2M-x)\big]\,\mathrm{d}x \\ &\le \frac{1}{2y}\int_{M-y}^{M+y} f(x)\,\mathrm{d}x \\ &\le L\big[f(M-y), f(M+y)\big], \end{aligned} \tag{1.10}$$

where $M = ra + (1-r)b$ and the logarithmic mean $L(p,q) = \frac{p-q}{\ln p - \ln q}$.

Corollary 3. *Let $g : I \to \mathbb{R}$ be a convex mapping on I, $0 \le r \le 1$ and $a, b \in I$ with $a < b$, then we have*

$$\exp g[A(r;a,b)] \le \exp\left[\frac{1}{2y}\int_{M-y}^{M+y} g(x)\,\mathrm{d}x\right] \le \frac{1}{2y}\int_{M-y}^{M+y} \exp\left[\frac{g(x)+g(2M-x)}{2}\right]\mathrm{d}x \le \frac{1}{2y}\int_{M-y}^{M+y} \exp g(x)\,\mathrm{d}x \le E\big[f(M-y), f(M+y)\big]. \tag{1.11}$$

where $M = ra + (1-r)b$ and the exponential mean $E(p,q) = \frac{\exp p - \exp q}{p-q}$.

2. Proofs of Theorems

Proof of Theorem 1. It is clear that

$$\int_{M-y}^{M+y} f(x)\,\mathrm{d}x = \int_{M-y}^{M+y} f(2M-x)\,\mathrm{d}x.$$

Applying $A(r;a,b) = ra+(1-r)b$ with $r = \frac{p}{p+q}$ to inequality(1.2), we obtain the inequalities (1.6) immediately.

Since f is log-convex, we have

$$f[ta+(1-t)b] \le [f(a)]^t[f(b)]^{1-t},$$

$$f[(1-t)a+b] \le [f(a)]^{1-t}[f(b)]^t$$

for all $t \in [0,1]$.

If we multiply the above inequalities and take weighted geometric mean, we obtain

$$G\big[r; f(ta+(1-t)b), f((1-t)a+tb)\big] \le G\big[r; f(a), f(b)\big]$$

for all $t \in [0,1]$.

Integrating this inequality on [0,1], we get

$$\int_0^1 G\big[r; f(ta+(1-t)b), f((1-t)a+tb)\big]\,\mathrm{d}t \le G\big[r; f(a), f(b)\big].$$

If we change the variable $x := ta+(1-t)b$, $t \in [0,1]$, we obtain

$$\int_0^1 G\big[r; f(ta+(1-t)b), f((1-t)a+tb)\big]\,\mathrm{d}t = \frac{1}{b-a}\int_a^b G[r; f(x), f(a+b-x)]\,\mathrm{d}x$$

and the second inequality in (1.7) is proved.

Let us reconsider the generalization of Hermite-Hadamard's inequalities (1.2). Applying the log-convex functions $f : I \to [0,\infty)$ to (1.2), we have

$$\ln\left[f\left(\frac{pa+qb}{p+q}\right)\right] \le \frac{1}{2y}\int_{M-y}^{M+y} \ln f(x)\,\mathrm{d}x \le \frac{p\ln f(a) + q\ln f(b)}{p+q}.$$

Hence, we get

$$f\left(\frac{pa+qb}{p+q}\right) \le \exp\left[\frac{1}{2y}\int_{M-y}^{M+y} \ln f(x)\,\mathrm{d}x\right] \le \sqrt[p+q]{f(a)^p f(b)^q}. \tag{2.1}$$

From inequality (2.1), we have

$$f\left(\frac{p\alpha+q\beta}{p+q}\right) \leq \sqrt[p+q]{f(\alpha)^p f(\beta)^q}$$

for all $\alpha, \beta \in I$.

If we choose $\alpha = ta + (1-t)b$, $\beta = (1-t)a + tb$, then we get the inequality

$$f[A(r;a,b)] \leq G[r; f(ta+(1-t)b), f((1-t)a+tb)] \tag{2.2}$$

for all $t \in [0,1]$.

Integrating this inequality (2.2) on $[0,1]$ over t, we obtain the first inequality in (1.7). The theorem 1 is proved completely. ∎

Proof of Corollary 1. Since f is nondecreasing on I, and $A(r;a,b) \geq G(r;a,b)$, we get

$$f[A(r;a,b)] \geq f[G(r;a,b)].$$

So the inequalities (1.8) hold. ∎

Proof of Corollary 2. Define the mapping $f :\rightarrow (0,\infty)$, $f(x) = \exp g(x)$, which is clearly log-convex on I. Using theorem 1, we obtain

$$\begin{aligned}\exp g[A(r;a,b)] &\leq \frac{1}{b-a}\int_a^b G[r; \exp g(x), \exp g(a+b-x)]\,\mathrm{d}x \\ &\leq G[r; \exp g(a), \exp g(b)].\end{aligned}$$

By taking the logarithm in both sides, we get the inequalities (1.9). ∎

Proof of Theorem 2. The first inequality is proved in (2.1).

We now have

$$G\big[r; f(x), f(2M-x)\big] = \exp\left[\ln\big(G(r; f(x), f(2M-x))\big)\right]$$

for all $x \in [a,b]$.

Integrating this equality on $[a,b]$ and using the well known Jensen's integral inequality for the convex mapping exp(.), we have

$$\begin{aligned}&\frac{1}{2y}\int_{M-y}^{M+y} G\big[r; f(x), f(2M-x)\big]\,\mathrm{d}x \\ &= \frac{1}{2y}\int_{M-y}^{M+y} \exp\left[\ln\big(G(r; f(x), f(2M-x))\big)\right]\mathrm{d}x \\ &\geq \exp\left[\frac{1}{2y}\int_{M-y}^{M+y} \ln\big(G(r; f(x), f(2M-x))\big)\,\mathrm{d}x\right] \\ &= \exp\left[\frac{1}{2y}\int_{M-y}^{M+y} A\big(r; f(x), f(2M-x)\big)\,\mathrm{d}x\right] \\ &= \exp\left[\frac{1}{2y}\int_{M-y}^{M+y} \ln f(x)\,\mathrm{d}x\right].\end{aligned}$$

Since it is obvious that

$$\int_{M-y}^{M+y} \ln f(x)\,\mathrm{d}x = \int_{M-y}^{M+y} \ln f(2M-x)\,\mathrm{d}x,$$

the second inequality in (1.10) is proved.

By the monotonicity of weighted power mean, we have the inequality

$$G\big[r; f(x), f(2M-x)\big] \leq A\big[r; f(x), f(2M-x)\big]$$

for $x \in [a,b]$. By integration, we obtain

$$\frac{1}{2y}\int_{M-y}^{M+y} G\big[r; f(x), f(2M-x)\big]\,\mathrm{d}x \leq \frac{1}{2y}\int_{M-y}^{M+y} f(x)\,\mathrm{d}x \tag{2.3}$$

and the third inequality (1.10) is proved.

To prove the last inequality, we use the log-convexity of f. Thus, we have

$$f\big[t(M-y)+(1-t)(M+y)\big] \leq [f(M-y)]^t[f(M+y)]^{1-t} \tag{2.4}$$

for all $t \in [0,1]$.

Integrating (2.4) over t on $[0,1]$, we get

$$\int_0^1 f\big[t(M-y)+(1-t)(M+y)\big]\,\mathrm{d}t \leq \int_0^1 [f(M-y)]^t[f(M+y)]^{1-t}\,\mathrm{d}t. \tag{2.5}$$

As we know,

$$\int_0^1 f\big[t(M-y)+(1-t)(M+y)\big]\,\mathrm{d}t = \frac{1}{2y}\int_{M-y}^{M+y} f(x)\,\mathrm{d}x$$

and

$$\int_0^1 [f(M-y)]^t[f(M+y)]^{1-t}\,\mathrm{d}t = L\big[f(M-y), f(M+y)\big].$$

Applying these two equations to the inequality (2.5), we obtain the fourth inequality.

The proof is complete. ∎

Proof of Corollary 3. Using the same method of proof of theorem 2 with $\exp g(x)$ instead of $f(x)$, we obtain the inequalities (1.11). The details will be omitted. ∎

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