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PRODUCT FORMULAS INVOLVING GAUSS HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. New formulas for a product of two Gauss hypergeometric functions are derived. Applications to special functions, with emphasis on Jacobi polynomials, Jacobi functions, and Bessel functions of the first kind, are included. Most of the results are obtained with the aid of the double Dirichlet average of a univariate function.

Key words and phrases: Product formulas, Gauss hypergeometric function, Jacobi polynomials, Jacobi functions, Bessel functions of the first kind, Dirichlet averages.

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1. INTRODUCTION AND DEFINITIONS

Product formulas for special functions and orthogonal polynomials are of great importance in theory of special functions and their applications (see, e.g., [1], [7, 8, 9, 10]). Most of the product formulas have been obtained with the aid of either analytic or algebraic methods.

Let us introduce some notation which will be used throughout the sequel. In what follows the symbols $\mathbb{R}_>$ and $\mathbb{C}_>$ will stand for the positive semiaxis and the set of all complex numbers with a positive real part, respectively. Also, let $U = \{z \in \mathbb{C} : z \neq 0, -1, \ldots\}$. As usual, the symbol ${}_2F_1$ will stand for the Gauss hypergeometric function. For $\mu, \alpha, \beta \in \mathbb{C}$ ($\alpha, \beta \neq -1, -2, \ldots$) we let $\lambda = \alpha + \beta + 1$. Also, let $x > -1$ and $y < 1$ when $\mu \in U$ and let $x, y \in \mathbb{R}$ if $-\mu \in \mathbb{N}$ – set of all nonnegative integers. The product discussed in this paper is defined as follows

$$\pi(x,y) = {}_2F_1\left(-\mu, \lambda+\mu; \alpha+1; \frac{1-x}{2}\right) {}_2F_1\left(-\mu, \lambda+\mu; \beta+1; \frac{1+y}{2}\right). \tag{1.1}$$

It is demonstrated (see Theorem 2.1) that $\pi(x,y)$ can be expressed as the double Dirichlet average of the normalized Gegenbauer function $C_{2\mu}^{\lambda}$ (see (2.4)). Also, the product in question can be expressed as a double integral (see (2.6)) and also as the infinite series of Jacobi polynomials (see (2.8)). For the related results the interested reader is referred to [8, 9, 10].

For the reader's convenience, we recall definitions of Dirichlet average and double Dirichlet average of a univariate function. We need more notation. For $0 \le u, v \le 1$, let $\bar{u} = [u, 1-u]$, $\bar{v} = [v, 1-v]$, $b = [b_1, b_2] \in \mathbb{C}^2_>$, and let $d = [d_1, d_2] \in \mathbb{C}^2_>$. The Dirichlet measure μ_b on the unit interval $[0,1]$ is defined as [7]

$$\mu_b(u) = \frac{1}{B(b_1, b_2)} u^{b_1-1}(1-u)^{b_2-1}, \tag{1.2}$$

where $B(\cdot,\cdot)$ stands for the beta function.

Definition 1.1. [7, (5.2-1)] Dirichlet average of a holomorphic function f defined on conv$(\bar{x})$ – the convex hull of $\bar{x}$ is given by

$$F(b;\bar{x}) = \int_0^1 f(\bar{u}\cdot\bar{x})\mu_b(u)\,du, \tag{1.3}$$

where $\bar{u}\cdot\bar{x}$ is the dot product of vectors $\bar{u}$ and $\bar{x}$. In what follows the symbol R_a will stand for the Dirichlet average of the power function t^a ($t > 0$).

In [4] B. C. Carlson has introduced the notion of the double Dirichlet average of a function of one variable. Let A be a 2-by-2 real or complex matrix with entries a_{ij} ($1 \le i, j \le 2$) and let conv(A) denote the convex hull of A. In what follows the symbol $\bar{u}\cdot A\cdot\bar{v}$ will stand for the vector-matrix-vector product, i.e.,

$$u\cdot A\cdot v = ua_{11}v + ua_{12}(1-v) + (1-u)a_{21}v + (1-u)a_{22}(1-v)$$

and μ_d will denote the Dirichlet measure on $[0,1]$ with parameters $d \in \mathbb{C}^2_>$ (see (1.2)).

Definition 1.2. [4, p. 421] Let f be a holomorphic function defined on a domain D and assume that conv$(A) \subset D$. The double Dirichlet average of f, denoted by $\mathcal{F}(b;A;d)$, is defined by

$$\mathcal{F}(b;A;d) = \int_0^1\int_0^1 f(\bar{u}\cdot A\cdot\bar{v})\mu_b(u)\mu_d(v)\,du\,dv. \tag{1.4}$$

The double Dirichlet averages of t^a ($t > 0$) and e^t will be denoted by $\mathcal{R}_a$ and $\mathcal{S}$, respectively.

Among numerous properties of the double Dirichlet average the following ones are of special interest:

(a) Transposition property: $\mathcal{F}(b;A;d) = \mathcal{F}(d;A^T;b)$.

(b) The average $\mathcal{F}$ is holomorphic in the elements of b, A and d on its domain of definition.
(c) $\mathcal{F}$ can be continued analytically in the parameters of b and d and variables a_{ij} as long as $b_1 + b_2$, $d_1 + d_2 \in U$ and all variables remain in D, provided D is simply connected.

This paper is organized as follows. The main result is presented in Section 2. Applications to Jacobi polynomials, Jacobi functions, and the Bessel functions of the first kind are obtained in Section 3. Some of the previously established results are also included in this section.

2. MAIN RESULT

Before we shall state and prove the main result of this paper let us introduce more notation. For $\alpha + \frac{1}{2}$, $\beta + \frac{1}{2} \in \mathbb{C}_>$ let

$$b = \left[\alpha + \frac{1}{2}, \alpha + \frac{1}{2}\right], \quad d = \left[\beta + \frac{1}{2}, \beta + \frac{1}{2}\right]. \tag{2.1}$$

Also, let

$$X = \begin{bmatrix} r+s & r-s \\ -r+s & -r-s \end{bmatrix}, \tag{2.2}$$

where

$$r^2 = (1-x)(1-y)/4, \quad s^2 = (1+x)(1+y)/4. \tag{2.3}$$

Recall (see Section 1) that $x > -1$ and $y < 1$ when $\mu \in U$ and $x, y \in \mathbb{R}$ when $-\mu \in \mathbb{N}$. Also, the symbol $C^{\lambda}_{2\mu}$ will stand for the Gegenbauer function of degree 2μ with parameter λ.

Theorem 2.1. *Let the numbers μ, α, β, x, and y satisfy conditions stated above. Then*

$$\pi(x, y) = \mathcal{F}(b; X; d), \tag{2.4}$$

where $\lambda = \alpha + \beta + 1$. Here $\mathcal{F}$ is the double Dirichlet average of

$$f(z) = C^{\lambda}_{2\mu}(z)/C^{\lambda}_{2\mu}(0). \tag{2.5}$$

Moreover, the following formulas

$$\pi(x, y) = \int_0^{\pi}\int_0^{\pi} \left[C^{\lambda}_{2\mu}(r\cos\psi + s\cos\theta)/C^{\lambda}_{2\mu}(0)\right] dm(\psi, \theta), \tag{2.6}$$

where

$$dm(\psi, \theta) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\pi\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)}(\sin\psi)^{2\alpha}(\sin\theta)^{2\beta} d\psi\, d\theta \tag{2.7}$$

and

$$\pi(x, y) = \sum_{m=0}^{\infty} \frac{(-\mu, m)(\lambda+\mu, m)}{(\alpha+1, m)(\beta+1, m)} \left(\frac{x+y}{2}\right)^m P^{(\alpha,\beta)}_m\left(\frac{1+xy}{x+y}\right) \tag{2.8}$$

hold true. Here $(a, 0) = 1$, $(a, m) = a(a+1)\cdot\ldots\cdot(a+m-1)$ is the Appell symbol (see, e.g., [7, Chap. 2]*) and $P^{(\alpha,\beta)}_m$ stands for the* m*th Jacobi polynomial of order (α, β).*

Proof. In order to establish the product formula (2.4) we shall use Appell's hypergeometric function F_4. Recall that

$$F_4(a, a'; c, c'; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a, m+n)(a', m+n)}{(c, m)(c', m)m!n!} x^m y^n$$

($a, a', x, y \in \mathbb{C}$, $c, c' \in U$, $|x|^{1/2} + |y|^{1/2} < 1$). (See, e.g., [7, Ex. 7.1-13]). B.C. Carlson [6, (4.1)] has shown that

$$\begin{aligned} &F_4\big(a, a'; c, 1+a+a'-c; z(1-w), w(1-z)\big) \\ &\qquad = R_{-a}(a', c-a'; 1-z, 1) R_{-a}(a', 1+a-c; 1-w, 1). \end{aligned}$$

Letting $a = -\mu$, $a' = \lambda + \mu$, $c = \alpha + 1$, $z = (1-x)/2$, $w = (1+y)/2$ and using the following formula [7, (5.9-11)]

$$R_{-a}(a', c-a'; 1-z, 1) = {}_2F_1(a, a'; c; z)$$

(Re $c >$ Re $a' > 0$, $a \in U$, $z < 1$) followed by application of (1.1) we obtain

$$\pi(x, y) = F_4(-\mu, \lambda + \mu; \alpha + 1, \beta + 1; r^2, s^2), \tag{2.9}$$

where r^2 and s^2 are defined in (2.3). On the other hand, letting $a = -\mu$, $a' = \lambda + \mu$, $c = \alpha + 1$, $c' = \beta + 1$, $x = r$, and $y = s$ in

$$\begin{aligned} &F_4(a, a'; c, c'; x^2, y^2) \\ &\quad = \sum_{m=0}^{\infty} \frac{(a, m)(a', m)}{\left(\frac{1}{2}, m\right) m!} \mathcal{R}_{2m}\left(c - \frac{1}{2}, c - \frac{1}{2}; W; c' - \frac{1}{2}, c' - \frac{1}{2}\right), \end{aligned}$$

where

$$W = \begin{bmatrix} x + y & x - y \\ -x + y & -x - y \end{bmatrix}$$

(see [6, (2.9)]) we obtain

$$F_4(-\mu, \lambda + \mu; \alpha + 1, \beta + 1; r^2, s^2) = \sum_{m=0}^{\infty} \frac{(-\mu, m)(\lambda + \mu, m)}{\left(\frac{1}{2}, m\right) m!} \mathcal{R}_{2m}(b; X; d), \tag{2.10}$$

where b and d are defined in (2.1) and the matrix X is given in (2.2). To complete the proof of (2.4) it suffices to show that the right side of (2.10) is equal to $\mathcal{F}(b; X; d)$. Using a series expansion of the function $f(z)$ defined in (2.5) we have

$$\begin{aligned} f(z) = C^{\lambda}_{2\mu}(z)/C^{\lambda}_{2\mu}(0) &= {}_2F_1\left(-\mu, \lambda + \mu; \frac{1}{2}, z^2\right) \\ &= \sum_{m=0}^{\infty} \frac{(-\mu, m)(\lambda + \mu, m)}{\left(\frac{1}{2}, m\right) m!} z^2 m. \end{aligned}$$

Averaging the first and last members we obtain

$$\mathcal{F}(b; X; d) = \sum_{m=0}^{\infty} \frac{(-\mu, m)(\lambda + \mu, m)}{\left(\frac{1}{2}, m\right) m!} \mathcal{R}_{2m}(b; X; d). \tag{2.11}$$

This in conjunction with (2.10) gives

$$F_4(-\mu, \lambda + \mu; \alpha + 1, \beta + 1; r^2, s^2) = \mathcal{F}(b; X; d).$$

Combining this with (2.9) gives the desired result (2.4). In order to establish formula (2.6) we apply (1.4) to the right side of (2.4) to obtain

$$\mathcal{F}(b; X; d) = \int_0^1 \int_0^1 f\big[(2u-1)r + (2v-1)s\big] \mu_b(u) \mu_d(v)\, du\, dv,$$

where f is defined in (2.5),

$$\mu_b(u) = \big[u(1-u)\big]^{\alpha - \frac{1}{2}} / B\left(\alpha + \frac{1}{2}, \alpha + \frac{1}{2}\right),$$

and

$$\mu_d(v) = \left[v(1-v)\right]^{\beta-\frac{1}{2}}/B\left(\beta+\frac{1}{2},\beta+\frac{1}{2}\right).$$

Letting $2u-1=\cos\psi$, $2v-1=\cos\theta$ ($0\le\psi,\theta\le\pi$) and next using $1/B\left(\alpha+\frac{1}{2},\alpha+\frac{1}{2}\right) = \Gamma(2\alpha+1)/\left[\Gamma\left(\alpha+\frac{1}{2}\right)\right]^2$ followed by application of Legendre's duplication formula

$$\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)/\sqrt{\pi}$$

($2z\in U$) we obtain

$$\mathcal{F}(b;X;d) = \int_0^\pi\int_0^\pi f(r\cos\psi+s\cos\theta)\,dm(\psi,\theta), \tag{2.12}$$

where $dm(\psi,\theta)$ is defined in (2.7). This completes the proof of (2.6). The product formula (2.8) follows immediately by use of

$$\mathcal{R}_{2m}(b;X;d) = \frac{\left(\frac{1}{2},m\right)m!}{(\alpha+1,m)(\beta+1,m)}\left(\frac{x+y}{2}\right)^m P_m^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right)$$

(see [12, (3.19)]) on (2.11). The proof is complete. ∎

3. APPLICATIONS

In this section we shall show, among other things, how some known product formulas can be derived from the theorem contained in Section 2.

Using a group theoretic method, Dijksma and Koornwinder [8] have proven the following.

Corollary 3.1. *Let* $\alpha+\frac{1}{2}$, $\beta+\frac{1}{2}\in\mathbb{C}_>$, $x,y\in\mathbb{R}$. *Then*

$$\begin{aligned}&\frac{P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)P_n^{(\alpha,\beta)}(-1)}\\ &\qquad = \int_0^\pi\int_0^\pi\left[C_{2n}^\lambda(r\cos\psi+s\cos\theta)/C_{2n}^\lambda(0)\right]dm(\psi,\theta),\end{aligned} \tag{3.1}$$

where C_{2n}^λ *is the Gegenbauer polynomial of degree* $2n$ *with parameter* $\lambda=\alpha+\beta+1$ *and* $dm(\psi,\theta)$ *is defined in* (2.7).

Proof. Apply

$$P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1) = {}_2F_1\left(-n,\lambda+n;\alpha+1;\frac{1-x}{2}\right) \tag{3.2}$$

and

$$P_n^{(\alpha,\beta)}(y)/P_n^{(\alpha,\beta)}(-1) = {}_2F_1\left(-n,\lambda+n;\beta+1;\frac{1+y}{2}\right) \tag{3.3}$$

to (2.6) to obtain the assertion (3.1). ∎

An analytic proof of (3.1) appears in [10]. In his proof the author used, among other things, a formula which is due to Bateman [3] and is contained in the following.

Corollary 3.2. *We have*

$$\begin{aligned}\frac{P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)P_n^{(\alpha,\beta)}(-1)} \\ = \sum_{m=0}^{n} \frac{(-n,m)(\lambda+n,m)}{(\alpha+1,m)(\beta+1,m)}\left(\frac{x+y}{2}\right)^m P_m^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right).\end{aligned} \tag{3.4}$$

Proof. In (1.1) and (2.8) put $\mu = n$ and next use (3.2) and (3.3). ∎

We shall now deal with the Jacobi functions $\phi_t^{(\alpha,\beta)}$. For $t \in \mathbb{C}$ and $\alpha + \frac{1}{2}$, $\beta + \frac{1}{2} \in \mathbb{C}_>$ they are defined in terms of Gauss' hypergeometric function

$$\phi_t^{(\alpha,\beta)}(p) = {}_2F_1\left(\frac{\lambda+it}{2}, \frac{\lambda-it}{2}; \alpha+1; -(\sinh p)^2\right), \tag{3.5}$$

where $p \geq 0$ and $\lambda = \alpha + \beta + 1$ (see, e.g., [1, (2.48)], [9, (2.1)]). Jacobi functions form a continuous orthogonal system of functions on $[0, \infty)$ with respect to the weight function $(\sinh t)^{2\alpha+1}(\cosh t)^{2\beta+1}$. For later use we define a product

$$\pi_t^{(\alpha,\beta)}(p,q) = \phi_t^{(\alpha,\beta)}(p)\phi_t^{(\beta,\alpha)}(q) \tag{3.6}$$

and two numbers

$$r = i(\sinh p)(\cosh q), \quad s = i(\cosh p)(\sinh q). \tag{3.7}$$

In what follows the symbol X will stand for the matrix defined in (2.2) with r and s as defined in (3.7).

We are in a position to state and prove the following.

Corollary 3.3. *The following formulas*

$$\pi_t^{(\alpha,\beta)}(p,q) = \mathcal{F}(b; X; d), \tag{3.8}$$

where $\mathcal{F}$ stands for the double Dirichlet average of the function

$$f(z) = C^{\lambda}_{-(\lambda+it)}(z)/C^{\lambda}_{-(\lambda+it)}(0) = {}_2F_1\left(\frac{\lambda-it}{2}, \frac{\lambda+it}{2}; \frac{1}{2}; z^2\right),$$

$$\pi_t^{(\alpha,\beta)}(p,q) = \int_0^{\pi}\int_0^{\pi} f(r\cos\psi + s\cos\theta)\, dm(\psi,\theta), \tag{3.9}$$

where $dm(\psi,\theta)$ is defined in (2.7), *and*

$$\begin{aligned}&\pi_t^{(\alpha,\beta)}(p,q) \\ &= \sum_{m=0}^{\infty} \frac{\big((\lambda+it)/2, m\big)\big((\lambda-it)/2, m\big)}{(\alpha+1,m)(\beta+1,m)}(s^2-r^2)^m P_m^{(\alpha,\beta)}\left(\frac{s^2+r^2}{s^2-r^2}\right)\end{aligned} \tag{3.10}$$

($p \neq q$) are valid.

Proof. In order to prove the product formula (3.8) we substitute $\mu = -(\lambda+it)/2$ into (1.1) and (2.5). Letting $x = 1 + 2(\sinh p)^2$ and $y = -1 - 2(\sinh q)^2$ in (2.3) we obtain (3.7). Thus (3.8) is a special case of (2.4). With μ and f as defined earlier, we see that formula (3.9) follows immediately from (2.6). With the same substitution for μ, x, and y one easily obtains using (3.7), the desired result (3.10). ∎

The Mehler-Dirichlet formula for the Jacobi functions is a special case of (3.9). We have the following.

Corollary 3.4. *Let* $\alpha+\frac{1}{2}, \beta+\frac{1}{2} \in \mathbb{C}_>$, $t \in \mathbb{C}$, *and* $p \geq 0$. *Then*

$$
\begin{aligned}
\phi_t^{(\alpha,\beta)}(p) &= \frac{2}{B\left(\alpha+\frac{1}{2},\frac{1}{2}\right)}(\sinh p)^{-2\alpha} \\
&\cdot \int_0^p \frac{\cosh x}{(\sinh^2 p - \sinh^2 x)^{\frac{1}{2}-\alpha}}\, {}_2F_1\left(\frac{\lambda+it}{2}, \frac{\lambda-it}{2}; \frac{1}{2}; -(\sinh x)^2\right) dx.
\end{aligned} \tag{3.11}
$$

Proof. First, we let $q=0$ in (3.6) and next use (3.5) to obtain

$$
\pi_t^{(\alpha,\beta)}(p,0) = \phi_t^{(\alpha,\beta)}(p). \tag{3.12}
$$

Letting $q=0$ in (3.7) we obtain $r = i\sinh p$ and $s=0$. It follows from (2.7) that

$$
dm(\psi,\theta) = \frac{1}{B\left(\alpha+\frac{1}{2},\frac{1}{2}\right)B\left(\beta+\frac{1}{2},\frac{1}{2}\right)}(\sin\psi)^{2\alpha}(\sin\theta)^{2\beta}d\psi\, d\theta.
$$

This in conjunction with (3.12) and (3.9) gives

$$
\begin{aligned}
\pi_t^{(\alpha,\beta)}(p) &= \frac{1}{B\left(\beta+\frac{1}{2},\frac{1}{2}\right)}\int_0^\pi (\sin\theta)^{2\beta}d\theta \\
&\cdot \frac{1}{B\left(\alpha+\frac{1}{2},\frac{1}{2}\right)}\int_0^\pi {}_2F_1\left(\frac{\lambda+it}{2}, \frac{\lambda-it}{2}; \frac{1}{2}; -(\sinh p\cos\psi)^2\right)(\sin\psi)^{2\alpha}d\psi.
\end{aligned}
$$

Taking into account that

$$
\int_0^\pi (\sin\theta)^{2\beta}d\theta = B\left(\beta+\frac{1}{2},\frac{1}{2}\right)
$$

and introducing a new variable of integration x, where $\sinh x = \sinh p\cos\psi$ we obtain, after a little algebra, the desired formula (3.11). ∎

A third product discussed in this section involves Bessel functions of the first kind. For $x, y \in \mathbb{C}$ let

$$
\pi_J(x,y) = x^{-\alpha}J_\alpha(x)y^{-\beta}J_\beta(y). \tag{3.13}
$$

For later use, let

$$
c(\alpha,\beta) = \left[2^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)\right]^{-1}
$$

and

$$
X = \begin{bmatrix} x+y & x-y \\ -x+y & -x-y \end{bmatrix}.
$$

Corollary 3.5. *Let* $f(z) = \cos z$. *The following product formulas*

$$
\pi_J(x,y) = c(\alpha,\beta)\mathcal{F}(b;X;d) \tag{3.14}
$$

and

$$
\begin{aligned}
\pi_J(x,y) &= \frac{1}{\pi 2^{\alpha+\beta}\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)} \\
&\cdot \int_0^\pi\int_0^\pi f(x\cos\psi + y\cos\theta)(\sin\psi)^{2\alpha}(\sin\theta)^{2\beta}d\psi\, d\theta
\end{aligned} \tag{3.15}
$$

($\alpha+\frac{1}{2}, \beta+\frac{1}{2} \in \mathbb{C}_>$) *are valid.*

Proof. The following result

$$\pi_J(x,y) = c(\alpha,\beta)\mathcal{S}(b;Y;d), \tag{3.16}$$

where $\mathcal{S}$ stands for the double Dirichlet average of $s(z) = \exp(z)$ and $Y = iX$ is mentioned in [4, p. 427]. Application of

$$\mathcal{S}(b;Y;d) = \sum_{m=0}^{\infty} \frac{1}{m!}\mathcal{R}_m(b;Y;d)$$

(see [4, (6.4)]) to (3.16) together with the use of $\mathcal{R}_{2m+1}(b;Y;d) = 0$ (see [5, (5.8)]) and $\mathcal{R}_{2m}(b;Y;d) = (-1)^m\mathcal{R}_{2m}(b;X;d)$ gives

$$\pi_J(x,y) = c(\alpha,\beta)\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!}\mathcal{R}_{2m}(b;X;d) = c(\alpha,\beta)\mathcal{F}(b;X;d), \tag{3.17}$$

where in the last step we have used the fact that the double Dirichlet average of

$$f(z) = \cos z = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!}z^{2m}$$

is equal to $\mathcal{F}(b;X;d)$. Koornwinder's formula (3.15) (see [10, (4.4]) follows immediately from (3.14) and (2.12). ∎

We close this section with a new proof of the following result of Bateman [2, p. 113–114].

Corollary 3.6. *Let $c(\alpha,\beta)$ be the same as in Corollary 3.5 and let I_α stand for the modified Bessel function of the first kind of order α. Then for $x, y \in \mathbb{C}$ ($x, y \neq 0$)*

$$\begin{aligned} &x^{-\beta}J_\beta(x)y^{-\alpha}I_\alpha(y) \\ &= c(\alpha,\beta)\sum_{m=0}^{\infty}(-1)^m \frac{(x^2+y^2)^m}{(\alpha+1,m)(\beta+1,m)2^{2m}}P_m^{(\alpha,\beta)}\left(\frac{x^2-y^2}{x^2+y^2}\right). \end{aligned} \tag{3.18}$$

Proof. In (3.13) we interchange α with β and next we let $y := iy$. Use of

$$(iy)^{-\alpha}J_\alpha(iy) = y^{-\alpha}I_\alpha(y)$$

together with (3.14) and (3.17) yields

$$x^{-\beta}J_\beta(x)y^{-\alpha}I_\alpha(y) = c(\alpha,\beta)\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!}\mathcal{R}_{2m}(b;Z;d), \tag{3.19}$$

where

$$Z = \begin{bmatrix} x+iy & -x+iy \\ x-iy & -x-iy \end{bmatrix}.$$

Here we have used the transposition property (a). Since $\mathcal{R}_{2m}$ is homogeneous of degree $2m$ in its variables

$$\mathcal{R}_{2m}(b;Z;d) = (x^2+y^2)^m\mathcal{R}_{2m}(b;W;d), \tag{3.20}$$

where

$$W = \begin{bmatrix} e^{i\theta/2} & -e^{i\theta/2} \\ e^{i\theta/2} & -e^{i\theta/2} \end{bmatrix},$$

with $\cos(\theta/2) = x/(x^2+y^2)^{1/2}$ and $\sin(\theta/2) = y/(x^2+y^2)^{1/2}$. Application of

$$\mathcal{R}_{2m}(b;W;d) = \frac{\left(\frac{1}{2},m\right)m!}{(\alpha+1,m)(\beta+1,m)}P_m^{(\alpha,\beta)}\left(\frac{x^2-y^2}{x^2+y^2}\right)$$

(see [5, (5.10)]) to the right side of (3.20) followed by use of (3.19) and

$$\frac{\left(\frac{1}{2}, m\right) m!}{(2m)!} = \frac{1}{2^{2m}}$$

completes the proof of Bateman's formula (3.18). ∎

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