The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 4, Issue 1, Article x, pp. 1-7, 2007

GENERALIZED QUASILINEARIZATION METHOD FOR THE FORCED DÜFFING EQUATION

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Received 1 May, 2006; accepted 8 November, 2006; published 26 April, 2007.

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ABSTRACT. A generalized quasilinearization method for the periodic problem related to the forced Düffing equation is developed and a sequence of approximate solutions converging monotonically and quadratically to the solution of the given problem is presented.

Key words and phrases: Düffing equation, Periodic problem, Quasilinearization, Quadratic convergence.

2000 *Mathematics Subject Classification.* Primary: 34A45, 34B15.

ISSN (electronic): 1449-5910

The author would like to thank the referees for their valuable comments on this work. This research is completed while the author was awarded the Junior Research Fellowship at the Department of Pure Mathematics and Mathematical Statistics, at the University of Cambridge, so the author would like to express his sincere thanks and gratitude to the Department of Pure Mathematics and Mathematical Statistics, at the University of Cambridge.

1. INTRODUCTION

The method of quasilinearization [1] offers an excellent tool for obtaining approximate solutions of nonlinear differential equations. This technique works fruitfully only for the problems involving convex/concave functions and gives the sequence of approximate solutions converging monotonically and quadratically to the solution. Later after that the convexity assumption was relaxed and the method was generalized and extended in various directions to make it applicable to a large class of problems [5, 6, 7, 8, 9]. The generalized quasilinearization method was discussed for second order boundary value problems [11, 12, 13, 14, 15]. A quasilinearization method was presented for the periodic problem for the forced Düffing equation [2].

The Düffing equation is a well known nonlinear equation of applied science which is used as a powerful tool to discuss some important practical phenomena, for example, periodic orbit extraction, nonuniformity caused by an infinite domain, nonlinear mechanical oscillators, etc. This paper consider and study the periodic problem for the forced Düffing equation to find periodic solutions without requiring the nonlinear force function involved to be convex/concave, and obtain a sequence of approximate solutions converging quadratically to a solution of the problem. Moreover, this paper discusses and shows that the case where the damping part in the forced Düffing equation vanishes is just a special case.

2. PRELIMINARIES

The periodic problem for the Düffing equation

$$\begin{aligned} -\psi''(t) - k\psi'(t) &= \lambda\psi(t), \quad t \in J = [0,\pi], \\ \psi(0) &= \psi(\pi); \;\; \psi'(0) = \psi'(\pi), \end{aligned}$$

has a nontrivial solution if and only if the damping part is vanishing and $\lambda = (2m)^2 \;\; (m = 1, 2, 3\cdots)$. Now, define $\gamma = \frac{\sqrt{k^2-4\lambda}}{2}$ and $\delta = \frac{-k}{2}$, where $k \in R$. For $\gamma \neq 2m \;\; (m = 1, 2, 3\cdots)$, and $\omega(t) \in C[0,\pi]$, the unique solution of the periodic boundary value problem

$$-\psi''(t) - k\psi'(t) - \lambda\psi = \omega(t), \quad t \in J = [0,\pi],$$

$$\psi(0) = \psi(\pi); \;\; \psi'(0) = \psi'(\pi), \tag{2.1}$$

is given by

$$\psi(t) = \int_0^\pi G_\lambda(t,\upsilon)\omega(\upsilon)d\upsilon. \tag{2.2}$$

Here, $G_\lambda(t,\upsilon)$ is the Green's function, where $G_\lambda(t,\upsilon)$ for $\lambda > \frac{k^2}{4}$, is given by

$$\begin{aligned} & \frac{\sin\gamma\pi}{(\gamma A + 2\delta)\sin\gamma\pi + \gamma(1-e^{\delta\pi})} e^{\delta(\pi+\upsilon-t)} \left[\cos\gamma(\pi-t) + A\sin\gamma(\pi-t)\right] \\ & \times \left[\cos\gamma\upsilon + B\sin\gamma\upsilon\right], \\ 0 \;\leq\; & \upsilon \leq t \leq \pi, \\ & \frac{\sin\gamma\pi}{(\gamma A + 2\delta)\sin\gamma\pi + \gamma(1-e^{\delta\pi})} e^{\delta(\pi-\upsilon+t)} \left[\cos\gamma(\pi-\upsilon) + A\sin\gamma(\pi-\upsilon)\right] \\ & \times \left[\cos\gamma t + B\sin\gamma t\right], \\ 0 \;\leq\; & t \leq \upsilon \leq \pi, \end{aligned}$$

where

$$A = \frac{e^{\delta\pi}\left[\gamma\sin\gamma\pi - \delta\cos\gamma\pi\right] + \delta}{e^{\delta\pi}\left[\gamma\cos\gamma\pi + \delta\sin\gamma\pi\right] - \gamma} \quad , \quad B = \frac{e^{-\delta\pi} - \cos\gamma\pi}{\sin\gamma\pi}.$$

And $G_\lambda(t,\upsilon)$ for $\lambda < \frac{k^2}{4}$, is given by

$$\begin{aligned} & \frac{\sinh\gamma\pi}{(\gamma A+2\delta)\sinh\gamma\pi+\gamma(1-e^{\delta\pi})}e^{\delta(\pi+\upsilon-t)}\left[\cosh\gamma(\pi-t)+A\sinh\gamma(\pi-t)\right] \\ & \times[\cosh\gamma\upsilon+B\sinh\gamma\upsilon], \\ 0 \quad \leq \quad & \upsilon\leq t\leq\pi, \\ & \frac{\sinh\gamma\pi}{(\gamma A+2\delta)\sinh\gamma\pi+\gamma(1-e^{\delta\pi})}e^{\delta(\pi-\upsilon+t)}\left[\cosh\gamma(\pi-\upsilon)+A\sinh\gamma(\pi-\upsilon)\right] \\ & \times[\cosh\gamma t+B\sinh\gamma t], \\ 0 \quad \leq \quad & t\leq\upsilon\leq\pi, \end{aligned}$$

where

$$A=\frac{\delta-e^{\delta\pi}\left[\gamma\sinh\gamma\pi+\delta\cosh\gamma\pi\right]}{e^{\delta\pi}\left[\gamma\cosh\gamma\pi+\delta\sinh\gamma\pi\right]-\gamma} \quad , \quad B=\frac{e^{-\delta\pi}-\cosh\gamma\pi}{\sinh\gamma\pi}.$$

When $\lambda=\frac{k^2}{4}$, then $G_{\frac{k^2}{4}}(t,\upsilon)$ is given by

$$\begin{aligned} & (\pi-t)\left[\cosh\delta(\pi-t)+A\sinh\delta(\pi-t)\right]\left[\cosh\delta\upsilon+B\sinh\delta\upsilon\right], \\ 0 \quad \leq \quad & \upsilon\leq t\leq\pi, \\ & (\pi-\upsilon)\left[\cosh\delta(\pi-\upsilon)+A\sinh\delta(\pi-\upsilon)\right]\left[\cosh\delta t+B\sinh\delta t\right], \\ 0 \quad \leq \quad & t\leq\upsilon\leq\pi, \end{aligned}$$

where

$$A=\frac{1-\left[\cosh\delta\pi+\pi\sinh\delta\pi\right]}{\sinh\delta\pi+\pi\cosh\delta\pi} \quad , \quad B=\frac{1-\cosh\delta\pi}{\sinh\delta\pi}.$$

Note that, putting $k=0$ in the given PBVP, then the solution given above is a solution for the special case where the damping part is vanish.

Now, consider the following nonlinear PBVP

$$-u''(t)-ku'(t)=f(t,u(t)), \quad t\in J=[0,\pi],$$

$$u(0)=u(\pi);\;\; u'(0)=u'(\pi), \tag{2.3}$$

where $f: J\times R\to R$ is a continuous real valued function. A function $\alpha\in C^2[J,R]$ is a lower solution of (2.3) if

$$\begin{aligned} -\alpha''(t)-k\alpha'(t) &\leqslant f(t,\alpha(t)), \quad t\in J, \\ \alpha(0) &= \alpha(\pi);\;\; \alpha'(0)=\alpha'(\pi), \end{aligned}$$

and $\beta\in C^2[J,R]$ is an upper solution of (2.3) if

$$\begin{aligned} -\beta''(t)-k\beta'(t) &\geqslant f(t,\beta(t)), \quad t\in J, \\ \beta(0) &= \beta(\pi);\;\; \beta'(0)=\beta'(\pi). \end{aligned}$$

The following lemma plays a crucial role in the sequel and we sketch its proof for the sake of completeness.

Lemma 2.1. *Assume that $\alpha,\beta\in C^2[J,R]$ are lower and upper solutions of (2.3), respectively, such that $\alpha(t)\leqslant\beta(t)$ for every $t\in J$. Then there exists a solution $u(t)$ of (2.3) such that $\alpha(t)\leqslant u(t)\leqslant\beta(t)$ for $t\in J$.*

The proof of Lemma 2.1. is very standard proof in the use of this quasilinearization method, so the proof is omitted, for more details about the proof see [4, 12].

3. MAIN RESULT

Theorem 3.1. *Assume that*

(A$_1$) $\alpha_0, \beta_0 \in C^2[J, R]$ *are lower and upper solutions of (2.3), respectively, such that* $\alpha_0(t) \leqslant \beta_0(t)$ *on* J,

(A$_2$) $f \in C[\Omega, R]$ *is such that* $f_u(t,u), f_{uu}(t,u)$ *exist and are continuous for every* $(t,u) \in \Omega$, *where*

$$\Omega = \{(t,u) \in J \times R : \alpha_0(t) \leqslant u(t) \leqslant \beta_0(t)\},$$

(A$_3$) $f_u(t,u) < 0$ *for every* $(t,u) \in \Omega$.

Then there exists monotone non decreasing sequence $\{\alpha_n\}$ *which converges uniformly to a solution of* (2.3) *and the convergence is quadratic.*

Proof. Let $F : J \times R \to R$ is such that $F(t,u)$, $F_u(t,u)$ and $F_{uu}(t,u)$ are continuous on $J \times R$ and

$$F_{uu}(t,u) \geqslant 0, \quad (t,u) \in J \times R. \tag{3.1}$$

Motivated by Eloe and Zhang [3], take $\Phi(t,u) = F(t,u) - f(t,u)$ on $J \times R$. In view of (3.1), we see that

$$F(t,u) \geqslant F(t,v) + F_u(t,v)(u-v)$$

for $u \geqslant v$ and therefore

$$f(t,u) \geqslant f(t,v) + F_u(t,v)(u-v) - [\Phi(t,u) - \Phi(t,v)]. \tag{3.2}$$

Now, consider the PBVP

$$\begin{aligned} -u''(t) - ku'(t) &= g(t,u;\alpha_0) = f(t,\alpha_0) + F_u(t,\alpha_0)(u-\alpha_0) \\ &\quad -[\Phi(t,u) - \Phi(t,\alpha_0)], \end{aligned}$$

$$u(0) = u(\pi); \;\; u'(0) = u'(\pi). \tag{3.4}$$

The inequality (3.2) and (A$_1$) imply

$$-\alpha_0''(t) - k\alpha_0'(t) \leqslant f(t,\alpha_0(t)) = g(t,\alpha_0;\alpha_0),$$

$$\begin{aligned} -\beta_0''(t) - k\beta_0'(t) &\geqslant f(t,\beta_0(t)) \geqslant f(t,\alpha_0) + F_u(t,\alpha_0)(\beta_0 - \alpha_0) \\ &\quad -[\Phi(t,\beta_0) - \Phi(t,\alpha_0)] \\ &= g(t,\beta_0;\alpha_0). \end{aligned}$$

By Lemma 2.1, there exists a solution α_1 of (3.4) such that $\alpha_0(t) \leqslant \alpha_1(t) \leqslant \beta_0(t)$ on J. Next, consider the PBVP

$$-u''(t) - ku'(t) = g(t,u;\alpha_1),$$

$$u(0) = u(\pi); \;\; u'(0) = u'(\pi). \tag{3.5}$$

Observe that

$$\begin{aligned} -\alpha_1''(t) - k\alpha_1'(t) &= g(t,\alpha_1;\alpha_0) \\ &= f(t,\alpha_0) + F_u(t,\alpha_0)(\alpha_1 - \alpha_0) - [\Phi(t,\alpha_1) - \Phi(t,\alpha_0)] \\ &\leqslant f(t,\alpha_1) = g(t,\alpha_1;\alpha_1), \end{aligned}$$

$$\begin{aligned} -\beta_0''(t) - k\beta_0'(t) &\geqslant f(t,\beta_0(t)) \geqslant f(t,\alpha_1) + F_u(t,\alpha_1)(\beta_0 - \alpha_1) \\ &\quad -[\Phi(t,\beta_0) - \Phi(t,\alpha_1)] \\ &= g(t,\beta_0;\alpha_1), \end{aligned}$$

in view of (3.2). It follows from Lemma 2.1 that there exists a solution α_2 such that $\alpha_1(t) \leqslant \alpha_2(t) \leqslant \beta_0(t)$ on J. Thus, $\alpha_0(t) \leqslant \alpha_1(t) \leqslant \alpha_2(t) \leqslant \beta_0(t)$ on J. Employing the same arguments successively, we conclude

$$\alpha_0(t) \leqslant \alpha_1(t) \leqslant \alpha_2(t) \cdots \leqslant \alpha_n(t) \leqslant \beta_0(t) \quad \text{on } J,$$

where the elements of the monotone sequence $\{\alpha_n(t)\}$ are the solutions of the PBVP

$$\begin{aligned} -u''(t) - ku'(t) &= g(t,u;\alpha_{n-1}) = f(t,\alpha_{n-1}) + F_u(t,\alpha_{n-1})(u-\alpha_{n-1}) \\ &\quad -[\Phi(t,u) - \Phi(t,\alpha_{n-1})], \\ u(0) &= u(\pi);\ \ u'(0) = u'(\pi). \end{aligned}$$

The monotonicity of the sequence $\{\alpha_n(t)\}$ ensures the existence of its (pointwise) limit u.

Next we are interested to find the solution u of (2.3), so in order to do that we will consider the following linear PBVP

$$-u''(t) - ku'(t) - \frac{k^2}{4}u(t) = f_n(t),$$

(3.6)
$$u(0) = u(\pi);\ \ u'(0) = u'(\pi),$$

where

$$f_n(t) = g(t,\alpha_n(t);\alpha_{n-1}(t)), \quad \text{on } t \in J.$$

The continuity of g on Ω implies that the sequence $\{f_n\}$ is bounded in $C[J,R]$ and so

$$\lim_{n\to\infty} f_n(t) = f(t,u(t)), \quad t \in J.$$

We have mentioned in Section 2 that the PBVP (2.1) has a solution which given by (2.2). Here (3.6) is a linear PBVP and has the term $\frac{k^2}{4}u(t)$ instead of $\lambda u(t)$, because the Duffing equation has a damping part which given by $ku'(t)$, so λ should has the value $\frac{k^2}{4}$ in order to get the solution u for (2.3) in terms of the Green's function. Here

$$\alpha_n(t) = \int_0^\pi G_{\frac{k^2}{4}}(t,s) f_n(s)\,ds,$$

is a solution of (3.6). Thus $\{\alpha_n(t)\}$ is bounded in $C^2[J,R]$ and $\{\alpha_n(t)\} \uparrow u$ uniformly on J. Consequently,

$$u(t) = \int_0^\pi G_{\frac{k^2}{4}}(t,s) f(s,u(s))\,ds, \quad t \in J.$$

Hence u is a solution of (2.3).

For quadratic convergence, we set the error as $p_n(t) = u(t) - \alpha_n(t)$. Using the mean value theorem repeatedly, we obtain

$$\begin{aligned}
-p_n''(t) - kp_n'(t) &= f(t,u(t)) - g(t,\alpha_n(t);\alpha_{n-1}(t)) \\
&= f(t,u(t)) - f(t,\alpha_{n-1}(t)) - F_u(t,\alpha_{n-1}(t))[\alpha_n(t) - \alpha_{n-1}(t)] \\
&\quad +[\Phi(t,\alpha_n(t)) - \Phi(t,\alpha_{n-1}(t))] \\
&= F(t,u(t)) - F(t,\alpha_{n-1}(t)) - F_u(t,\alpha_{n-1}(t))[\alpha_n(t) - \alpha_{n-1}(t)] \\
&\quad +[\Phi(t,\alpha_n(t)) - \Phi(t,u(t))] \\
&= F_u(t,\xi)[u(t) - \alpha_{n-1}(t)] - F_u(t,\alpha_{n-1}(t))[\alpha_n(t) - \alpha_{n-1}(t)] \\
&\quad +[\Phi(t,\alpha_n(t)) - \Phi(t,u(t))] \\
&= [F_u(t,\xi) - F_u(t,\alpha_{n-1}(t))][u(t) - \alpha_{n-1}(t)] + F_u(t,\alpha_{n-1}(t)) \\
&\quad \times[u(t) - \alpha_n(t)] + [\Phi(t,\alpha_n(t)) - \Phi(t,u(t))] \\
&= F_{uu}(t,\sigma)[\xi - \alpha_{n-1}(t)][u(t) - \alpha_{n-1}(t)] + F_u(t,\alpha_{n-1}(t)) \\
&\quad \times[u(t) - \alpha_n(t)] - \Phi_u(t,\eta)[u(t) - \alpha_n(t)]
\end{aligned}$$

where

$$\alpha_{n-1}(t) \leqslant \xi \leqslant \sigma \leqslant u(t) \ \text{ and } \ \alpha_n(t) \leqslant \eta \leqslant u(t).$$

Set

$$h_n(t) = F_u(t,\alpha_{n-1}(t)) - \Phi_u(t,\eta),$$

and

$$l_n(t) = F_{uu}(t,\sigma)[\xi - \alpha_{n-1}(t)][u(t) - \alpha_{n-1}(t)] - Mp_{n-1}^2(t),$$

where $0 \leqslant F_{uu}(t,v) \leqslant M$, $(t,v) \in \Omega$. Clearly $l_n(t) \leqslant 0$. Since F_u is nondecreasing and $\alpha_{n-1}(t) \leqslant \eta$, it follows by ($A_3$) that there exists $\lambda < \frac{k^2}{4}$ and an integer N such that $h_n(t) \leqslant \lambda$, $t \in J$ for $n \geqslant N$. thus the error p_n satisfies the PBVP

$$\begin{aligned}
-p_n''(t) - kp_n'(t) - \lambda p_n(t) &= [h_n(t) - \lambda]p_n(t) + Mp_{n-1}^2(t) + l_n(t) \\
p_n(0) &= p_n(\pi); \ \ p_n'(0) = p_n'(\pi).
\end{aligned}$$

This implies that

$$p_n(t) = \int_0^\pi G_\lambda(t,s)\left([h_n(s) - \lambda]p_n(s) + Mp_{n-1}^2(s) + l_n(s)\right) ds\ ,$$

which gives

$$p_n(t) \leqslant M\int_0^\pi G_\lambda(t,s)p_{n-1}^2(s)ds, \qquad n \geqslant N.$$

Hence, there exists a constant $\delta > 0$ such that

$$\| p_n \| \leqslant \delta \| p_{n-1} \|^2, \qquad n \geqslant N,$$

where $\| u \| = \max\{| u(t) | : t \in J\}$ is the usual uniform norm on $C[J,R]$. ∎

Remark 3.1. One can also construct the sequence of upper solutions in such a way that it converges to the solution of the mentioned periodic boundary value problem.

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