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NUMERICAL STUDIES ON DYNAMICAL SYSTEMS METHOD FOR SOLVING ILL-POSED PROBLEMS WITH NOISE

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ABSTRACT. In this paper, we apply the dynamical systems method proposed by A. G. RAMM, and the the variational regularization method to obtain numerical solution to some ill-posed problems with noise. The results obtained are compared to exact solutions. It is found that the dynamical systems method is preferable because it is easier to apply, highly stable, robust, and it always converges to the solution even for large size models.

Key words and phrases: Dynamical systems method, Variational regularization method, Ill-posed problems.

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1. INTRODUCTION

Let $A : H \to H$ be a closed, densely defined linear operator on a Hilbert space H. Consider the following linear equations in H

$$Au = f, \tag{1.1}$$

and assume that the range $R(A)$ is not closed, so that problem (1.1) is ill-posed. Let us assume that $f = Ay$ where y is the minimal-norm solution of (1.1), and that noisy data f_δ are given such that $\|f - f_\delta\| \le \delta$. Discretizing problem (1.1), one often deals with a finite-dimensional problem of solving ill-conditioned linear algebraic system. Problem (1.1) is called discrete ill-posed problem if the matrix A is ill- conditioned, that is the condition number is large and the singular values of A decay gradually to zero. Also the inverse of A may not exist or may be unbounded. Our goal in this paper is to compute a stable approximation to y, given f_δ . Discrete ill posed problems arise in a variety of applications such as astronomy see [3], computerized tomography see [9], electrocardiography see [4], mathematical physics see [24] and other fields. The classical example of an ill-posed problem is encountered in the linear Fredholm integral equation of the first kind with a square integrable kernel:

$$\int_a^b K(s,t)u(t)dt = g(s), \quad c \le s \le d, \tag{1.2}$$

where the right-hand side g and the kernel K are given functions and u is an unknown function. By using discretization techniques like Galerkin method with an orthonormal basis or quadrature method see [2], [5] and [6], equation (1.2) can be written as a linear system $Ku = g$, with K integral operator mapping u to g. Since the kernel is square integrable over $[a,b] \times [c,d]$, then it is a classical result that K is a compact operator from $L^2[a,b]$ into $L^2[c,d]$. Regularization methods are often used to obtain stable and smooth solutions to such ill-posed problem. The most common and well known technique for regularizing ill-posed problems is the variational regularization method see [7], [23] and [24]. This method attempts to provide a good estimate of the solution of (1.1) by a solution $u_{\alpha,\delta}$ of the problem

$$\min\{\|Au - f_\delta\|^2 + \alpha\|u\|^2\}, \tag{1.3}$$

where α is the regularization parameter and $u_{\alpha,\delta}$ is the regularization solution. The success of the variational regularization method depends on making a good choice of the regularization parameter which is not easy to find. The reason is that $u_{\alpha,\delta}$ is too sensitive to perturbations in f, i.e., a small change in f may produce a large change in u_α .

In this paper, we will consider two methods for solving numerically some ill-posed models with noise. The first method is the dynamical systems method (DSM) which is proposed by A. G. RAMM see [11]- [21] and the references therein. The DSM is based on an analysis of the solution of Cauchy problem for nonlinear differential equations in Hilbert space. Such an analysis was done for well-posed and some ill-posed problems see [11], and the references therein, using some integral inequalities. The DSM has several attractive properties; it is fast convergent, can be easily designed and no need to calculate the inverse of large condition number matrices. In section 2, a brief description of the analysis of the DSM is presented.

The second method is the variational regularization method see [8], [23] and [24]. This method consists of finding a global minimizer of (1.3), where f_δ is a noisy data and $\|f - f_\delta\| \le \delta$. The global minimizer of the quadratic functional (1.3) is the unique solution to the linear system $(A^*A + \alpha I)u_{\alpha,\delta} = A^* f_\delta$, where I is the unit matrix. This system has a unique solution

$u_{\alpha,\delta} = (A^*A + \alpha I)^{-1}A^*f_\delta$. To determine the suitable α, let $u_{\alpha(\delta),\delta}$ be a solution of (1.3) and consider the equation

$$\|Au_{\alpha,\delta} - f_\delta\| = \tau\delta, \tag{1.4}$$

where $\tau \in]1, 2[$. Equation (1.4) is the usual discrepancy principle. One can prove that equation (1.4) determines $\alpha = \alpha(\delta)$ uniquely, $\alpha(\delta) \to 0$ as $\delta \to 0$, and $u_\delta := u_{\alpha(\delta),\delta} \to y$ where y is the minimal-norm solution to (1.1) as $\delta \to 0$. This justifies the usual discrepancy principle for choosing the regularization parameter see [8]. For more details on the theory of variational regularization method see e.g., [11, Chapter 2].

The accomplishment of the paper will be explained in the following manner. In section 2, a brief description of the analysis of the DSM is presented. In section 3, numerical experiments and comparisons are made for the regularized solutions chosen by DSM and variational regularization method. We end in section 4 with the conclusions.

2. ANALYSIS OF THE DSM

In this section, we will give a brief description of the analysis of the DSM and for more details on the analysis of DSM see [11]- [21] and the references therein. The DSM analysis is bases on a construction of a dynamical systems with the trajectory; by using Cauchy problem for nonlinear differential equations in a Hilbert space; starting from an initial approximation point and having a solution to problem (1.1) as a limiting point. It is proved in see [11] that if equation (1.1) is solvable and $\|f - f_\delta\| \leq \delta$, the following results hold:

Theorem 2.1. *Assume that $f = Ay$, $y \perp N(A)$, A is a linear operator, closed and densely defined in H. Consider the problem*

$$\frac{du}{dt} = -u + T^{-1}_{\epsilon(t)}A^*f \ , \ \ u(0) = u_0, \tag{2.1}$$

$N(A) := \{u : Au - f = 0\}$, $u_0 \in H$ is arbitrary, $T_\epsilon = T + \epsilon(t)$, $T = A^*A$, $\epsilon = \epsilon(t)$ is a continuous function monotonically decaying to zero at $t \to \infty$ and $\int_0^\infty \epsilon(s)ds = \infty$. Then problem (2.1) has a unique solution $u(t)$ defined on $[0, \infty)$, and the following limit exists:

$$lim_{t\to(\infty)}u(t) := u(\infty) \ \ and \ \ u(\infty) = y \ .$$

It is pointed out in [11] that if f_δ is given in place of the exact solution f, calculate its solution $u_\delta(t)$ as $t = t_\delta$, it can be proved that

$$lim_{\delta\to(\infty)}\|u_\delta(t_\delta) - y\| = 0.$$

If t_δ is suitable chosen. The stopping time t_δ can be uniquely determined, for example by a discrepancy principle, see [17], for bounded operators A. Also, it is pointed out that the argument in see [11] remains valid in the case of unbounded A without any change.

3. THE DYNAMICAL SYSTEMS ALGORITHM

The DSM is a stable regularized algorithm for solving (1.1), especially when f is replaced by the noise data f_δ The algorithm can be applied by using the following steps:
Step 1. Solve the following ordinary differential equation:

$$\frac{du}{dt} = \Phi(u, t), \qquad u(0) = u_0, \tag{3.1}$$

where

$$\Phi(u,t) = -u + (A^*A + \alpha I)^{-1}A^* f_\delta, \quad u_0 = 0, \tag{3.2}$$

and the discretization is based on an explicit Runge-Kutta formula.
Step 2. The stopping time t_δ is defined by using the following generalization of the discrepancy principle:
when $\tau \in]1,2[$, the stopping time is chosen by the formula

$$\|Au_\delta(t_\delta) - f_\delta\| = \tau\delta \tag{3.3}$$

and we assume that

$$\tau\delta < \|Au_\delta(t) - f_\delta\| \quad \text{for all times} \quad t < t_\delta \tag{3.4}$$

i.e., t_δ is the first moment t, at which the discrepancy is equal to $\tau\delta$. If

$$\|Au_0 - f_\delta\| > \tau\delta,$$

then formulas (3.3) and (3.4) determine uniquely $t_\delta > 0$, see [17].

4. NUMERICAL EXPERIMENTS

In the following, three extremely unstable test examples from the literature are presented. Comparisons are made for the regularized solutions chosen by DSM and by the variational regularization method. Table (4.1) displays the results of these ill-posed examples for which the exact solutions are known. In each example; the size of the coefficient matrix A is taken as 20×20 and the noise term is $\delta = 0.02$ and $\tau = 1.9$, and $\epsilon(t) = \frac{0.1}{\exp(t)}$.

Example 4.1. (PHILIPS example [10]):
Consider Fredholm integral equation of the first kind (1.2), where $a = -6$, $b = 6$ and

$$K(s,t) = \begin{cases} 1 + \cos(\frac{\pi(s-t)}{s}), & |s-t| < 3. \\ 0, & |s-t| \geq 3. \end{cases}$$

$$g(s) = (6 - |s|)\left(1 + \frac{1}{2}\cos\left(\frac{\pi s}{3}\right)\right) + \frac{9}{2\pi}\sin\left(\frac{\pi|s|}{3}\right)$$

and the exact solution $u(t)$ is given by

$$u(t) = \begin{cases} 1 + \cos(\frac{\pi t}{3}), & |t| < 3. \\ 0, & |t| \geq 3. \end{cases}$$

By using Galerkin method for discretization with orthonormal box functions as basis functions (see [5], chapter 7), where both integration intervals are [-6,6]. Then the Galerkin method leads to a linear ill-posed system of equations $Au = f$ where the condition number of the matrix A is equal to $3.95818402e^3$. Perturbed the right-hand side vector f; by adding a noise term δ to the last row in f; in order to have f_δ, then we have an extremely unstable system.

Example 4.2. (SHAW example [22]):
Consider Fredholm integral equation of the first kind (1.2), where $a = \frac{-\pi}{2}$, $b = \frac{\pi}{2}$ and

$$K(s,t) = (\cos(s) + \cos(t))^2 \left(\frac{\sin\omega}{\omega}\right)^2, \quad \omega = \pi(\sin(s) + \sin(t)),$$

and the exact solution is given by

$$u(t) = a_1 \exp(-c_1(t-t_1)^2) + a_2 \exp(-c_2(t-t_2)^2).$$

By using quadrature method (see [5], chapter 6), where both integration intervals are $[\frac{-\pi}{2}, \frac{\pi}{2}]$, hence, we obtain a linear ill-posed system of equations $Au = f$, where the condition number of the matrix A is equal to $3.74711237e^{14}$. Perturbed the right-hand side vector f; by adding a noise term δ to the last row in f; in order to have f_δ, then we have an extremely unstable system. The integral equation in this example represents a one-dimensional model of an image reconstruction problem from see [1]. The kernel K is the point spread function for an infinitely long slit. The parameters a_1, a_2, etc., are constants that determine the shape of the solution $u(t)$; in this implementation we use $a_1 = 2, a_2 = 1, c_1 = 6, c_2 = 2, t_1 = 0.8, t_2 = -0.5$, giving an $u(t)$ with two different "humps".

Example 4.3. (Inverse Laplace transformation (ilaplace) example [25]):
Consider Fredholm integral equation of the first kind (1.2), in the interval $[0, \infty)$. The kernel K and the the corresponding right-hand side are given by

$$K(s,t) = \exp(-st), \quad g(s) = \frac{1}{s + (1/2)},$$

the exact solution is given by

$$u(t) = \exp(-t/2).$$

Discretization of the inverse Laplace transformation (i.e., equation (1.2) in this example) by using Gauss-Laguerre quadrature method, see [25], where both integration intervals are $[0, \infty)$, hence, we obtain a linear ill posed system of equations $Au = f$, where the condition number of the matrix A is equal to $3.79743094e^{30}$. Perturbed the right-hand side vector f by adding a noise term δ to the last row in f in order to have f_δ, then we have an extremely unstable system.

Problem	Method	Rerr	t_δ, α
Phillips (20)	variational regularization	2.37e-2	$\alpha = 0.0457$
	DSM	7.55e-2	$t_\delta = 6$
Shaw (20)	variational regularization	1.44e-1	$\alpha = 0.00679$
	DSM	8.69e-2	$t_\delta = 5.7$
Ilaplace (20)	variational regularization	4.18e-2	$\alpha = 0.02$
	DSM	3.17e-2	$t_\delta = 3.9$

Table 4.1: Comparison between results of the ill-posed examples.

The third column in Table (4.1) gives the relative error $Rerr := \frac{\|u^{exact} - u^{approx}\|}{\|u^{exact}\|}$, and the last column gives the values of the stopping time t_δ and the regularization parameter α.

5. HILBERT MATRIX EXAMPLE

Consider problem (1.1) where the matrix A is a Hilbert matrix:

$$A = \begin{pmatrix} 1 & 1/2 & ... & 1/n \\ 1/2 & 1/3 & ... & 1/(n+1) \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 1/n & 1/(n+1) & ... & 1/(2n-1) \end{pmatrix}$$

The condition number of the 20×20 Hilbert matrix A is equal to $1.908432e^{18}$. We will consider here two different choices of the right-hand side vector f of (1.1). The exact solution is generated by some mathematical formula as $u_{exact} = F(t_i)$; $t_i = 0.5i$. The right-hand side f is then produced as $f = Au_{exact}$. Perturbed the right-hand side vector f; by adding a noise term δ to the last row in f; in order to have f_δ, then we have an extremely unstable system, where $\delta = 0.02,\ \ \tau = 1.9$ and $\epsilon(t) = \frac{0.1}{\exp(t)}$. The results listed in Table (5.1) show that the higher accuracy is obtained by DSM method.

$u_{exact} = F(t_i)$	Method	Rerr	t_δ, α
$\sin(t_i)$	variational regularization	9.15e-1	$\alpha = 0.0087$
	DSM	8.96e-1	$t_\delta = 4.35$
$\frac{1}{t_i^3 + t_i^2 + t_i + 1}$	variational regularization	1.22e-1	$\alpha = 0.0565$
	DSM	7.62e-2	$t_\delta = 3.5$
$\exp(-t_i)$	variational regularization	1.61e-1	$\alpha = 0.0501$
	DSM	8.02e-2	$t_\delta = 4.9$

Table 5.1: Comparison between results of Hilber matrix examples.

6. CONCLUSIONS

In this paper, the dynamical systems method which is proposed by A. G. RAMM is applied to solve numerically some ill-posed models. Three test examples taken from the literature are tested, the PHILLIPS example, the SHAW example, and the inverse Laplace transformation (ilaplace) example. Also artificial examples by using the Hilbert matrix are tested. Comparisons are made between the DSM and the variational regularization method. For all test problems with noise considered in this paper, the DSM has many advantages than the other, it is easier to apply, can choose a regularized solution that is as good as and frequently better than the regularized solution chosen by the variational regularization method. The main difficulty in variational regularization method is the inversion of the matrix $A^*A + \alpha I$ which is numerically difficult if α is small, because the condition number of the matrix A^*A is much larger than the condition number of the matrix A. We noted that in all tested examples the relative errors by using the DSM are smaller than the relative errors by using the variational regularization method or it is of the same order of magnitude.

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