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A REVERSE OF THE TRIANGLE INEQUALITY IN INNER PRODUCT SPACES AND APPLICATIONS FOR POLYNOMIALS

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ABSTRACT. A reverse of the triangle inequality in inner product spaces related to the celebrated Diaz-Metcalf inequality with applications for complex polynomials is given.

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1. INTRODUCTION

In 1966, J. B. Diaz and F. T. Metcalf [3] proved the following reverse of the triangle inequality:

Theorem 1.1 (Diaz-Metcalf, 1966). *Let a be a unit vector in the inner product space $(H;\langle\cdot,\cdot\rangle)$ over the real or complex number field $\mathbb{K}$. Suppose that the vectors $x_i \in H\setminus\{0\}$, $i\in\{1,\dots,n\}$ satisfy*

$$0 \le r \le \frac{\operatorname{Re}\langle x_i, a\rangle}{\|x_i\|}, \quad i\in\{1,\dots,n\}. \tag{1.1}$$

Then

$$r\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n} x_i\right\|, \tag{1.2}$$

where equality holds if and only if

$$\sum_{i=1}^{n} x_i = r\left(\sum_{i=1}^{n}\|x_i\|\right)a. \tag{1.3}$$

In an effort to find other sufficient conditions for the vectors $x_1,\dots,x_n$ such that a reverse of the generalised triangle inequality would hold, S. S. Dragomir obtained in [4] the following results:

Theorem 1.2 (Dragomir, 2004)**.** *Let a be as in Theorem 1.1 and $\rho\in(0,1)$. If $x_i\in H$, $i\in\{1,\dots,n\}$ are such that*

$$\|x_i - a\| \le \rho \quad \text{for each } i\in\{1,\dots,n\}, \tag{1.4}$$

then

$$\sqrt{1-\rho^2}\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n} x_i\right\|, \tag{1.5}$$

with equality if and only if

$$\sum_{i=1}^{n} x_i = \sqrt{1-\rho^2}\left(\sum_{i=1}^{n}\|x_i\|\right)a. \tag{1.6}$$

Theorem 1.3 (Dragomir, 2004)**.** *Let a be as above and $M\ge m>0$. If $x_i\in H$, $i\in\{1,\dots,n\}$ are such that either*

$$\operatorname{Re}\langle Ma - x_i, x_i - ma\rangle \ge 0 \tag{1.7}$$

or, equivalently,

$$\left\|x_i - \frac{M+m}{2}a\right\| \le \frac{1}{2}(M-m) \tag{1.8}$$

holds for each $i\in\{1,\dots,n\}$, then

$$\frac{2\sqrt{mM}}{m+M}\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n} x_i\right\|, \tag{1.9}$$

with equality if and only if

$$\sum_{i=1}^{n} x_i = \frac{2\sqrt{mM}}{m+M}\left(\sum_{i=1}^{n}\|x_i\|\right)a. \tag{1.10}$$

The aim of this paper is to provide refinements for the inequalities (1.5) and (1.9) under slightly more general assumptions for the vectors involved $x_1, \dots, x_n$. Applications in the spirit of Wilf [9] for polynomials with complex coefficients are also given.

For some classical results concerning reverses for the generalised triangle inequality, see [6], [7] and the Chapter XIII of the book [8]. For recent results, see [1], [2] and [5].

2. REVERSES OF THE TRIANGLE INEQUALITY

The following result holds:

Theorem 2.1. *Let $(H; \langle\cdot,\cdot\rangle)$ be a real or complex inner product space $\beta \in \mathbb{K}(\mathbb{C},\mathbb{R})$ and $\gamma > 0$ such that $|\beta| \geq \gamma$. If $e, x_j \in H$, $j \in \{1,\dots,n\}$ are such that $\|e\| = 1$ and*

$$\|x_j - \beta e\| \leq \left(|\beta|^2 - \gamma^2\right)^{\frac{1}{2}} \quad \textit{for each } j \in \{1,\dots,n\}, \tag{2.1}$$

then

$$\sum_{j=1}^{n} \|x_j\| \leq \frac{|\beta|}{\gamma} \left| \left\langle \sum_{j=1}^{n} x_j, e \right\rangle \right| \leq \frac{|\beta|}{\gamma} \left\| \sum_{j=1}^{n} x_j \right\|. \tag{2.2}$$

The equality holds in all inequalities from (2.2) simultaneously if and only if

$$\|x_j\| = \gamma \quad \textit{for any } j \in \{1,\dots,n\} \tag{2.3}$$

and

$$\frac{1}{n}\sum_{j=1}^{n} x_j = \frac{\beta\gamma^2}{|\beta|^2} \cdot e. \tag{2.4}$$

Proof. The condition (2.1), on taking the square, is equivalent to

$$\|x_j\|^2 + \gamma^2 \leq 2\left[\operatorname{Re}\beta \operatorname{Re}\langle x_j, e\rangle + \operatorname{Im}\beta \operatorname{Im}\langle x_j, e\rangle\right] \tag{2.5}$$

for any $j \in \{1,\dots,n\}$ with equality if and only if the equality case holds in (2.1) for $j \in \{1,\dots,n\}$.

Since

$$2\gamma \|x_j\| \leq \|x_j\|^2 + \gamma^2 \quad \text{for each } j \in \{1,\dots,n\} \tag{2.6}$$

with equality iff $\|x_j\| = \gamma$ for $j \in \{1,\dots,n\}$, hence, by (2.5) and (2.6) we have

$$\|x_j\| \leq \frac{\operatorname{Re}\beta \operatorname{Re}\langle x_j, e\rangle + \operatorname{Im}\beta \operatorname{Im}\langle x_j, e\rangle}{\gamma} \quad \text{for } j \in \{1,\dots,n\} \tag{2.7}$$

with equality iff the case of equality holds in (2.1) and $\|x_j\| = \gamma$ for $j \in \{1,\dots,n\}$.

Summing (2.7) over j from 1 to n we get

$$\sum_{j=1}^{n} \|x_j\| \leq \frac{\operatorname{Re}\beta \operatorname{Re}\left\langle \sum_{j=1}^{n} x_j, e\right\rangle + \operatorname{Im}\beta \operatorname{Im}\left\langle \sum_{j=1}^{n} x_j, e\right\rangle}{\gamma} \tag{2.8}$$

with equality if and only if the equality case holds in (2.1) for each $j \in \{1,\dots,n\}$ and $\|x_j\| = \gamma$ for every $j \in \{1,\dots,n\}$.

Utilising the Cauchy-Buniakowsky-Schwarz inequality

$$\left(m^2 + p^2\right)\left(M^2 + P^2\right) \geq (mM + pP)^2, \quad m, p, M, P \in \mathbb{R}$$

with equality if and only if $pM = mP$, we can state that

$$\operatorname{Re}\beta \cdot \operatorname{Re}\left\langle \sum_{j=1}^{n} x_j, e \right\rangle + \operatorname{Im}\beta \cdot \operatorname{Im}\left\langle \sum_{j=1}^{n} x_j, e \right\rangle \leq |\beta| \left| \left\langle \sum_{j=1}^{n} x_j, e \right\rangle \right| \tag{2.9}$$

with equality iff

$$\operatorname{Re}(\beta) \cdot \operatorname{Im}\left\langle \sum_{j=1}^{n} x_j, e \right\rangle = \operatorname{Im}(\beta) \cdot \operatorname{Re}\left\langle \sum_{j=1}^{n} x_j, e \right\rangle. \tag{2.10}$$

Making use of the elementary fact that

$$\|y - \langle y, e\rangle e\|^2 = \|y\|^2 - |\langle y, e\rangle|^2,$$

where $y \in H$ and $e \in H$ with $\|e\| = 1$, we deduce, for $y = \sum_{j=1}^{n} x_j$, that

$$\left| \left\langle \sum_{j=1}^{n} x_j, e \right\rangle \right| \leq \left\| \sum_{j=1}^{n} x_j \right\| \tag{2.11}$$

with equality if and only if

$$\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \langle x_j, e\rangle e. \tag{2.12}$$

Now, by (2.8), (2.9) and (2.11) we deduce the desired inequality (2.2).

Further, if (2.3) and (2.4) hold true, then, obviously, the equality case is realised simultaneously in all inequalities from (2.2).

Conversely, if the case of equality is realised in both inequalities in (2.2), it must hold in all the inequalities used to prove them. Therefore, $\|x_j\| = \gamma$ for each $j \in \{1, \dots, n\}$. Moreover, we must have

$$\|x_j - \beta e\| = \left(|\beta|^2 - \gamma^2\right)^{\frac{1}{2}} \quad \text{for any } j \in \{1, \dots, n\}, \tag{2.13}$$

and (2.10) and (2.12). Squaring (2.13) and utilising (2.3) we deduce $\operatorname{Re}\left[\bar{\beta}\langle x_j, e\rangle\right] = \gamma^2$ for each $j \in \{1, \dots, n\}$, which is clearly equivalent to

$$\operatorname{Re}\beta \cdot \operatorname{Re}\langle x_j, e\rangle + \operatorname{Im}\beta \cdot \operatorname{Im}\langle x_j, e\rangle = \gamma^2 \tag{2.14}$$

for every $j \in \{1, \dots, n\}$. Summation over j from 1 to n in (2.14) produces

$$\operatorname{Re}\beta \cdot \operatorname{Re}\left\langle \sum_{j=1}^{n} x_j, e \right\rangle + \operatorname{Im}\beta \cdot \operatorname{Im}\left\langle \sum_{j=1}^{n} x_j, e \right\rangle = n\gamma^2. \tag{2.15}$$

A simple calculation shows that the equations (2.10) and (2.15) give

$$\operatorname{Re}\left\langle \sum_{j=1}^{n} x_j, e \right\rangle = n\gamma^2 \cdot \frac{\operatorname{Re}\beta}{|\beta|^2} \quad \text{and} \quad \operatorname{Im}\left\langle \sum_{j=1}^{n} x_j, e \right\rangle = n\gamma^2 \cdot \frac{\operatorname{Im}\beta}{|\beta|^2}, \tag{2.16}$$

which, by (2.12) imply (2.4).

The proof is complete. ∎

The following corollary may be stated.

Corollary 2.2. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be as above and* $\delta,\Delta\in\mathbb{K}\ (\mathbb{C},\mathbb{R})$ *such that* $\operatorname{Re}\left(\Delta\bar{\delta}\right)>0$. *If* $e,x_j\in H$, $j\in\{1,\dots,n\}$ *are such that either*

$$\operatorname{Re}\langle\Delta e-x_j,x_j-\delta e\rangle\geq 0\quad\text{for each } j\in\{1,\dots,n\}\tag{2.17}$$

or, equivalently,

$$\left\|x_j-\frac{\delta+\Delta}{2}\cdot e\right\|\leq\frac{1}{2}\left|\Delta-\delta\right|\quad\text{for each } j\in\{1,\dots,n\},\tag{2.18}$$

then

$$\sum_{j=1}^{n}\|x_j\|\leq\frac{1}{2}\cdot\frac{|\Delta+\delta|}{\sqrt{\operatorname{Re}\left(\Delta\bar{\delta}\right)}}\left|\left\langle\sum_{j=1}^{n}x_j,e\right\rangle\right|\leq\frac{1}{2}\cdot\frac{|\Delta+\delta|}{\sqrt{\operatorname{Re}\left(\Delta\bar{\delta}\right)}}\left\|\sum_{j=1}^{n}x_j\right\|.\tag{2.19}$$

The equality holds in all inequalities (2.19) simultaneously if and only if

$$\|x_j\|=\sqrt{\operatorname{Re}\left(\Delta\bar{\delta}\right)}\quad\text{for each } j\in\{1,\dots,n\}\tag{2.20}$$

and

$$\frac{1}{n}\sum_{j=1}^{n}x_j=\frac{2\left(\Delta+\delta\right)\operatorname{Re}\left(\Delta\bar{\delta}\right)}{|\Delta+\delta|^2}\cdot e.\tag{2.21}$$

Proof. Let $\beta:=\frac{\Delta+\delta}{2}$ and $\gamma=\left[\operatorname{Re}\left(\Delta\bar{\delta}\right)\right]^{\frac{1}{2}}$. Then it is clear that $|\beta|\geq\gamma$ and applying Theorem 1.1 for these choices, we deduce the desired result. ∎

Remark 2.1. If $\Delta=M\geq m=\delta>0$, then (2.19) provides a refinement of (1.9).

Another particular case of interest is incorporated in the following corollary:

Corollary 2.3. *Let* $\varepsilon\in(0,1)$. *If* $e,x_j\in H$ *are such that* $\|e\|=1$ *and*

$$\|x_j-e\|\leq\varepsilon\quad\text{for each } j\in\{1,\dots,n\},\tag{2.22}$$

then

$$\sum_{j=1}^{n}\|x_j\|\leq\frac{1}{\sqrt{1-\varepsilon^2}}\left|\left\langle\sum_{j=1}^{n}x_j,e\right\rangle\right|\leq\frac{1}{\sqrt{1-\varepsilon^2}}\left\|\sum_{j=1}^{n}x_j\right\|.\tag{2.23}$$

The equality holds in all inequalities (2.23) simultaneously if and only if

$$\|x_j\|=\sqrt{1-\varepsilon^2}\quad\text{for each } j\in\{1,\dots,n\}\tag{2.24}$$

and

$$\frac{1}{n}\sum_{j=1}^{n}x_j=\left(1-\varepsilon^2\right)\cdot e.\tag{2.25}$$

The proof is obvious by Theorem 1.1 on choosing $\beta=1$ and $\gamma=\sqrt{1-\varepsilon^2}$.

Remark 2.2. The above Corollary 2.3 provides a refinement of the result in Theorem 1.2.

3. APPLICATIONS FOR POLYNOMIALS

In 1963, H. S. Wilf obtained the following inequality of arithmetic mean – geometric mean type for complex numbers:

Theorem 3.1 (Wilf, 1963)**.** *Suppose the complex numbers z_i satisfy*

$$|\arg z_i| \leq \psi < \frac{\pi}{2}, \quad i = 1, \dots, n. \tag{3.1}$$

Then

$$|z_1 z_2 \cdots z_n|^{\frac{1}{2}} < (\sec \psi) \frac{1}{n} |z_1 + \cdots + z_n|, \tag{3.2}$$

unless n is even and $z_1 = \cdots = z_{n/2} = \bar{z}_{(n/2)+1} = \cdots = \bar{z}_n = re^{i\Psi} \quad (r > 0)$, in which case equality holds.

Let

$$P(z) = a_0 + a_1 z + \cdots + a_n t^n = a_n (z - z_1) \cdots (z - z_n) \tag{3.3}$$

be given, where $z_1, \dots, z_n$ are the zeros of $P(z)$.

Utilising Theorem 3.1, Wilf proved the following interesting inequalities for the polynomial $P(z)$.

Theorem 3.2 (Wilf, 1963)**.** *Let z be a point from which the convex hull of the zeros of the polynomial $P(z)$ subtends an angle $2\psi < \pi$. Then*

$$|P'(z)| \geq n |a_n|^{\frac{1}{n}} |P(z)|^{\frac{n-1}{n}} \cos \psi. \tag{3.4}$$

If ρ denotes the distance from z to the centre of gravity of the zeros of $P(z)$, then

$$|P(z)| \leq |a_n| (\rho \sec \psi)^n. \tag{3.5}$$

Now, utlising the well-known arithmetic mean - geometric mean inequality

$$\left(\prod_{i=1}^{n} a_i \right)^{\frac{1}{2}} \leq \frac{1}{n} \sum_{i=1}^{n} a_i, \qquad a_i > 0, \ i \in \{1, \dots, n\}, \tag{3.6}$$

with equality if and only if $a_1 = \cdots = a_n$, we can state the following result.

Theorem 3.3. *Let $(H; \langle \cdot, \cdot \rangle)$, β, γ, e and x_j, $j \in \{1, \dots, n\}$ be as in Theorem 1.1. Then*

$$\left(\prod_{j=1}^{n} \|x_j\| \right)^{\frac{1}{n}} \leq \frac{|\beta|}{\gamma} \left\| \frac{1}{n} \sum_{j=1}^{n} x_j \right\|, \tag{3.7}$$

with equality if and only if (2.3) and (2.4) hold true.

Remark 3.1. It is obvious that the inequalities (2.19) and (2.23) can generate the following multiplicative weaker versions

$$\left(\prod_{j=1}^{n} \|x_j\| \right)^{\frac{1}{n}} \leq \frac{1}{2} \cdot \frac{|\Delta + \delta|}{\sqrt{\operatorname{Re}(\Delta \bar{\delta})}} \left\| \frac{1}{n} \sum_{j=1}^{n} x_j \right\|. \tag{3.8}$$

provided x_j, e, δ and Δ satisfy the hypotheses of Corollary 2.2; and

$$\left(\prod_{j=1}^{n} \|x_j\| \right)^{\frac{1}{n}} \leq \frac{1}{\sqrt{1 - \varepsilon^2}} \left\| \frac{1}{n} \sum_{j=1}^{n} x_j \right\|, \tag{3.9}$$

where x_j, e and ε satisfy the hypothesis of Corollary 2.3.

If one is interested to state a Wilf-type inequality for the arithmetic mean – geometric mean of complex numbers as in (3.2), then one can obtain the following result:

Proposition 3.4. *Let $\xi \in \mathbb{C}$ with $|\xi| = 1$ and $\varepsilon \in (0,1)$. If $u_1, \dots, u_n \in \mathbb{C}$ are such that*

$$|u_j - \xi| \leq \varepsilon \quad \text{for each } j \in \{1, \dots, n\} \tag{3.10}$$

then

$$\left| \prod_{j=1}^{n} u_j \right|^{\frac{1}{n}} \leq \frac{1}{\sqrt{1-\varepsilon^2}} \left| \frac{1}{n} \sum_{j=1}^{n} u_j \right| \tag{3.11}$$

with equality if and only if

$$|u_j| = \sqrt{1-\varepsilon^2} \quad \text{for each } j \in \{1, \dots, n\} \tag{3.12}$$

and

$$\frac{1}{n} \sum_{j=1}^{n} u_j = \left(1-\varepsilon^2\right) \xi. \tag{3.13}$$

Now, applying Proposition 3.4 to the complex numbers

$$\frac{1}{z-z_1}, \dots, \frac{1}{z-z_n}$$

and proceeding as in Wilf's paper, we can state the following result:

Proposition 3.5. *Let P and $z_1, \dots, z_n$ be as above. If $z \in \mathbb{C}$ is such that for $\xi \in \mathbb{C}$ with $|\xi| = 1$ and $\varepsilon \in (0,1)$,*

$$\left| \frac{1}{z-z_k} - \xi \right| \leq \varepsilon \quad \text{for each } k \in \{1, \dots, n\}, \tag{3.14}$$

then

$$|P'(z)| \geq n |a_n|^{\frac{1}{n}} \sqrt{1-\varepsilon^2} \, |P(z)|^{\frac{n-1}{n}}, \tag{3.15}$$

with equality if

$$|z - z_n| = \frac{1}{\sqrt{1-\varepsilon^2}} \quad \text{for each } k \in \{1, \dots, n\}$$

and

$$\frac{1}{n} \cdot \frac{P'(z)}{P(z)} = \left(1-\varepsilon^2\right) \xi.$$

Finally, if one applies Proposition 3.4 to the complex numbers

$$z - z_1, \dots, z - z_n,$$

then one can state the following result:

Proposition 3.6. *Let P and $z_1, \dots, z_n$ be as above. If $z \in \mathbb{C}$ is such that there exists $\xi \in \mathbb{C}$, $|\xi| = 1$ and $\varepsilon \in (0,1)$ with*

$$|z - z_k - \xi| \leq \varepsilon \quad \text{for each } k \in \{1, \dots, n\}, \tag{3.16}$$

then

$$|P(z)| \leq \frac{|a_n|}{(1-\varepsilon^2)^{\frac{n}{2}}} \left| z - \frac{1}{n} \sum_{k=1}^{n} z_k \right|^{n}. \tag{3.17}$$

The equality holds in (3.17) if and only if

$$|z - z_k| = \sqrt{1 - \varepsilon^2} \quad \text{for each } \ k \in \{1, \ldots, n\}$$

and

$$z = \frac{1}{n} \sum_{k=1}^{n} z_k + \left(1 - \varepsilon^2\right) \xi.$$

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