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ON A GENERALIZED BIHARMONIC EQUATION IN PLANE POLARS WITH APPLICATIONS TO FUNCTIONALLY GRADED MATERIALS

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ABSTRACT. In this paper we consider a generalized biharmonic equation modelling a two–dimensional inhomogeneous elastic state in the curvilinear rectangle $a \leq r \leq b$, $0 \leq \theta \leq \alpha$, where (r, θ) denote plane polar coordinates. Such an arch–like region is maintained in equilibrium under self–equilibrated traction applied on the edge $\theta = 0$, while the other three edges $r = a$, $r = b$ and $\theta = \alpha$ are traction free. Our aim is to derive some explicit spatial exponential decay bounds for the specific Airy stress function and its derivatives. Two types of smoothly varying inhomogeneity are considered: (i) the elastic moduli vary smoothly with the polar angle, (ii) they vary smoothly with the polar distance. Such types of smoothly varying inhomogeneous elastic materials provide a model for technological important functionally graded materials. The results of the present paper prove how the spatial decay rate varies with the constitutive profile.

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1. INTRODUCTION

We consider a curvilinear strip of the form of an arch–like region S, which in plane polar coordinates θ and r is described by $S : a \le r \le b$, $0 \le \theta \le \alpha$, where a, b and $\alpha(< 2\pi)$ are prescribed positive constants. Such a region is assumed to be filled with an inhomogeneous and isotropic elastic material subjected to no body force. The arch–like region is maintained in equilibrium under self–equilibrated traction on the edge $\theta = 0$, while the other three edges $r = a$, $r = b$ and $\theta = \alpha$ are traction free. Then, the specific Airy stress function ϕ is the solution of the following boundary–value problem $\mathcal{P}$ (see the Appendix A for the bases leading to this equation)

$$\mathcal{L}\phi \equiv L\phi + L_0\phi = 0 \quad \text{in} \quad S, \tag{1.1}$$

with the boundary conditions

$$\phi(a,\theta) = \frac{\partial \phi}{\partial r}(a,\theta) = 0, \quad \phi(b,\theta) = \frac{\partial \phi}{\partial r}(b,\theta) = 0 \quad \text{for} \quad \theta \in [0,\alpha], \tag{1.2}$$

$$\phi(r,\alpha) = \frac{\partial \phi}{\partial \theta}(r,\alpha) = 0 \quad \text{for} \quad r \in [a,b], \tag{1.3}$$

and

$$\phi(r,0) = \varphi_1(r), \quad \frac{\partial \phi}{\partial \theta}(r,0) = \varphi_2(r) \quad \text{for} \quad r \in [a,b], \tag{1.4}$$

where $\varphi_1(r)$ and $\varphi_2(r)$ are appropriate prescribed functions over $[a,b]$. Moreover, the operators L and L_0 are defined as follows

$$\begin{aligned} L\phi \equiv\ & \frac{\partial^2}{\partial r^2}\left(\varepsilon r \frac{\partial^2 \phi}{\partial r^2}\right) + \frac{\partial^2}{\partial \theta^2}\left(\frac{\varepsilon}{r^3}\frac{\partial^2 \phi}{\partial \theta^2}\right) + 2\frac{\partial^2}{\partial r \partial \theta}\left(\frac{\varepsilon}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}\right) - \\ & -\frac{\partial}{\partial r}\left(\frac{\varepsilon}{r}\frac{\partial \phi}{\partial r}\right) + \frac{4\varepsilon}{r^3}\frac{\partial^2 \phi}{\partial \theta^2}, \end{aligned} \tag{1.5}$$

$$\begin{aligned} L_0\phi \equiv\ & -\left(\frac{3}{r^2}\frac{\partial \varepsilon}{\partial r} + \frac{1}{r}\frac{\partial^2 \bar{\varepsilon}}{\partial r^2}\right)\frac{\partial^2 \phi}{\partial \theta^2} + 2\left[\frac{1}{r^2}\frac{\partial \varepsilon}{\partial \theta} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \bar{\varepsilon}}{\partial \theta}\right)\right]\frac{\partial^2 \phi}{\partial r \partial \theta} + \\ & + \left(-\frac{1}{r}\frac{\partial^2 \bar{\varepsilon}}{\partial \theta^2} + \frac{\partial \bar{\varepsilon}}{\partial r}\right)\frac{\partial^2 \phi}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial \theta}(\varepsilon + \bar{\varepsilon})\right]\frac{\partial \phi}{\partial \theta} + \\ & + \left(\frac{1}{r^2}\frac{\partial^2 \varepsilon}{\partial \theta^2} - \frac{\partial^2 \bar{\varepsilon}}{\partial r^2}\right)\frac{\partial \phi}{\partial r}, \end{aligned} \tag{1.6}$$

where $\varepsilon = \varepsilon(r,\theta)$ and $\bar{\varepsilon} = \bar{\varepsilon}(r,\theta)$ are prescribed characteristics of the elastic material which are related to the well–known Lamé coefficients $\lambda(r,\theta)$ and $\mu(r,\theta)$ by means of

$$\varepsilon = \frac{\lambda + 2\mu}{4\mu(\lambda + \mu)}, \quad \bar{\varepsilon} = \frac{\lambda}{4\mu(\lambda + \mu)}. \tag{1.7}$$

It is assumed throughout that (see [1])

$$\mu > 0, \quad 3\lambda + 2\mu > 0, \tag{1.8}$$

and therefore, we have

$$\epsilon > 0. \tag{1.9}$$

We have to outline that, in the case of a homogeneous elastic material when $\varepsilon =$constant and $\bar{\varepsilon} =$constant, the operator L is related to the biharmonic operator in plane polar coordinates, while $L_0 \equiv 0$.

We are interested in the spatial decay bounds for the solution of the boundary–value problem $\mathcal{P}$ when the following two types of smoothly varying inhomogeneity are considered:

(i) the elastic coefficients vary smoothly with the polar angle θ, that is

$$\varepsilon = \varepsilon(\theta), \quad \bar{\varepsilon} = \bar{\varepsilon}(\theta), \tag{1.10}$$

(ii) they vary smoothly with the polar distance, that is

$$\varepsilon = \varepsilon(r), \quad \bar{\varepsilon} = \bar{\varepsilon}(r). \tag{1.11}$$

It should be noted that the above classes of smoothly varying inhomogeneous elastic materials provide a model for technologically important functionally graded materials. These materials have received considerable attention in recent literature (see, for example, the fundamental research developed in the papers [2, 3, 4, 5, 6]). Our aim is to derive sufficient conditions on the elastic coefficients ε and $\bar{\varepsilon}$ which will allow us to introduce appropriate measures concerning the Airy stress function and moreover, under such conditions we will obtain some second–order differential inequalities whose integration furnish some explicit spatial exponential decay results. The study is exemplified on the well–known class of inhomogeneous (isotropic) elastic materials occuring in literature [7] and characterized by

$$\varepsilon(r) = \varepsilon_0 r^p, \quad \bar{\varepsilon}(r) = f\varepsilon(r), \tag{1.12}$$

where $\varepsilon_0 > 0$, p and f are constants and f is such that $0 < f < 1$.

Decay bounds of the solution of a biharmonic equation in an inhomogeneous rectangular strip have been investigated in the last years by other authors as, for instance, [2, 8, 9, 4].

2. Main Results

Throughout this paper we will denote by S_θ the curvilinear rectangle bounded by the straight lines corresponding to the arbitrary angular variable θ and $\theta = \alpha$, as well as those defined by $r = a$ and $r = b$.

2.1. Spatial decay bounds for angular inhomogeneity.

We assume that the elastic coefficients vary smoothly with the polar angle θ and hence (1.10) holds true. We set

$$\dot{\varepsilon} = \frac{d\varepsilon}{d\theta}, \quad \ddot{\varepsilon} = \frac{d^2\varepsilon}{d\theta^2}, \quad \dot{\bar{\varepsilon}} = \frac{d\bar{\varepsilon}}{d\theta}, \quad \ddot{\bar{\varepsilon}} = \frac{d^2\bar{\varepsilon}}{d\theta^2} \tag{2.1}$$

and then we define

$$\begin{aligned} E_1(\theta) = \iint_{S_\theta} \varepsilon \Bigg\{ & r\left(\frac{\partial^2\phi}{\partial r^2}\right)^2 + \frac{1}{r}\left(\frac{1}{r}\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial\phi}{\partial r}\right)^2 + \frac{2}{r}\left(\frac{\partial^2\phi}{\partial r\partial\eta} - \frac{1}{r}\frac{\partial\phi}{\partial\eta}\right)^2 + \\ & + \frac{\ddot{\bar{\varepsilon}}}{\varepsilon r}\left(\frac{\partial\phi}{\partial r}\right)^2 - \frac{2\dot{\bar{\varepsilon}}}{\varepsilon r^2}\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial\eta}\Bigg\} dr d\eta, \quad \theta \in [0,\alpha]. \end{aligned} \tag{2.2}$$

With a view to establish sufficient conditions that $E_1(\theta)$ to be positive definite in ϕ we apply the identity

$$-\frac{4\varepsilon}{r^2}\frac{\partial\phi}{\partial\theta}\frac{\partial^2\phi}{\partial r\partial\theta} = -\frac{\partial}{\partial r}\left[\frac{2\varepsilon}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2\right] - \frac{4\varepsilon}{r^3}\left(\frac{\partial\phi}{\partial\theta}\right)^2 \tag{2.3}$$

and then use the boundary condition (1.2) to write the relation (2.2) in the following form

$$E_1(\theta) = \iint_{S_\theta} \varepsilon \left\{ r\left(\frac{\partial^2\phi}{\partial r^2}\right)^2 + \frac{1}{r}\left(\frac{1}{r}\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial\phi}{\partial r}\right)^2 + \frac{2}{r}\left(\frac{\partial^2\phi}{\partial r\partial\eta}\right)^2 + \right. \\ \left. -\frac{2}{r^3}\left(\frac{\partial\phi}{\partial\eta}\right)^2 + \frac{\ddot{\bar{\varepsilon}}}{\varepsilon r}\left(\frac{\partial\phi}{\partial r}\right)^2 - \frac{2\dot{\bar{\varepsilon}}}{\varepsilon r^2}\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial\eta}\right\} da, \quad \theta \in [0,\alpha]. \tag{2.4}$$

Furthermore, we apply the inequality (7.2a) – given in the Appendix B – with $\psi = \phi$ and the inequality (7.1a) with $\psi = \frac{\partial\phi}{\partial\theta}$ (noting that the relevant boundary conditions are satisfied in view of the relation (1.2)), so that we have the following inequalities

$$\int_a^b r\left(\frac{\partial^2\phi}{\partial r^2}\right)^2 dr \geq k\int_a^b \frac{1}{r}\left(\frac{\partial\phi}{\partial r}\right)^2 dr \tag{2.5}$$

and

$$\int_a^b \frac{1}{r}\left(\frac{\partial^2\phi}{\partial r\partial\theta}\right)^2 dr \geq \left[1 + \frac{\pi^2}{(\ln\frac{b}{a})^2}\right]\int_a^b \frac{1}{r^3}\left(\frac{\partial\phi}{\partial\theta}\right)^2 dr \tag{2.6}$$

where

$$k \geq \frac{\pi^2}{(\ln\frac{b}{a})^2}. \tag{2.7}$$

Thus, if we use the relations (2.5) and (2.6) in (2.4), then we obtain

$$E_1(\theta) \geq \iint_{S_\theta} \frac{\varepsilon}{r}\left\{\left(k + \frac{\ddot{\bar{\varepsilon}}}{\varepsilon}\right)\left(\frac{\partial\phi}{\partial r}\right)^2 - \frac{2\dot{\bar{\varepsilon}}}{\varepsilon r}\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial\eta} + \frac{2\pi^2}{(\ln\frac{b}{a})^2}\frac{1}{r^2}\left(\frac{\partial\phi}{\partial\eta}\right)^2\right\} dr d\eta + \\ + \iint_{S_\theta} \frac{\varepsilon}{r}\left(\frac{1}{r}\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial\phi}{\partial r}\right)^2 dr d\eta. \tag{2.8}$$

Let us assume henceforward that

$$k + \frac{\ddot{\bar{\varepsilon}}}{\varepsilon} > 0, \quad \max_{\theta\in[0,\alpha]}\left[\left(\frac{\dot{\bar{\varepsilon}}}{\varepsilon}\right)^2\right] < \frac{2\pi^2}{(\ln\frac{b}{a})^2}\min_{\theta\in[0,\alpha]}\left(k + \frac{\ddot{\bar{\varepsilon}}}{\varepsilon}\right). \tag{2.9}$$

In these circumstances it is clear that $E_1(\theta)$ represents a global measure of the magnitude of the stress function ϕ in S_θ. Consequently we can introduce the following measure

$$F_1(\theta) = \int_\theta^\alpha E_1(\eta)\, d\eta = \\ = \int_\theta^\alpha (\eta - \theta)\,\varepsilon \int_a^b \left\{ r\left(\frac{\partial^2\phi}{\partial r^2}\right)^2 + \frac{1}{r}\left(\frac{1}{r}\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial\phi}{\partial r}\right)^2 + \right. \\ \left. + \frac{2}{r}\left(\frac{\partial^2\phi}{\partial r\partial\eta} - \frac{1}{r}\frac{\partial\phi}{\partial\eta}\right)^2 + \frac{\ddot{\bar{\varepsilon}}}{\varepsilon r}\left(\frac{\partial\phi}{\partial r}\right)^2 - \frac{2\dot{\bar{\varepsilon}}}{\varepsilon r^2}\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial\eta}\right\} dr d\eta, \quad \theta\in[0,\alpha]. \tag{2.10}$$

The main result of the paper concerning the present type of inhomogeneity is expressed in the following theorem.

Theorem 2.1. *Consider an inhomogeneous arch–like region with ε and $\bar{\varepsilon}$ satisfying the relations (1.10) and (2.9). Suppose that the Airy stress function $\phi \in C^1\left(\bar{S}\right) \cap C^4(S)$ satisfies the equation*

$$L\phi + \frac{2}{r^2}\left(\dot{\varepsilon} - \dot{\bar{\varepsilon}}\right)\frac{\partial^2\phi}{\partial r\partial\theta} - \frac{\ddot{\bar{\varepsilon}}}{r}\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r^3}\left(\dot{\varepsilon} + \dot{\bar{\varepsilon}}\right)\frac{\partial\phi}{\partial\theta} + \frac{\ddot{\varepsilon}}{r^2}\frac{\partial\phi}{\partial r} = 0 \quad \textit{in} \quad S \tag{2.11}$$

and the boundary conditions described by the relations (1.2) and (1.3). Then there exists a positive constant $\varkappa_1$ such that

$$0 \leq F_1(\theta) \leq F_1(0)\, e^{-\varkappa_1 \theta} \quad \textit{for all} \quad \theta \in [0, \alpha]. \tag{2.12}$$

2.2. **Spatial decay bounds for radial inhomogeneity.** In this subsection we assume that the elastic coefficients vary smoothly with the polar distance r so that (1.11) holds true. Then we set

$$\varepsilon' = \frac{d\varepsilon}{dr}, \quad \varepsilon'' = \frac{d^2\varepsilon}{dr^2}, \quad \bar{\varepsilon}' = \frac{d\bar{\varepsilon}}{dr}, \quad \bar{\varepsilon}'' = \frac{d^2\bar{\varepsilon}}{dr^2} \tag{2.13}$$

and introduce the function

$$\begin{aligned} E_2(\theta) \;=\; & \iint_{S_\theta} \varepsilon \left\{ r \left(\frac{\partial^2 \phi}{\partial r^2} \right)^2 + \frac{1}{r} \left(\frac{1}{r} \frac{\partial^2 \phi}{\partial \eta^2} + \frac{\partial \phi}{\partial r} \right)^2 + \right. \\ & + \frac{2}{r} \left(\frac{\partial^2 \phi}{\partial r \partial \eta} - \frac{1}{r} \frac{\partial \phi}{\partial \eta} \right)^2 + \frac{\bar{\varepsilon}''}{\varepsilon r} \left(\frac{\partial \phi}{\partial \eta} \right)^2 + \\ & \left. + \frac{\bar{\varepsilon}'}{\varepsilon} \left(\frac{\partial \phi}{\partial r} \right)^2 \right\} dr d\eta, \quad \theta \in [0, \alpha]. \end{aligned} \tag{2.14}$$

It is worth to outline that $E_2(\theta)$ represents a measure of the Airy stress function, provided $\bar{\varepsilon}'' > 0$ and $\bar{\varepsilon}' > 0$. We can relax these restrictions if we use the identity

$$-\frac{4\varepsilon}{r^2} \frac{\partial \phi}{\partial \theta} \frac{\partial^2 \phi}{\partial r \partial \theta} = -\frac{\partial}{\partial r} \left[\frac{2\varepsilon}{r^2} \left(\frac{\partial \phi}{\partial \theta} \right)^2 \right] + \frac{2}{r^3} (r\varepsilon' - 2\varepsilon) \left(\frac{\partial \phi}{\partial \theta} \right)^2 \tag{2.15}$$

in order to write the relation (2.14) in the following form

$$\begin{aligned} E_2(\theta) \;=\; & \iint_{S_\theta} \varepsilon \left\{ r \left(\frac{\partial^2 \phi}{\partial r^2} \right)^2 + \frac{1}{r} \left(\frac{1}{r} \frac{\partial^2 \phi}{\partial \eta^2} + \frac{\partial \phi}{\partial r} \right)^2 + \frac{2}{r} \left(\frac{\partial^2 \phi}{\partial r \partial \eta} \right)^2 + \right. \\ & \left. + \frac{\bar{\varepsilon}'}{\varepsilon} \left(\frac{\partial \phi}{\partial r} \right)^2 + \left(\frac{r^2 \bar{\varepsilon}''}{\varepsilon} + \frac{2r\varepsilon'}{\varepsilon} - 2 \right) \frac{1}{r^3} \left(\frac{\partial \phi}{\partial \eta} \right)^2 \right\} dr d\eta. \end{aligned} \tag{2.16}$$

Furthermore, we use the boundary condition (1.2) in order to apply the inequality (8.1) – given in the Appendix C – with $\psi = r\frac{\partial \phi}{\partial r}$ and then with $\psi = \frac{\partial \phi}{\partial \theta}$. Thus, we can write the following inequalities

$$\int_a^b \varepsilon r \left(\frac{\partial^2 \phi}{\partial r^2} \right)^2 dr \geq \int_a^b \left(\lambda_1 + \frac{r\varepsilon'}{\varepsilon} \right) \frac{\varepsilon}{r} \left(\frac{\partial \phi}{\partial r} \right)^2 dr, \tag{2.17}$$

$$\int_a^b \frac{\varepsilon}{r} \left(\frac{\partial^2 \phi}{\partial r \partial \theta} \right)^2 dr \geq (\lambda_1 + 1) \int_a^b \frac{\varepsilon}{r^3} \left(\frac{\partial \phi}{\partial \theta} \right)^2 dr, \tag{2.18}$$

where λ_1 is the lowest eigenvalue defined in the Appendix C. So, if we use the estimates (2.17) and (2.18) in the relation (2.16), then we get

$$\begin{aligned} E_2(\theta) \;\geq\; & \iint_{S_\theta} \varepsilon \left\{ \frac{1}{r} \left(\frac{1}{r} \frac{\partial^2 \phi}{\partial \eta^2} + \frac{\partial \phi}{\partial r} \right)^2 + \left[\lambda_1 + \frac{r}{\varepsilon} (\varepsilon' + \bar{\varepsilon}') \right] \frac{1}{r} \left(\frac{\partial \phi}{\partial r} \right)^2 + \right. \\ & \left. + \left(\frac{r^2 \bar{\varepsilon}''}{\varepsilon} + \frac{2r\varepsilon'}{\varepsilon} + 2\lambda_1 \right) \frac{1}{r^3} \left(\frac{\partial \phi}{\partial \eta} \right)^2 \right\} dr d\eta. \end{aligned} \tag{2.19}$$

We now assume for henceforward that

$$\lambda_1 + \frac{r}{\varepsilon}\left(\varepsilon' + \bar{\varepsilon}'\right) > 0, \quad \frac{r^2\bar{\varepsilon}''}{\varepsilon} + \frac{2r\varepsilon'}{\varepsilon} + 2\lambda_1 > 0, \tag{2.20}$$

so that $E_2(\theta)$ represents a global measure of the Airy stress function ϕ in S_θ. Thus, we can introduce the following measure

$$\begin{aligned} F_2(\theta) &= \int_\theta^\alpha E_2(\eta)\, d\eta = \\ &= \int_\theta^\alpha \int_a^b (\eta-\theta)\,\varepsilon \left\{ r\left(\frac{\partial^2\phi}{\partial r^2}\right)^2 + \frac{1}{r}\left(\frac{1}{r}\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial\phi}{\partial r}\right)^2 + \right. \\ &\quad \left. + \frac{2}{r}\left(\frac{\partial^2\phi}{\partial r\partial\eta} - \frac{1}{r}\frac{\partial\phi}{\partial\eta}\right)^2 + \frac{\bar{\varepsilon}''}{\varepsilon r}\left(\frac{\partial\phi}{\partial\eta}\right)^2 + \frac{\bar{\varepsilon}'}{\varepsilon}\left(\frac{\partial\phi}{\partial r}\right)^2 \right\} dr d\eta, \quad \theta\in[0,\alpha]. \end{aligned} \tag{2.21}$$

The main result concerning this type of inhomogeneity is expressed in the following theorem.

Theorem 2.2. *Consider an inhomogeneous arch–like region with ε and $\bar{\varepsilon}$ satisfying the relations (1.11) and (2.20). Suppose that the Airy stress function $\phi \in C^1\left(\bar{S}\right) \cap C^4(S)$ satisfies the equation*

$$L\phi - \left(\frac{3}{r^2}\varepsilon' + \frac{1}{r}\bar{\varepsilon}''\right)\frac{\partial^2\phi}{\partial\theta^2} + \bar{\varepsilon}'\frac{\partial^2\phi}{\partial r^2} - \bar{\varepsilon}''\frac{\partial\phi}{\partial r} = 0 \quad \textit{in} \quad S \tag{2.22}$$

and the boundary conditions described by the relations (1.2) and (1.3). Then there is a positive constant $\varkappa_2$ such that

$$0 \le F_2(\theta) \le F_2(0)\, e^{-\varkappa_2\theta} \quad \textit{for all} \ \theta\in[0,\alpha]. \tag{2.23}$$

3. THE PROOF OF THE THEOREM 2.1

Let us consider ϕ a solution of the equation (2.11) and let us introduce the function

$$\begin{aligned} I_1(\theta) &= \int_a^b \left[-\frac{\varepsilon}{r^3}\phi\frac{\partial^2\phi}{\partial\theta^2} + \frac{\varepsilon}{r^3}\left(\frac{\partial\phi}{\partial\theta}\right)^2 + \frac{\varepsilon}{r}\left(\frac{\partial\phi}{\partial r}\right)^2 - \frac{2\varepsilon}{r^3}\phi^2 \right] dr - \\ &\quad - \iint_{S_\theta} \left[-\frac{\dot{\varepsilon}}{r^3}\left(\frac{\partial\phi}{\partial\eta}\right)^2 - \frac{\dot{\varepsilon}}{r}\left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{\dot{\varepsilon} + \dot{\bar{\varepsilon}}}{r^3}\phi^2 + \right. \\ &\quad \left. + \frac{2\varepsilon}{r^2}\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial\eta} \right] dr d\eta, \quad \theta\in[0,\alpha]. \end{aligned} \tag{3.1}$$

By successive differentiations with respect to θ we obtain

$$\begin{aligned} \frac{dI_1}{d\theta}(\theta) &= \int_a^b \left[-\phi\frac{\partial}{\partial\theta}\left(\frac{\varepsilon}{r^3}\frac{\partial^2\phi}{\partial\theta^2}\right) + \frac{\varepsilon}{r^3}\frac{\partial\phi}{\partial\theta}\frac{\partial^2\phi}{\partial\theta^2} + \frac{2\varepsilon}{r}\frac{\partial\phi}{\partial r}\frac{\partial^2\phi}{\partial r\partial\theta} + \right. \\ &\quad \left. + \frac{2\varepsilon}{r^2}\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial\theta} - \frac{4\varepsilon}{r^3}\phi\frac{\partial\phi}{\partial\theta} + \frac{\dot{\bar{\varepsilon}} - \dot{\varepsilon}}{r^3}\phi^2 \right] dr, \end{aligned} \tag{3.2}$$

and

$$
\begin{aligned}
\frac{d^2 I_1}{d\theta^2}(\theta) &= \int_a^b \left[-\phi \frac{\partial^2}{\partial\theta^2}\left(\frac{\varepsilon}{r^3}\frac{\partial^2\phi}{\partial\theta^2}\right) + \frac{\varepsilon}{r^3}\left(\frac{\partial^2\phi}{\partial\theta^2}\right)^2 + \frac{2\varepsilon}{r}\left(\frac{\partial^2\phi}{\partial r\partial\theta}\right)^2 + \right. \\
&\quad + \frac{\partial\phi}{\partial r}\frac{\partial}{\partial\theta}\left(\frac{2\varepsilon}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\right) + \frac{2\varepsilon}{r^2}\frac{\partial\phi}{\partial\theta}\frac{\partial^2\phi}{\partial r\partial\theta} + \frac{2\varepsilon}{r^2}\frac{\partial\phi}{\partial r}\frac{\partial^2\phi}{\partial\theta^2} - \\
&\quad - \frac{4\varepsilon}{r^3}\left(\frac{\partial\phi}{\partial\theta}\right)^2 + \frac{2\dot{\varepsilon}}{r^2}\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial\theta} - \frac{4\varepsilon}{r^3}\phi\frac{\partial^2\phi}{\partial\theta^2} + \frac{2\left(\dddot{\varepsilon} - 3\dot{\varepsilon}\right)}{r^3}\phi\frac{\partial\phi}{\partial\theta} + \\
&\quad \left. + \frac{\ddddot{\varepsilon} - \ddot{\varepsilon}}{r^3}\phi^2 \right] dr.
\end{aligned} \tag{3.3}
$$

Further, by using the equation (2.11), the integration by parts and the boundary condition (1.2), we obtain that

$$
\begin{aligned}
\frac{d^2 I_1}{d\theta^2}(\theta) &= \int_a^b \varepsilon \left\{ r\left(\frac{\partial^2\phi}{\partial r^2}\right)^2 + \frac{1}{r}\left(\frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial\phi}{\partial r}\right)^2 + \frac{2}{r}\left(\frac{\partial^2\phi}{\partial r\partial\theta}\right)^2 + \right. \\
&\quad \left. - \frac{2}{r^3}\left(\frac{\partial\phi}{\partial\theta}\right)^2 + \frac{\ddot{\varepsilon}}{\varepsilon r}\left(\frac{\partial\phi}{\partial r}\right)^2 - \frac{2\dot{\varepsilon}}{\varepsilon r^2}\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial\theta} \right\} dr
\end{aligned} \tag{3.4}
$$

and hence, by means of the relations (2.5) and (2.6), we deduce that

$$
\frac{d^2 I_1}{d\theta^2}(\theta) \geq \sigma_m \int_a^b \frac{\varepsilon}{r}\left[\left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2\right] dr + \int_a^b \frac{\varepsilon}{r}\left(\frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial\phi}{\partial r}\right)^2 dr \geq 0, \tag{3.5}
$$

where

$$
\begin{aligned}
\mu_1 &= \min_{\theta\in[0,\alpha]}\left(k + \frac{\ddot{\varepsilon}}{\varepsilon}\right), \quad \mu_2 = \max_{\theta\in[0,\alpha]}\left|\frac{\dot{\varepsilon}}{\varepsilon}\right|, \\
\sigma_m &= \frac{1}{\mu_1}\left[\mu_2 - \sqrt{\mu_2^2 - \frac{2\pi^2}{\left(\ln\frac{b}{a}\right)^2}\mu_1}\right].
\end{aligned} \tag{3.6}
$$

On the other hand, by means of the boundary condition (1.3), from the relations (3.1) and (3.2), we obtain

$$
I_1(\alpha) = 0, \quad \frac{dI_1}{d\theta}(\alpha) = 0. \tag{3.7}
$$

Thus, the relations (3.5) and (3.7) give

$$
\frac{dI_1}{d\theta}(\theta) \leq \frac{dI_1}{d\theta}(\alpha) = 0 \quad \text{for all } \theta\in[0,\alpha], \tag{3.8}
$$

$$
I_1(\theta) \geq I_1(\alpha) = 0 \quad \text{for all } \theta\in[0,\alpha]. \tag{3.9}
$$

Moreover, by integrating the relations (3.4) and (3.5) successively with respect to θ over $[0,\alpha]$ and by using the relations (3.7) and (3.8), we get

$$
\begin{aligned}
-\frac{dI_1}{d\theta}(\theta) &= E_1(\theta) \geq \sigma_m \iint_{S_\theta} \frac{\varepsilon}{r}\left[\left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi}{\partial\eta}\right)^2\right] dr d\eta + \\
&\quad + \iint_{S_\theta} \frac{\varepsilon}{r}\left(\frac{1}{r}\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial\phi}{\partial r}\right)^2 dr d\eta,
\end{aligned} \tag{3.10}
$$

$$
I_1(\theta) = F_1(\theta) \quad \text{for all } \theta\in[0,\alpha]. \tag{3.11}
$$

We proceed now to establish a second–order differential inequality in terms of the measure $I_1(\theta) = F_1(\theta)$. We will determine the positive constants α_1 and β_1 such that

$$\frac{d^2 I_1}{d\theta^2}(\theta) - \alpha_1 \frac{dI_1}{d\theta}(\theta) - \beta_1 I_1(\theta) \geq 0 \quad \text{for all} \quad \theta \in [0,\alpha]. \tag{3.12}$$

To this end we combine the relations (3.1), (3.5) and (3.10) to obtain

$$\begin{aligned}
\frac{d^2 I_1}{d\theta^2}(\theta) - \alpha_1 \frac{dI_1}{d\theta}(\theta) - \beta_1 I_1(\theta) \geq & \int_a^b \left[(\sigma_m + 1 - \beta_1 - \gamma) \frac{\varepsilon}{r} \left(\frac{\partial \phi}{\partial r} \right)^2 + \right. \\
& + (\sigma_m - \beta_1) \frac{\varepsilon}{r^3} \left(\frac{\partial \phi}{\partial \theta} \right)^2 + \left(1 - \frac{\beta_1}{8} - \frac{1}{\gamma} \right) \frac{\varepsilon}{r^3} \left(\frac{\partial^2 \phi}{\partial \theta^2} \right)^2 + \\
& \left. + 2\beta_1 \frac{\varepsilon}{r^3} \left(\phi + \frac{1}{4} \frac{\partial^2 \phi}{\partial \theta^2} \right)^2 + \frac{\varepsilon}{r} \left(\sqrt{\gamma} \frac{\partial \phi}{\partial r} + \frac{1}{\sqrt{\gamma} r} \frac{\partial^2 \phi}{\partial \theta^2} \right)^2 \right] dr + \\
& + \iint_{S_\theta} \left\{ \left(\alpha_1 \sigma_m - \beta_1 - \beta_1 \frac{\dot{\varepsilon}}{\varepsilon} \right) \frac{\varepsilon}{r} \left[\left(\frac{\partial \phi}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \phi}{\partial \eta} \right)^2 \right] + \right. \\
& + \alpha_1 \frac{\varepsilon}{r^3} \left(\frac{\partial^2 \phi}{\partial \eta^2} + r \frac{\partial \phi}{\partial r} \right)^2 + \beta_1 \frac{\varepsilon}{r} \left(\frac{\partial \phi}{\partial r} + \frac{1}{r} \frac{\partial \phi}{\partial \eta} \right)^2 + \\
& \left. + \beta_1 \frac{\dot{\varepsilon} + \dot{\bar{\varepsilon}}}{\varepsilon} \frac{\varepsilon}{r^3} \phi^2 \right] dr d\eta,
\end{aligned} \tag{3.13}$$

where γ is a positive parameter at our disposal. Now we set

$$\frac{1}{\gamma} = 1 - \frac{\beta_1}{8}, \quad \beta_1 = \min \left\{ 8, \sigma_m, \frac{1}{2} \left[9 + \sigma_m - \sqrt{(\sigma_m - 7)^2 + 32} \right] \right\} \tag{3.14}$$

and moreover, we introduce the notation

$$\mu_3 = \max_{\theta \in [0,\alpha]} \left| \frac{\dot{\varepsilon}}{\varepsilon} + 1 \right|, \tag{3.15}$$

so that the relation (3.13) yields

$$\begin{aligned}
\frac{d^2 I_1}{d\theta^2}(\theta) - \alpha_1 \frac{dI_1}{d\theta}(\theta) - \beta_1 I_1(\theta) \geq & \iint_{S_\theta} \left[(\alpha_1 \sigma_m - \beta_1 \mu_3) \frac{\varepsilon}{r} \left(\frac{\partial \phi}{\partial r} \right)^2 + \right. \\
& \left. + (\alpha_1 \sigma_m - \beta_1 \mu_3) \frac{\varepsilon}{r^3} \left(\frac{\partial \phi}{\partial \eta} \right)^2 + \beta_1 \frac{\dot{\varepsilon} + \dot{\bar{\varepsilon}}}{\varepsilon} \frac{\varepsilon}{r^3} \phi^2 \right] dr d\eta.
\end{aligned} \tag{3.16}$$

Furthermore, we set

$$\mu_4 = \max_{\theta \in [0,\alpha]} \left| \frac{\dot{\varepsilon} + \dot{\bar{\varepsilon}}}{\varepsilon} \right| \tag{3.17}$$

and then we take

$$\alpha_1 = \frac{\beta_1}{\sigma_m} \left[\mu_3 + \mu_4 \left(1 + \frac{\pi^2}{\left(\ln \frac{b}{a} \right)^2} \right)^{-1} \right], \tag{3.18}$$

so that, by making use of the inequality (7.1a), from (3.16) we obtain the second–order differential inequality (3.12).

We now have all the preliminary material to prove the Theorem 2.1. In fact, by the inequality (3.12) and a well–known Comparison Principle (a generalization of the curve under chord property for convex functions) it follows that $I_1(\theta)$ is bounded above by the solution of the differential equation corresponding to the differential inequality (3.12) with the same boundary conditions, that is the function $G_1(\theta)$ satisfying

$$\frac{d^2G_1}{d\theta^2}(\theta)-\alpha_1\frac{dG_1}{d\theta}(\theta)-\beta_1G_1(\theta)=0 \quad \text{for all} \quad \theta\in[0,\alpha] \tag{3.19}$$

with

$$G_1(0)=I_1(0), \quad G_1(\alpha)=0. \tag{3.20}$$

On this basis we get

$$0\le I_1(\theta)\le\frac{1-e^{-(\nu_1+\nu_2)(\alpha-\theta)}}{1-e^{-(\nu_1+\nu_2)\alpha}}I_1(0)\,e^{-\nu_2\theta} \quad \text{for all} \quad \theta\in[0,\alpha], \tag{3.21}$$

where

$$\nu_1=\frac{1}{2}\left(\alpha_1+\sqrt{\alpha_1^2+4\beta_1}\right), \quad \nu_2=\frac{1}{2}\left(-\alpha_1+\sqrt{\alpha_1^2+4\beta_1}\right). \tag{3.22}$$

This result yields

$$0\le I_1(\theta)\le I_1(0)\,e^{-\nu_2\theta} \quad \text{for all} \quad \theta\in[0,\alpha], \tag{3.23}$$

which when combined with the relation (3.11) furnishes the estimate (2.12) with $\varkappa_1=\nu_2$ and so the proof is complete.

4. THE PROOF OF THE THEOREM 2.2

With the solution of the equation (2.22) we associate the following function

$$\begin{aligned} I_2(\theta) &= \int_a^b\left[-\frac{\varepsilon}{r^3}\phi\frac{\partial^2\phi}{\partial\theta^2}+\frac{\varepsilon}{r^3}\left(\frac{\partial\phi}{\partial\theta}\right)^2+\frac{\varepsilon}{r}\left(\frac{\partial\phi}{\partial r}\right)^2+\right.\\ &\left.+\left(-\frac{2\varepsilon}{r^3}+\frac{3\varepsilon'}{2r^2}+\frac{\bar{\varepsilon}''}{2r}\right)\phi^2\right]dr-\iint_{S_\theta}\frac{2\varepsilon}{r^2}\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial\eta}dr\,d\eta, \quad \theta\in[0,\alpha] \end{aligned} \tag{4.1}$$

and note that, by successive differentiations with respect to θ, we obtain

$$\begin{aligned} \frac{dI_2}{d\theta}(\theta) &= \int_a^b\left[-\phi\frac{\partial}{\partial\theta}\left(\frac{\varepsilon}{r^3}\frac{\partial^2\phi}{\partial\theta^2}\right)+\frac{\varepsilon}{r^3}\frac{\partial\phi}{\partial\theta}\frac{\partial^2\phi}{\partial\theta^2}+\frac{2\varepsilon}{r}\frac{\partial\phi}{\partial r}\frac{\partial^2\phi}{\partial r\partial\theta}+\right.\\ &\left.+\left(-\frac{4\varepsilon}{r^3}+\frac{3\varepsilon'}{r^2}+\frac{\bar{\varepsilon}''}{r}\right)\phi\frac{\partial\phi}{\partial\theta}+\frac{2\varepsilon}{r^2}\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial\theta}\right]dr, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \frac{d^2I_2}{d\theta^2}(\theta) &= \int_a^b\left\{-\phi\frac{\partial^2}{\partial\theta^2}\left(\frac{\varepsilon}{r^3}\frac{\partial^2\phi}{\partial\theta^2}\right)+\frac{\varepsilon}{r^3}\left(\frac{\partial^2\phi}{\partial\theta^2}\right)^2+\frac{2\varepsilon}{r}\left(\frac{\partial^2\phi}{\partial r\partial\theta}\right)^2+\right.\\ &+\frac{\partial\phi}{\partial r}\frac{\partial}{\partial\theta}\left(\frac{2\varepsilon}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\right)+\left(-\frac{4\varepsilon}{r^3}+\frac{3\varepsilon'}{r^2}+\frac{\bar{\varepsilon}''}{r}\right)\left[\left(\frac{\partial\phi}{\partial\theta}\right)^2+\phi\frac{\partial^2\phi}{\partial\theta^2}\right]+\\ &\left.+\frac{2\varepsilon}{r^2}\frac{\partial\phi}{\partial\theta}\frac{\partial^2\phi}{\partial r\partial\theta}+\frac{2\varepsilon}{r^2}\frac{\partial\phi}{\partial r}\frac{\partial^2\phi}{\partial\theta^2}\right\}dr. \end{aligned} \tag{4.3}$$

By using the equation (2.22), the integration by parts and the boundary condition (1.2), we can write

$$
\begin{aligned}
\frac{d^2 I_2}{d\theta^2}(\theta) = & \int_a^b \left\{ \varepsilon r \left(\frac{\partial^2\phi}{\partial r^2}\right)^2 + \frac{\varepsilon}{r^3}\left(\frac{\partial^2\phi}{\partial\theta^2}\right)^2 + \frac{\varepsilon}{r}\left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{2\varepsilon}{r}\left(\frac{\partial^2\phi}{\partial r\partial\theta}\right)^2 + \right. \\
& + \frac{2\varepsilon}{r^2}\frac{\partial\phi}{\partial r}\frac{\partial^2\phi}{\partial\theta^2} - \frac{4\varepsilon}{r^2}\frac{\partial\phi}{\partial\theta}\frac{\partial^2\phi}{\partial r\partial\theta} + \frac{2\varepsilon}{r^3}\left(\frac{\partial\phi}{\partial\theta}\right)^2 + \\
& \left. + \frac{\bar{\varepsilon}''}{r}\left(\frac{\partial\phi}{\partial\theta}\right)^2 + \bar{\varepsilon}'\left(\frac{\partial\phi}{\partial r}\right)^2 \right\} dr,
\end{aligned}
\tag{4.4}
$$

or, by means of the identity (2.15),

$$
\begin{aligned}
\frac{d^2 I_2}{d\theta^2}(\theta) = & \int_a^b \left\{ \frac{\varepsilon}{r}\left(\frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial\phi}{\partial r}\right)^2 + \left[\varepsilon r\left(\frac{\partial^2\phi}{\partial r^2}\right)^2 + \bar{\varepsilon}'\left(\frac{\partial\phi}{\partial r}\right)^2\right] + \right. \\
& \left. + \left[\frac{2\varepsilon}{r}\left(\frac{\partial^2\phi}{\partial r\partial\theta}\right)^2 + \frac{1}{r}\left(-\frac{2\varepsilon}{r^2} + \frac{2\varepsilon'}{r} + \bar{\varepsilon}''\right)\left(\frac{\partial\phi}{\partial\theta}\right)^2\right]\right\} dr.
\end{aligned}
\tag{4.5}
$$

Therefore, if we use the relations (2.17) and (2.18) into (4.5), then we can write

$$
\begin{aligned}
\frac{d^2 I_2}{d\theta^2}(\theta) \geq & \int_a^b \left\{ \frac{\varepsilon}{r}\left(\frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial\phi}{\partial r}\right)^2 + \left[\lambda_1 + \frac{r}{\varepsilon}(\varepsilon' + \bar{\varepsilon}')\right]\frac{\varepsilon}{r}\left(\frac{\partial\phi}{\partial r}\right)^2 + \right. \\
& \left. + \left(\frac{r^2}{\varepsilon}\bar{\varepsilon}'' + \frac{2r\varepsilon'}{\varepsilon} + 2\lambda_1\right)\frac{\varepsilon}{r^3}\left(\frac{\partial\phi}{\partial\theta}\right)^2\right\} dr
\end{aligned}
\tag{4.6}
$$

and hence we have

$$
\frac{d^2 I_2}{d\theta^2}(\theta) \geq \int_a^b \left[\frac{\varepsilon}{r}\left(\frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial\phi}{\partial r}\right)^2 + \xi_1\frac{\varepsilon}{r}\left(\frac{\partial\phi}{\partial r}\right)^2 + \xi_2\frac{\varepsilon}{r^3}\left(\frac{\partial\phi}{\partial\theta}\right)^2\right] dr \geq 0,
\tag{4.7}
$$

where

$$
\xi_1 = \min_{r\in[a,b]}\left[\lambda_1 + \frac{r}{\varepsilon}(\varepsilon' + \bar{\varepsilon}')\right], \quad \xi_2 = \min_{r\in[a,b]}\left(\frac{r^2}{\varepsilon}\bar{\varepsilon}'' + \frac{2r\varepsilon'}{\varepsilon} + 2\lambda_1\right).
\tag{4.8}
$$

On the other hand, the relations (1.3), (4.1) and (4.2) furnish

$$
I_2(\alpha) = 0, \quad \frac{dI_2}{d\theta}(\alpha) = 0
\tag{4.9}
$$

and hence, by means of (4.7), we deduce that

$$
\frac{dI_2}{d\theta}(\theta) \leq 0, \quad I_2(\theta) \geq 0 \quad \text{for all } \theta \in [0,\alpha].
\tag{4.10}
$$

Further, by integrating the relation (4.7) successively with respect to θ over $[\theta,\alpha]$ and by using the relations (2.16) and (4.9), we get

$$
-\frac{dI_2}{d\theta}(\theta) = E_2(\theta) \geq \iint_{S_\theta}\left[\frac{\varepsilon}{r}\left(\frac{1}{r}\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial\phi}{\partial r}\right)^2 + \xi_1\frac{\varepsilon}{r}\left(\frac{\partial\phi}{\partial r}\right)^2 + \xi_2\frac{\varepsilon}{r^3}\left(\frac{\partial\phi}{\partial\eta}\right)^2\right] dr d\eta,
\tag{4.11}
$$

$$
I_2(\theta) = F_2(\theta) \quad \text{for all } \theta \in [0,\alpha].
\tag{4.12}
$$

In what follows we prove that it is possible to determine the positive parameters α_2 and β_2 such that

$$\frac{d^2 I_2}{d\theta^2}(\theta) - \alpha_2 \frac{dI_2}{d\theta}(\theta) - \beta_2 I_2(\theta) \geq 0 \quad \text{for all} \quad \theta \in [0, \alpha]. \tag{4.13}$$

To this end we combine the relations (4.1), (4.7) and (4.11) to obtain

$$\begin{aligned} \frac{d^2 I_2}{d\theta^2}(\theta) - \alpha_2 \frac{dI_2}{d\theta}(\theta) - \beta_2 I_2(\theta) \geq & \iint_{S_\theta} \left[\alpha_2 \xi_1 \frac{\varepsilon}{r} \left(\frac{\partial \phi}{\partial r} \right)^2 + \alpha_2 \xi_2 \frac{\varepsilon}{r^3} \left(\frac{\partial \phi}{\partial \eta} \right)^2 + \right. \\ & \left. + 2\beta_2 \frac{\varepsilon}{r^2} \frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial \eta} \right] dr d\eta + \int_a^b \left[\frac{\varepsilon}{r^3} \left(\frac{\partial^2 \phi}{\partial \theta^2} \right)^2 + \right. \\ & + \beta_2 \frac{\varepsilon}{r^3} \phi \frac{\partial^2 \phi}{\partial \theta^2} + \frac{2\varepsilon}{r^2} \frac{\partial \phi}{\partial r} \frac{\partial^2 \phi}{\partial \theta^2} + \\ & + (\xi_1 + 1 - \beta_2) \frac{\varepsilon}{r} \left(\frac{\partial \phi}{\partial r} \right)^2 + (\xi_2 - \beta_2) \frac{\varepsilon}{r^3} \left(\frac{\partial \phi}{\partial \theta} \right)^2 + \\ & \left. + \beta_2 \left(2 - \frac{3r\varepsilon'}{2\varepsilon} - \frac{r^2 \overline{\varepsilon}''}{2\varepsilon} \right) \frac{\varepsilon}{r^3} \phi^2 \right] dr. \end{aligned} \tag{4.14}$$

At this instant we set

$$\beta_2 = \alpha_2 \sqrt{\xi_1 \xi_2}, \tag{4.15}$$

so that the relation (4.14) furnishes

$$\begin{aligned} \frac{d^2 I_2}{d\theta^2}(\theta) - \alpha_2 \frac{dI_2}{d\theta}(\theta) - \beta_2 I_2(\theta) \geq & \int_a^b \left[\left(1 - \frac{\beta_2}{8} \right) \frac{\varepsilon}{r^3} \left(\frac{\partial^2 \phi}{\partial \theta^2} \right)^2 + \right. \\ & + \frac{2\varepsilon}{r^2} \frac{\partial \phi}{\partial r} \frac{\partial^2 \phi}{\partial \theta^2} + (\xi_1 + 1 - \beta_2) \frac{\varepsilon}{r} \left(\frac{\partial \phi}{\partial r} \right)^2 + \\ & + (\xi_2 - \beta_2) \frac{\varepsilon}{r^3} \left(\frac{\partial \phi}{\partial \theta} \right)^2 - \beta_2 m \frac{\varepsilon}{r^3} \phi^2 + \\ & \left. + 2\beta_2 \frac{\varepsilon}{r^3} \left(\phi + \frac{1}{4} \frac{\partial^2 \phi}{\partial \theta^2} \right)^2 \right] dr, \end{aligned} \tag{4.16}$$

where

$$m = \max_{r \in [a,b]} \left| \frac{3r\varepsilon'}{2\varepsilon} + \frac{r^2 \overline{\varepsilon}''}{2\varepsilon} \right|. \tag{4.17}$$

Further, we use the arithmetic–geometric mean inequality in (4.16) to obtain

$$\begin{aligned} \frac{d^2 I_2}{d\theta^2}(\theta) - \alpha_2 \frac{dI_2}{d\theta}(\theta) - \beta_2 I_2(\theta) \geq & \int_a^b \left[\left(1 - \frac{\beta_2}{8} - \frac{1}{\tau} \right) \frac{\varepsilon}{r^3} \left(\frac{\partial^2 \phi}{\partial \theta^2} \right)^2 + \right. \\ & + (\xi_1 + 1 - \beta_2 - \tau) \frac{\varepsilon}{r} \left(\frac{\partial \phi}{\partial r} \right)^2 + \\ & \left. + (\xi_2 - \beta_2) \frac{\varepsilon}{r^3} \left(\frac{\partial \phi}{\partial \theta} \right)^2 - \beta_2 m \frac{\varepsilon}{r^3} \phi^2 \right] dr, \end{aligned} \tag{4.18}$$

with τ a positive constant at our disposal. At this instant we set

$$\frac{1}{\tau}=1-\frac{\beta_2}{8},\quad \beta_2^*=\min\left\{8,\xi_2,\frac{1}{2}\left[\xi_1+9-\sqrt{(\xi_1-7)^2+32}\right]\right\},\quad \beta_2<\beta_2^*, \tag{4.19}$$

so that the relation (4.18) gives

$$\frac{d^2I_2}{d\theta^2}(\theta)-\alpha_2\frac{dI_2}{d\theta}(\theta)-\beta_2I_2(\theta)\ \geq\ \int_a^b\left[\left(\xi_1+1-\beta_2-\frac{8}{8-\beta_2}\right)\frac{\varepsilon}{r}\left(\frac{\partial\phi}{\partial r}\right)^2-\right.$$
$$\left.-\beta_2m\frac{\varepsilon}{r^3}\phi^2\right]dr. \tag{4.20}$$

By using the inequality (8.1), from (4.20) we obtain

$$\frac{d^2I_2}{d\theta^2}(\theta)-\alpha_2\frac{dI_2}{d\theta}(\theta)-\beta_2I_2(\theta)\geq\int_a^b\left[\left(\xi_1+1-\beta_2-\frac{8}{8-\beta_2}\right)(\lambda_1+1)-\beta_2m\right]\frac{\varepsilon}{r^3}\phi^2dr. \tag{4.21}$$

Thus, if we set

$$\beta_2=\min\left\{\beta_2^*,\hat{\beta}_2^*\right\}, \tag{4.22}$$

$$\hat{\beta}_2^*\ =\ \frac{1}{2(\lambda_1+1+m)}\left\{(\lambda_1+1)(\xi_1+9)+8m-\right.$$
$$\left.-\sqrt{[(\lambda_1+1)(\xi_1+9)+8m]^2-32\xi_1(\lambda_1+1)(\lambda_1+1+m)}\right\}, \tag{4.23}$$

then the relation (4.21) implies the second–order differential inequality (4.13).

The integration of this differential inequality leads to the estimate (2.23) with

$$\varkappa_2=\frac{1}{2}\left(-\alpha_2+\sqrt{\alpha_2^2+4\beta_2}\right) \tag{4.24}$$

and the proof is complete.

Let us consider now the class of elastic materials characterized by the relation (1.12). It is a straightforward task to see that the assumption (2.20) imposes upon the exponent p the following restrictions

$$\lambda_1+p(1+f)>0,\quad fp^2+(2-f)p+2\lambda_1>0. \tag{4.25}$$

Thus, we can see that the parameter p has to satisfy

$$p>-\frac{\lambda_1}{1+f} \tag{4.26}$$

for $(2-f)^2-8f\lambda_1<0$, or

$$p>\max\left\{-\frac{\lambda_1}{1+f},\frac{1}{2}\left[-2+f+\sqrt{(2-f)^2-8f\lambda_1}\right]\right\} \tag{4.27}$$

for $(2-f)^2-8f\lambda_1\geq 0$.

We note that, for this case, we have

$$\xi_1=\lambda_1+p(1+f),\quad \xi_2=\ fp^2+(2-f)p+2\lambda_1,\quad m=\frac{3}{2}p+\frac{1}{2}fp(p-1) \tag{4.28}$$

and so, by means of the relations (4.19), (4.23) and (4.24), we can obtain the explicit dependence of the decay rate $\varkappa_2$ with respect to the parameters p and f characterizing the considered class of elastic materials.

5. CONCLUDING REMARKS

The present paper establishes some decay bounds for a generalized biharmonic equation for an arch–like region when the characteristic coefficients are depending only on the polar angle or on the polar distance. It should be noted that the smoothly varying inhomogeneous elastic materials provide a model for technologically important functionally graded materials. These materials have received considerable attention in recent literature on solid mechanics. Our main results described by the theorems of the Section 2 offer explicit upper decay bounds for the measures $F_1(\theta)$ and $F_2(\theta)$ and these estimates can be regarded as a version of Saint Venant's principle for a curvilinear strip in the context of two dimensional inhomogeneous and isotropic elastostatics. It is worth to outline that from our results described by the estimates (2.12) and (2.23) we can rediscover the results established in [10] for a homogeneous arch–like region.

6. APPENDIX A

Denoting with u_r and u_θ the radial and transversal components of the displacement vector in a plane polar reference frame, the geometrical measures of deformation are

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{1}{r}u_\theta\right), \quad e_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right). \tag{6.1}$$

By eliminating u_r and u_θ in the above relation we obtain the Saint Venant compatibility condition

$$r\frac{\partial^2}{\partial r^2}\left(re_{\theta\theta}\right) + \left(\frac{\partial^2}{\partial \theta^2} - r\frac{\partial}{\partial r}\right)e_{rr} - 2\frac{\partial^2}{\partial r \partial \theta}\left(re_{r\theta}\right) = 0. \tag{6.2}$$

The constitutive equations for a plane strain state in an isotropic and inhomogeneous body are

$$\begin{aligned} \tau_{rr} &= \lambda\left(e_{rr} + e_{\theta\theta}\right) + 2\mu e_{rr}, \\ \tau_{\theta\theta} &= \lambda\left(e_{rr} + e_{\theta\theta}\right) + 2\mu e_{\theta\theta}, \\ \tau_{r\theta} &= 2\mu e_{r\theta}, \end{aligned} \tag{6.3}$$

or

$$e_{rr} = \varepsilon\tau_{rr} - \bar{\varepsilon}\tau_{\theta\theta}, \quad e_{\theta\theta} = -\bar{\varepsilon}\tau_{rr} + \varepsilon\tau_{\theta\theta}, \quad e_{r\theta} = \left(\varepsilon + \bar{\varepsilon}\right)\tau_{r\theta}, \tag{6.4}$$

where

$$\varepsilon = \frac{\lambda + 2\mu}{4\mu\left(\lambda + \mu\right)}, \quad \bar{\varepsilon} = \frac{\lambda}{4\mu\left(\lambda + \mu\right)}. \tag{6.5}$$

In the above relations τ_{rr}, $\tau_{\theta\theta}$ and $\tau_{r\theta}$ are the components of the plane stress in plane polars and λ and μ are the elastic Lamé coefficients depending on the variables r and θ.

The state of plane stress is represented in terms of the Airy stress function ϕ as

$$\tau_{rr} = \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}, \quad \tau_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}, \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) \tag{6.6}$$

and so the equilibrium equations are identically satisfied. If we substitute the relation (6.6) into (6.4) and the result in the relation (6.2), then we get the following equation for the Airy stress function

$$
\begin{aligned}
0 \;=\;& \frac{\partial^2}{\partial r^2}\left(\varepsilon r \frac{\partial^2 \phi}{\partial r^2}\right)+\frac{\partial^2}{\partial \theta^2}\left(\frac{\varepsilon}{r^3}\frac{\partial^2 \phi}{\partial \theta^2}\right)+2\frac{\partial^2}{\partial r \partial \theta}\left(\frac{\varepsilon}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}\right)-\frac{\partial}{\partial r}\left(\frac{\varepsilon}{r}\frac{\partial \phi}{\partial r}\right)+\frac{4\varepsilon}{r^3}\frac{\partial^2 \phi}{\partial \theta^2}- \\
&-\left(\frac{3}{r^2}\frac{\partial \varepsilon}{\partial r}+\frac{1}{r}\frac{\partial^2 \bar{\varepsilon}}{\partial r^2}\right)\frac{\partial^2 \phi}{\partial \theta^2}+2\left[\frac{1}{r^2}\frac{\partial \varepsilon}{\partial \theta}+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \bar{\varepsilon}}{\partial \theta}\right)\right]\frac{\partial^2 \phi}{\partial r \partial \theta}+ \\
&+\left(-\frac{1}{r}\frac{\partial^2 \bar{\varepsilon}}{\partial \theta^2}+\frac{\partial \bar{\varepsilon}}{\partial r}\right)\frac{\partial^2 \phi}{\partial r^2}-\frac{2}{r}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial \theta}\left(\varepsilon+\bar{\varepsilon}\right)\right]\frac{\partial \phi}{\partial \theta}+ \\
&+\left(\frac{1}{r^2}\frac{\partial^2 \varepsilon}{\partial \theta^2}-\frac{\partial^2 \bar{\varepsilon}}{\partial r^2}\right)\frac{\partial \phi}{\partial r}.
\end{aligned} \tag{6.7}
$$

7. APPENDIX B

Let $C_0^1([a,b])$ be the class of real–valued functions, each of which is continuously differentiable on the interval $[a,b]$ and vanishes at $r=a$ and $r=b$. Then for any function $\psi(r) \in C_0^1([a,b])$, one has (see [11])

$$\int_a^b \frac{1}{r}\left(\frac{d\psi}{dr}\right)^2 dr \geq \left[1+\frac{\pi^2}{\left(\ln \frac{b}{a}\right)^2}\right]\int_a^b \frac{1}{r^3}\psi^2 dr. \tag{7.1a}$$

Let $C_0^2([a,b])$ be the class of real–valued functions, each of which is twice continuously differentiable on the interval $[a,b]$ and vanishes together with its first derivative at $r=a$ and $r=b$. Then for any function $\psi(r) \in C_0^2([a,b])$, one has (see [11])

$$\int_a^b r\left(\frac{d^2\psi}{dr^2}\right)^2 dr \geq k\int_a^b \frac{1}{r}\left(\frac{d\psi}{dr}\right)^2 dr, \tag{7.2a}$$

where

$$k \geq \frac{\pi^2}{\left(\ln \frac{b}{a}\right)^2}. \tag{7.3}$$

8. APPENDIX C

Let $\varepsilon(r) > 0$ be a function continuous differentiable on the interval $[a,b]$. Any smooth function $\psi(r)$, $r \in [a,b]$, such that $\psi(a)=\psi(b)=0$, satisfies

$$\int_a^b \frac{\varepsilon}{r}\left(\frac{d\psi}{dr}\right)^2 dr \geq (\lambda_1+1)\int_a^b \frac{\varepsilon}{r^3}\psi^2 dr, \tag{8.1}$$

where λ_1 is the lowest eigenvalue of

$$\frac{d}{dr}\left[\frac{\varepsilon}{r}\frac{d\psi}{dr}\right]+(\lambda+1)\frac{\varepsilon}{r^3}\psi=0, \qquad \psi(a)=\psi(b)=0, \tag{8.2}$$

or (via the transformation $u(t)=\frac{1}{r}\sqrt{\varepsilon}\psi$, $r=e^t$) of

$$\frac{d^2u}{dt^2}+\left[\lambda+\frac{2}{\sqrt{\varepsilon}}\frac{d}{dt}\left(\sqrt{\varepsilon}\right)-\frac{1}{\sqrt{\varepsilon}}\frac{d^2}{dt^2}\left(\sqrt{\varepsilon}\right)\right]u=0, \quad u(\ln a)=u(\ln b)=0. \tag{8.3}$$

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