The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 3, Issue 2, Article 19, pp. 1-8, 2006

A STABILITY OF THE G-TYPE FUNCTIONAL EQUATION

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Received 12 April, 2006; accepted 4 August, 2006; published 28 December, 2006.

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ABSTRACT. We will investigate the stability in the sense of Găvruţă for the G-type functional equation $f(\varphi(x)) = \Gamma(x)f(x) + \psi(x)$ and the stability in the sense of Ger for the functional equation of the form $f(\varphi(x)) = \Gamma(x)f(x)$. As a consequence, we obtain a stability results for G-function equation.

Key words and phrases: G-function, Gamma function, Stability, Hyers-Ulam stability.

2000 *Mathematics Subject Classification.* Primary 39B82, 39B72.

ISSN (electronic): 1449-5910

This work was supported by Kangnam University Research Grant in 2005.

1. INTRODUCTION

In 1940, the stability problem raised by S. M. Ulam [12] was solved by D. H. Hyers in [4]. The result of Hyers has been generalized to the unbounded case by Th. M. Rassias [11], and Rassias's result also has been extended by P. Găvruţă [2] and R. Ger [3].

The gamma function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \quad (x > 0)$$

is a solution of the gamma functional equation $g(x+1) = xg(x)$, whose stability is researched in papers ([5], [6], [7], [8], [9], [10]).

The G-function introduced by E. W. Barnes [1]

$$G(z) = (2\pi)^{\frac{z-1}{2}} e^{-\frac{z(z-1)}{2}} e^{-\gamma\frac{(z-1)^2}{2}} \prod_{k=1}^{\infty} \left[\left(1 + \frac{z-1}{k}\right)^k e^{1-z+\frac{(z-1)^2}{2k}}\right]$$

does satisfy the equation $G(x+1) = \Gamma(x)G(x)$ and $\Gamma(1) = G(1) = 1$, where γ is the Euler-Mascheroni's constant defined by $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{k} - \log n\right) \cong 0.577215664\cdots$.

The properties and the availability of G-function depend on those of the gamma function. Since the double gamma function Γ_2 is defined by the reciprocal of the G-function (see [1]), $\Gamma_2(x) = 1/G(x)$, and its functional equation can be written in the form $\Gamma_2(x+1) = \Gamma_2(x)/\Gamma(x)$. Therefore the stability problem for the G-function is equivalent to the stability for the reciprocal of the double gamma function.

In this paper, we will investigate the stability in the sense of Găvruţă and Ger for the functional equations

$$f(\varphi(x)) = \Gamma(x)f(x) + \psi(x), \tag{1.1}$$

$$f(\varphi(x)) = \Gamma(x)f(x), \tag{1.2}$$

$$f(x+1) = \Gamma(x)f(x), \tag{1.3}$$

where φ, ψ are given functions, while f is the unknown function. The equation (1.3) will be called the G-functional equation because its solution is the G-function.

In section 2, we will study the stability in the sense of Găvruţă for the functional equations (1.1), (1.2).

In section 3, we will consider the stability in the sense of Ger for the functional equations (1.2), (1.3).

Throughout this paper, let R, R_+ and R_* denote the set of real numbers, the set of all positive real numbers and the set of all nonnegative real numbers, respectively. Each positive real number δ is fixed, n and k are natural numbers. Let $\varphi : R_+ \to R_+$ be strictly increasing function with $\varphi^0(x) = x$ and $\varphi^n(x) = \varphi(\varphi^{n-1}(x))$ for all x, and let $\psi : R_+ \to R_*$ and $\varepsilon : R_+ \to R_*$ be some function.

2. STABILITY IN THE SENSE OF GĂVRUŢĂ FOR THE EQUATION (1.1)

In this section, we will investigate the stability in the sense of Găvruţă for the equation (1.1). Therefore we can obtain the stability in the sense of Găvruţă for the equation (1.2) and the Hyers-Ulam stability of equations (1.1) and (1.2) as corollaries.

Theorem 2.1. *Let ε be a given function such that*

$$\omega(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(\varphi^k(x))}{\prod_{j=0}^{k} |\Gamma(\varphi^j(x))|} < \infty \qquad \forall x \in R_+. \tag{2.1}$$

If a function $f : R_+ \longrightarrow R_+$ *satisfies the inequality*

$$|f(\varphi(x)) - \Gamma(x)f(x) - \psi(x)| \leq \varepsilon(x) \qquad \forall x \in R_+, \tag{2.2}$$

then there exists a unique solution $g : R_+ \longrightarrow R_+$ *of the equation* (1.1) *with*

$$|g(x) - f(x)| \leq \omega(x) \qquad \forall x \in R_+. \tag{2.3}$$

Proof. For any $x \in R_+$ and for every positive integer n, let $\omega_n : R_+ \longrightarrow R_*$ and $g_n : R_+ \longrightarrow R_+$ be the functions defined by

$$\omega_n(x) := \sum_{k=0}^{n-1} \frac{\varepsilon(\varphi^k(x))}{\prod_{j=0}^{k} |\Gamma(\varphi^j(x))|}$$

and

$$g_n(x) := \frac{f(\varphi^n(x))}{\prod_{j=0}^{n-1} \Gamma(\varphi^j(x))} - \sum_{k=0}^{n-1} \frac{\psi(\varphi^k(x))}{\prod_{j=0}^{k} \Gamma(\varphi^j(x))}$$

for all $x \in R_+$, respectively.

By (2.2), it follows that

$$\left|\frac{f(\varphi(x))}{\Gamma(x)} - f(x) - \frac{\psi(x)}{\Gamma(x)}\right| \leq \frac{\varepsilon(x)}{|\Gamma(x)|} \qquad \text{for all} \quad x \in R_+.$$

Substituting x by $\varphi^n(x)$ in this inequality, and then dividing both sides of the obtained inequality by $\prod_{j=0}^{n-1} |\Gamma(\varphi^j(x))|$, we get

$$|g_{n+1}(x) - g_n(x)| = \frac{\varepsilon(\varphi^n(x))}{\prod_{j=0}^{n} |\Gamma(\varphi^j(x))|}. \tag{2.4}$$

By induction on n we prove that

$$|g_n(x) - f(x)| \leq \omega_n(x) \tag{2.5}$$

for all $x \in R_+$, and for all positive integers n. For the case $n = 1$, the inequality (2.5) is an immediate consequence of (2.2).

Assume that the inequality (2.5) holds true for some n. Then we obtain the inequality for $n + 1$ by (2.4) in the following way:

$$\begin{aligned} |g_{n+1}(x) - f(x)| &\leq |g_{n+1}(x) - g_n(x)| + |g_n(x) - f(x)| \\ &\leq \frac{\varepsilon(\varphi^n(x))}{\prod_{j=0}^{n} |\Gamma(\varphi^j(x))|} + \omega_n(x) \\ &= \omega_{n+1}(x). \end{aligned}$$

We claim that $\{g_n(x)\}$ is a Cauchy sequence. Indeed, by (2.4) and (2.1), we have for $n > m$ that

$$\begin{aligned} |g_n(x) - g_m(x)| &\leq \sum_{k=m}^{n-1} |g_{k+1}(x) - g_k(x)| \\ &\leq \sum_{k=m}^{n-1} \frac{\varepsilon(\varphi^k(x))}{\prod_{j=0}^{k} |\Gamma(\varphi^j(x))|} \longrightarrow 0 \end{aligned}$$

as $m \longrightarrow \infty$.

Hence, we can define a function $g : R_+ \longrightarrow R_+$ by

$$g(x) := \lim_{n\to\infty} g_n(x). \tag{2.6}$$

From the definition of g_n, we have $g_n(\varphi(x)) = \Gamma(x)g_{n+1}(x) + \psi(x)$, hence the function g satisfies (1.1).

We show from (2.5) that g satisfies the inequality (2.3) as follows:

$$|g(x) - f(x)| = \lim_{n\to\infty} |g_n(x) - f(x)| \le \lim_{n\to\infty} \omega_n(x) = \omega(x) \qquad \forall x \in R_+.$$

If $h : R_+ \longrightarrow R_+$ is another such function, which satisfies (1.1) and (2.3), then we have

$$\begin{aligned}
|g(x) - h(x)| &= |g(\varphi^n(x)) - h(\varphi^n(x))| \cdot \prod_{j=0}^{n-1} \frac{1}{|\Gamma(\varphi^j(x))|} \\
&\le 2\omega_n(\varphi^n(x)) \cdot \prod_{j=0}^{n-1} \frac{1}{|\Gamma(\varphi^j(x))|} \\
&= 2\Big(\sum_{k=0}^{\infty} \frac{\varepsilon(\varphi^{n+k}(x))}{\prod_{j=0}^{k} |\Gamma(\varphi^{n+j}(x))|}\Big) \cdot \prod_{j=0}^{n-1} \frac{1}{|\Gamma(\varphi^j(x))|} \\
&= 2\sum_{k=n}^{\infty} \frac{\varepsilon(\varphi^k(x))}{\prod_{j=0}^{k} |\Gamma(\varphi^j(x))|}
\end{aligned}$$

for all $x \in R_+$ and all positive integers n, which tends to zero as $n \to \infty$, since $\omega(x)$ is bounded. This implies the uniqueness of g. ∎

Corollary 2.2. *If a function $f : R_+ \longrightarrow R_+$ satisfies the inequality*

$$|f(\varphi(x)) - \Gamma(x)f(x) - \psi(x)| \le \delta \qquad \forall x \in R_+, \tag{2.7}$$

then there exists a unique solution $g : R_+ \longrightarrow R_+$ of the equation (1.1) *with*

$$|g(x) - f(x)| \le \delta\mu(x) \qquad \forall x \in R_+, \tag{2.8}$$

where the function $\mu(x) := \sum_{k=0}^{\infty} \prod_{j=0}^{k} \frac{1}{\Gamma(\varphi^j(x))}$ for all $x \in R_+$.

In particular, if $\varphi(x) > 2$ in the stability inequality (2.7)*, then there exists a unique solution $g : R_+ \longrightarrow R_+$ of the equation* (1.1) *with*

$$|g(x) - f(x)| \le \frac{\delta\Gamma(\varphi(x))}{\Gamma(x)\left(\Gamma(\varphi(x)) - 1\right)}. \tag{2.9}$$

Proof. Set $\varepsilon(x) = \delta$ in Theorem 2.1. The infinite series $\mu(x)$ satisfies the condition (2.1). Indeed, the sequence of partial sums $\{u_n(x)\}$ defined by

$$\mu_n(x) := \sum_{k=0}^{n} \prod_{j=0}^{k} \frac{1}{\Gamma(\varphi^j(x))}$$

is a Cauchy sequence with simple calculation.

In the case of $\varphi(x) > 2$, the defined function $\mu(x)$ implies

$$\begin{aligned}
\sum_{k=0}^{\infty} \prod_{j=0}^{k} \frac{1}{\Gamma(\varphi^j(x))} &\le \frac{1}{\Gamma(x)} \left(1 + \frac{1}{\Gamma(\varphi(x))} + \frac{1}{\Gamma(\varphi(x))^2} + \cdots\right) \\
&= \frac{\Gamma(\varphi(x))}{\Gamma(x)\left(\Gamma(\varphi(x)) - 1\right)}.
\end{aligned}$$

∎

Theorem 2.3 and Corollary 2.4 follow immediately from Theorem 2.1 and Corollary 2.2 with $\psi(x) = 0$.

Theorem 2.3. *Let the function* ε *satisfies the condition* (2.1). *If a function* $f : R_+ \longrightarrow R_+$ *satisfies the inequality*

$$|f(\varphi(x)) - \Gamma(x)f(x)| \leq \varepsilon(x) \qquad \forall x \in R_+,$$

then there exists a unique solution $g : R_+ \longrightarrow R_+$ *of the equation* (1.2) *satisfies the inequality* (2.3) *for all* $x \in R_+$.

Corollary 2.4. *If a function* $f : R_+ \longrightarrow R_+$ *satisfies the inequality*

$$|f(\varphi(x)) - \Gamma(x)f(x)| \leq \delta \qquad \forall x \in R_+, \tag{2.10}$$

then there exists a unique solution $g : R_+ \longrightarrow R_+$ *of the equation* (1.2) *satisfying* (2.8) *for all* $x \in R_+$.

In particular, if $\varphi(x) > 2$ *in the stability inequality* (2.10)*, then there exists a unique solution* $g : R_+ \longrightarrow R_+$ *of the equation* (1.2) *satisfying* (2.9)*.*

The case of $\varphi(x) = x + 1$ in Theorem 2.3 and Corollary 2.4 provide the stability in the sense of Găvruţă and the Hyers-Ulam stability for the G-functional equation (1.3), respectively. The latter is referred in paper [10].

Theorem 2.5. *Let the function* ε *satisfies the condition (2.1). If a function* $f : R_+ \longrightarrow R_+$ *satisfies the inequality*

$$|f(x+1) - \Gamma(x)f(x)| \leq \varepsilon(x) \qquad \forall x \in R_+,$$

then there exists a unique G-function $G : R_+ \longrightarrow R_+$ *such that*

$$|g(x) - f(x)| \leq \omega(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(x+k)}{\prod_{j=0}^{k} |\Gamma(x+j)|} < \infty \qquad \forall x \in R_+ \qquad \forall x \in R_+.$$

Corollary 2.6. *If a function* $f : R_+ \longrightarrow R_+$ *satisfies the inequality*

$$|f(x+1) - \Gamma(x)f(x)| \leq \delta \qquad \forall x \in R_+,$$

then there exists a unique G - function $G : R_+ \longrightarrow R_+$ *with*

$$|G(x) - f(x)| \leq \frac{\delta e}{\Gamma(x)}$$

for all $x \in R_+$.

Proof. Set $\varphi(x) = x+1$, $\psi(x) = 0$ in Corollary 2.2. Then the sequence of partial sums $\{u_n(x)\}$ defined by

$$\mu_n(x) := \sum_{k=0}^{n} \prod_{j=0}^{k} \frac{1}{\Gamma(x+j)} \leq \frac{\delta e}{\Gamma(x)}.$$

∎

3. STABILITY IN THE SENSE OF GER FOR THE EQUATION (1.2)

In this section, we will investigate the stability in the sense of Ger for the equation (1.2). Therefore we can obtain the stability in the sense of Ger for the G-functional equation (1.3) as a corollary.

Theorem 3.1. *Let a function* $f : R_+ \longrightarrow R_+$ *satisfies the inequality*

$$\left|\frac{f(\varphi(x))}{\Gamma(x)\ f(x)} - 1\right| \leq \varepsilon(x) \qquad \forall x \in R_+, \tag{3.1}$$

where $\varepsilon : R_+ \longrightarrow (0,1)$ *is a function such that*

$$\sum_{j=0}^{\infty} \varepsilon(\varphi^j(x)) < +\infty. \tag{3.2}$$

Then there exists a unique solution $g : R_+ \longrightarrow R_+$ *of the equation* (1.2) *with*

$$\alpha(x) \le \frac{g(x)}{f(x)} \le \beta(x), \tag{3.3}$$

where $\alpha(x) := \prod_{j=0}^{\infty}(1 - \varepsilon(\varphi^j(x)))$ *and* $\beta(x) := \prod_{j=0}^{\infty}(1 + \varepsilon(\varphi^j(x)))$ *for all* $x \in R_+$.

Proof. The condition (3.2) implies that $\prod_{j=0}^{\infty}(1 \pm \varepsilon(\varphi^j(x)))$ converges. Hence, we can define the functions α, β for all $x \in R_+$ such that $0 < \alpha(x) := \prod_{j=0}^{\infty}(1 - \varepsilon(\varphi^j(x))) < \prod_{j=0}^{\infty}(1 + \varepsilon(\varphi^j(x))) := \beta(x) < +\infty$, that is, these series are bounded.

For any $x \in R_+$ and for every positive integer n, we define

$$g_n(x) = \prod_{j=0}^{n-1} \frac{f(\varphi^n(x))}{\Gamma(\varphi^j(x))}. \tag{3.4}$$

For all positive integers m, n with $n > m$, we have

$$\frac{g_n(x)}{g_m(x)} = \frac{f(\varphi^{m+1}(x))}{\Gamma(\varphi^m(x))f(\varphi^m(x))} \cdot \frac{f(\varphi^{m+2}(x))}{\Gamma(\varphi^{m+1}(x))f(\varphi^{m+1}(x))} \cdots \frac{f(\varphi^n(x))}{\Gamma(\varphi^{n-1}(x))f(\varphi^{n-1}(x))}. \tag{3.5}$$

It also follows from (3.1) that

$$0 < 1 - \varepsilon(\varphi^j(x)) \le \frac{f(\varphi^{j+1}(x))}{\Gamma(\varphi^j(x))\; f(\varphi^j(x))} \le 1 + \varepsilon(\varphi^j(x)) \tag{3.6}$$

for all $x \in R_+$ and $j = 0, 1, 2, \cdots$. From (3.5) and (3.6), we get

$$\prod_{j=m}^{n-1}(1 - \varepsilon(\varphi^j(x))) \le \frac{g_n(x)}{g_m(x)} \le \prod_{j=m}^{n-1}(1 + \varepsilon(\varphi^j(x)))$$

or

$$\sum_{j=m}^{n-1} \log(1 - \varepsilon(\varphi^j(x))) \le \log g_n(x) - \log g_m(x) \le \sum_{j=m}^{n-1} \log(1 + \varepsilon(\varphi^j(x))).$$

Since $\sum_{j=0}^{\infty} \log(1 - \varepsilon(\varphi^j(x))) = \log \alpha(x)$ and $\sum_{j=0}^{\infty} \log(1 + \varepsilon(\varphi^j(x))) = \log \beta(x)$, it follows that $\lim_{m\to\infty} \sum_{j=m}^{\infty} \log(1 - \varepsilon(\varphi^j(x))) = \lim_{m\to\infty} \sum_{j=m}^{\infty} \log(1 + \varepsilon(\varphi^j(x))) = 0$ by boundedness of α, β. Hence, we note that $\{\log g_n(x)\}$ is a Cauchy sequence for all $x \in R_+$. It is reasonable to define a function $g : R_+ \to R_+$ by

$$g(x) = e^{L(x)} = \lim_{n\to\infty} g_n(x) \qquad \forall x \in R_+, \tag{3.7}$$

where $L(x) := \lim_{n\to\infty} \log g_n(x)$.

We get that

$$g(\varphi(x)) = \Gamma(x) g(x) \qquad \forall x \in R_+. \tag{3.8}$$

Since

$$\frac{g_n(x)}{f(x)} = \frac{f(\varphi(x))}{\Gamma(x)f(x)} \cdot \frac{f(x+2p)}{\Gamma(\varphi(x))f(\varphi(x))} \cdots \frac{f(\varphi^n(x))}{\Gamma(\varphi^{n-1}(x))\; f(\varphi^{n-1}(x))},$$

we get

$$\prod_{j=0}^{n-1}(1-\varepsilon(\varphi^j(x))) \leq \frac{g_n(x)}{f(x)} \leq \prod_{j=0}^{n-1}(1+\varepsilon(\varphi^j(x)))$$

for all $x \in R_+$. This inequality implies (3.3) with the definition of α, β as $n \longrightarrow \infty$.

Assume $h : R_+ \longrightarrow R_+$ is a solution of equation (3.8) which satisfies the inequality (3.3). By (3.8), we have

$$\frac{g(x)}{h(x)} = \frac{g(\varphi^n(x))}{h(\varphi^n(x))} = \frac{g(\varphi^n(x))}{f(\varphi^n(x))} \cdot \frac{f(\varphi^n(x))}{h(\varphi^n(x))}$$

for any $x \in R_+$ and for any natural number n.

Hence, we have

$$\frac{\alpha(\varphi^n(x))}{\beta(\varphi^n(x))} \leq \frac{g(x)}{h(x)} \leq \frac{\beta(\varphi^n(x))}{\alpha(\varphi^n(x))}$$

for any natural number n. By the boundedness of the series ε,

$$\alpha(\varphi^n(x)) = \prod_{j=n}^{\infty}(1-\varepsilon(\varphi^j(x))) \longrightarrow 1$$

as $n \longrightarrow \infty$. Similarly $\beta(\varphi^n(x)) \longrightarrow 1$ as $n \longrightarrow \infty$.

Therefore, it is obvious that $h(x) \equiv g(x)$. ∎

From the proof of Theorem 3.1, we can see that the assumption (3.2) is a weak condition for the convergence of α and β. The special case $\varphi(x) = x + 1$ has been considered in [10].

Corollary 3.2. *Let a function f satisfies inequality* (3.1), *in which $\varepsilon : R_+ \longrightarrow (0, 1)$ is a function such that*

$$\alpha(x) := \prod_{j=0}^{\infty}(1-\varepsilon(\varphi^j(x))) \quad \text{and} \quad \beta(x) := \prod_{j=0}^{\infty}(1+\varepsilon(\varphi^j(x)))$$

are bounded for all $x \in R_+$. Then there exists a unique solution $g : R_+ \longrightarrow R_+$ of the equation (1.2) *satisfying* (3.3) *for all $x \in R_+$.*

Corollary 3.3. *Let $\theta > 0$ be given. If a mapping $f : R_+ \longrightarrow R_+$ satisfies the inequality*

$$\left|\frac{f(x+1)}{\Gamma(x)f(x)} - 1\right| \leq \frac{\delta}{x^{1+\theta}} \qquad \forall x \in R_+,$$

then there exists a unique solution $g : R_+ \longrightarrow R_+$ of the gamma functional equation (1.3) *such that for any $x > \delta^{\frac{1}{1+\theta}}$ the following inequality is satisfied*

$$\alpha(x) \leq \frac{g(x)}{f(x)} \leq \beta(x),$$

where $\alpha(x) := \prod_{j=0}^{\infty}(1 - \frac{\delta}{(x+j)^{1+\theta}})$ and $\beta(x) := \prod_{j=0}^{\infty}(1 + \frac{\delta}{(x+j)^{1+\theta}})$.

Proof. Applying Theorem 3.1 with $\varphi(x) = x+1$, $\varepsilon(x) = \frac{\delta}{x^{1+\theta}}$, if $x > \delta^{\frac{1}{1+\theta}}$, then $\sum_{j=0}^{\infty} \frac{\delta}{(x+j)^{1+\theta}}$ converges by the p-series method. Hence, we get the desired result. ∎

4. EXAMPLES

We apply the result of Theorem 3.1 with $p = 1$.

Ex 1. $\varepsilon(1+i) = \frac{1}{(1+i)^q}$, for $q > 1$. Note that the series $\sum_{k=0}^{\infty} \frac{1}{k^q}$ in the case $q > 1$ converges.

Ex 2. $\varepsilon(1+i) = \frac{1}{(1+i)!}$. Note that $\sum_{i=0}^{\infty} \frac{1}{(1+i)!} = e - 1$.

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