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# A STRENGTHENED HARDY-HILBERT'S TYPE INEQUALITY

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ABSTRACT. By using the improved Euler-Maclaurin's summation formula and estimating the weight coefficient, we give a new strengthened version of the more accurate Hardy-Hilbert's type inequality. As applications, a strengthened version of the equivalent form is considered.

*Key words and phrases:* Hardy-Hilbert's type inequality, Weight coefficient, Hölder's inequality.

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## 1. INTRODUCTION

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n$,$b_n \geq 0$, such that $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(\frac{m}{n})a_m b_n}{m-n} < \left[\frac{\pi}{\sin(\frac{\pi}{p})}\right]^2 \left(\sum_{n=1}^{\infty} a_n^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}, \tag{1.1}$$

where the constant factor $\left[\frac{\pi}{\sin(\frac{\pi}{p})}\right]^2$ is the best possible [1]. Inequality (1.1) is called Hardy-Hilbert's type inequality, which is important in analysis and its applications [2]. In recent years, Pachpatte et al. [3]-[9] gave some extensions and improvements of (1.1). In 2005, by introducing a parameter $\lambda$, Yang [10] gave a extension of (1.1) as: If $0 < \lambda \leq \min\{p, q\}$, such that $0 < \sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln\left(\frac{m}{n}\right) a_m b_n}{m^\lambda - n^\lambda} < \left[\frac{\pi}{\lambda\sin\left(\frac{\pi}{p}\right)}\right]^2 \left[\sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)} b_n^q\right]^{\frac{1}{q}}, \tag{1.2}$$

where the constant factor $\left[\frac{\pi}{\lambda\sin\left(\frac{\pi}{p}\right)}\right]^2$ is the best possible. For $\lambda = 1$, inequality (1.2) reduces to (1.1). By introducing a parameter $\alpha$, Yang [11] gave a more accurate inequality of (1.1) as: If $\frac{1}{2} \leq \alpha \leq 1$, $a_n$,$b_n \geq 0$ satisfying $0 < \sum_{n=0}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=0}^{\infty} b_n^q < \infty$, then we have two equivalent inequalities as follows:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln\left(\frac{m+\alpha}{n+\alpha}\right) a_m b_n}{m-n} < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^2 \left(\sum_{n=0}^{\infty} a_n^p\right)^{\frac{1}{p}} \left(\sum_{n=0}^{\infty} b_n^q\right)^{\frac{1}{q}}; \tag{1.3}$$

$$\sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty}\frac{\ln\left(\frac{m+\alpha}{n+\alpha}\right) a_m}{m-n}\right]^p < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^{2p} \sum_{n=0}^{\infty} a_n^p, \tag{1.4}$$

where the constant factor $\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^2$ and $\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^{2p}$ are the best possible. For $\alpha = 1$, inequality (1.3) reduces to the equivalent form of (1.2).

This paper deals with some strengthened versions of (1.3) and (1.4) by estimating the weight coefficient. For this, we introduce some lemmas.

## 2. SOME LEMMAS

First, we need the following formula (see [1]):

$$\int_0^{\infty}\frac{\ln u}{u-1}u^{-\frac{1}{s}}du = \left[\frac{\pi}{\sin\left(\frac{\pi}{s}\right)}\right]^2 \quad (s > 1). \tag{2.1}$$

**Lemma 2.1.** (see [12]) *If* $f(x) > 0, f^{(2r-1)}(x) < 0, f^{(2r)}(x) \geq 0, x \in [0,\infty)(r = 1, 2), f^{(r)}(\infty) = 0(r = 0, 1, 2, 3, 4)$ *and* $\int_0^{\infty} f(x)dx < \infty$*, then*

$$\sum_{n=0}^{\infty} f(m) \leq \int_0^{\infty} f(x)dx + \frac{1}{2}f(0) - \frac{1}{12}f'(0). \tag{2.2}$$

**Lemma 2.2.** *For* $r > 1, \frac{1}{2} \le \alpha \le 1, n \in N_0$, *setting the weight coefficient* $W_\alpha(r,n)$ *as follows:*

$$W_\alpha(r,n) = \sum_{m=0}^{\infty} \frac{\ln\left(\frac{m+\alpha}{n+\alpha}\right)}{m-n}\left(\frac{n+\alpha}{m+\alpha}\right)^{\frac{1}{r}}, \tag{2.3}$$

*then we have*

$$W_\alpha(r,n) < \left[\frac{\pi}{\sin(\frac{\pi}{r})}\right]^2 - \frac{2}{3(r-1)} \cdot \frac{\ln(n+1)}{(2n+1)^{1-\frac{1}{r}}}. \tag{2.4}$$

*Proof.* Define the functions $g(u)$ and $f(x)$ as: $g(u) := \frac{\ln u}{u-1}, u \in (0,\infty)(g(1) := 1)$;

$$f(x) := g\left(\frac{x+\alpha}{n+\alpha}\right)\left(\frac{x+\alpha}{n+\alpha}\right)^{-\frac{1}{r}}, x \in (-\alpha,\infty)\ \left(\alpha \ge \frac{1}{2}, n \in N_0\right).$$

Then $f(x)$ satisfies the condition of (2.2)(see [11, Lemma 2.2]), we find

$$f(0) = g\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{-\frac{1}{r}};$$

$$f'(x) = \frac{1}{n+\alpha}g'\left(\frac{x+\alpha}{n+\alpha}\right)\left(\frac{x+\alpha}{n+\alpha}\right)^{-\frac{1}{r}} - \frac{1}{r(n+\alpha)}g\left(\frac{x+\alpha}{n+\alpha}\right)\left(\frac{x+\alpha}{n+\alpha}\right)^{-\frac{1}{r}-1},$$

$$f'(0) = \frac{1}{n+\alpha}g'\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{-\frac{1}{r}} - \frac{1}{r(n+\alpha)}g\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{-\frac{1}{r}-1};$$

$$\begin{aligned}\int_0^\infty f(x)dx &= \int_0^\infty g\left(\frac{x+\alpha}{n+\alpha}\right)\left(\frac{x+\alpha}{n+\alpha}\right)^{-\frac{1}{r}}dx \\ &= \int_{\frac{\alpha}{n+\alpha}}^\infty g(u)u^{-\frac{1}{r}}(n+\alpha)du \\ &= (n+\alpha)\left[\int_0^\infty g(u)u^{-\frac{1}{r}}du - \int_0^{\frac{\alpha}{n+\alpha}} g(u)u^{-\frac{1}{r}}du\right] \\ &= (n+\alpha)\left\{\left[\frac{\pi}{\sin\left(\frac{\pi}{r}\right)}\right]^2 - \int_0^{\frac{\alpha}{n+\alpha}} g(u)u^{-\frac{1}{r}}du\right\}.\end{aligned}$$

Integration by parts, since $g''(u) \ge 0$ (see [11, Lemma 2.2], ) we obtain

$$\begin{aligned}&\int_0^{\frac{\alpha}{n+\alpha}} g(u)u^{-\frac{1}{r}}du \\ =\ & \frac{r}{r-1}\int_0^{\frac{\alpha}{n+\alpha}} g(u)du^{1-\frac{1}{r}} \\ =\ & \frac{r}{r-1}g(u)u^{1-\frac{1}{r}}\Big|_0^{\frac{\alpha}{n+\alpha}} - \frac{r}{r-1}\int_0^{\frac{\alpha}{n+\alpha}} g'(u)u^{1-\frac{1}{r}}du \\ =\ & \frac{r}{r-1}g\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{1-\frac{1}{r}} - \frac{r^2}{(r-1)(2r-1)}\int_0^{\frac{\alpha}{n+\alpha}} g'(u)du^{2-\frac{1}{r}} \\ \ge\ & \frac{r}{r-1}g\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{1-\frac{1}{r}} - \frac{r^2}{(r-1)(2r-1)}g'\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{2-\frac{1}{r}}.\end{aligned}$$

Hence by (2.1), we have

$$\begin{aligned}
W_\alpha(r,n) &= \sum_{m=0}^{\infty}\frac{\ln\left(\frac{m+\alpha}{n+\alpha}\right)}{m-n}\left(\frac{n+\alpha}{m+\alpha}\right)^{\frac{1}{r}} = \frac{1}{n+\alpha}\sum_{m=0}^{\infty}f(m)\\
&\le \frac{1}{n+\alpha}\left[\int_0^\infty f(x)dx+\frac{1}{2}f(0)-\frac{1}{12}f'(0)\right]\\
&< \left[\frac{\pi}{\sin\left(\frac{\pi}{r}\right)}\right]^2-\frac{r}{r-1}g\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{1-\frac{1}{r}}\\
&\quad+\frac{r^2}{(r-1)(2r-1)}g'\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{2-\frac{1}{r}}\\
&\quad+\frac{1}{2\,(n+\alpha)}g\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{-\frac{1}{r}}\\
&\quad-\frac{1}{12\,(n+\alpha)^2}g'\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{-\frac{1}{r}}\\
&\quad-\frac{1}{12r\alpha(n+\alpha)}g\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{-\frac{1}{r}}\\
&= \left[\frac{\pi}{\sin\left(\frac{\pi}{r}\right)}\right]^2-g\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{1-\frac{1}{r}}\left(\frac{r}{r-1}-\frac{1}{2\alpha}-\frac{1}{12r\alpha^2}\right)\\
&\quad+g'\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{1-\frac{1}{r}}\left[\frac{r^2\alpha}{(r-1)(2r-1)(n+\alpha)}-\frac{1}{12\alpha(n+\alpha)}\right]\\
&= \left[\frac{\pi}{\sin\left(\frac{\pi}{r}\right)}\right]^2-\left(\frac{\alpha}{n+\alpha}\right)^{1-\frac{1}{r}}\left\{\left(\frac{r}{r-1}-\frac{1}{2\alpha}-\frac{1}{12r\alpha^2}\right)g\left(\frac{\alpha}{n+\alpha}\right)\right.\\
&\quad\left.+\frac{1}{n+\alpha}\left[\frac{r^2\alpha}{(r-1)(2r-1)}-\frac{1}{12\alpha}\right]\left(-g'\left(\frac{\alpha}{n+\alpha}\right)\right)\right\}.
\end{aligned}$$

Since $r>1$, $\frac{1}{2}\le\alpha\le 1$, $g'(u)\le 0\,(u\in(0,\infty))$(see [11, Lemma 2.2], ), we have

$$\left(\frac{\alpha}{n+\alpha}\right)^{1-\frac{1}{r}}\ge\left(\frac{\frac{1}{2}}{n+\frac{1}{2}}\right)^{1-\frac{1}{r}}=\frac{1}{(2n+1)^{1-\frac{1}{r}}},$$

$$g\left(\frac{\alpha}{n+\alpha}\right)\ge g\left(\frac{1}{n+1}\right)=\frac{n+1}{n}\cdot\ln(n+1)\ge\ln(n+1),$$

$$\frac{r}{r-1}-\frac{1}{2\alpha}-\frac{1}{12r\alpha^2}\ge 1+\frac{1}{r-1}-1-\frac{1}{3r}=\frac{2r+1}{3r(r-1)}>\frac{2}{3(r-1)},$$

$$\frac{r^2\alpha}{(r-1)(2r-1)}-\frac{1}{12\alpha}=\frac{2r^2\left(6\alpha^2-1\right)+3r-1}{12(2r-1)(r-1)\alpha}>0,$$

then, we find

$$W_\alpha(r,n)<\left[\frac{\pi}{\sin(\frac{\pi}{r})}\right]^2-\frac{2}{3(r-1)}\cdot\frac{\ln(n+1)}{(2n+1)^{1-\frac{1}{r}}}.$$

This proves the lemma. ∎

## 3. MAIN RESULTS AND APPLICATIONS

**Theorem 3.1.** *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{2} \le \alpha \le 1$, $a_n, b_n \ge 0$, *such that* $0 < \sum_{n=0}^{\infty} a_n^p < \infty$ *and* $0 < \sum_{n=0}^{\infty} b_n^q < \infty$, *then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\ln\left(\frac{m+\alpha}{n+\alpha}\right) a_m b_n}{m-n} < \left\{\sum_{n=0}^{\infty}\left[\left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right)^2 - \frac{2(p-1)\ln(n+1)}{3(2n+1)^{\frac{1}{p}}} a_n^p\right]\right\}^{\frac{1}{p}}$$

$$\times\left\{\sum_{n=0}^{\infty}\left[\left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right)^2 - \frac{2(q-1)\ln(n+1)}{3(2n+1)^{\frac{1}{q}}} b_n^q\right]\right\}^{\frac{1}{q}}. \tag{3.1}$$

*Proof.* By Hölder's inequality with weight (see [12]) and (2.3), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\ln\left(\frac{m+\alpha}{n+\alpha}\right) a_m b_n}{m-n} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\ln\left(\frac{m+\alpha}{n+\alpha}\right)}{m-n}\left[\left(\frac{m+\alpha}{n+\alpha}\right)^{\frac{1}{pq}} a_m\right]\left[\left(\frac{n+\alpha}{m+\alpha}\right)^{\frac{1}{pq}} b_n\right] \\
&\le \left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\ln\left(\frac{m+\alpha}{n+\alpha}\right)}{m-n}\left(\frac{m+\alpha}{n+\alpha}\right)^{\frac{1}{q}} a_m^p\right]^{\frac{1}{p}} \\
&\times \left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\ln\left(\frac{m+\alpha}{n+\alpha}\right)}{m-n}\left(\frac{n+\alpha}{m+\alpha}\right)^{\frac{1}{p}} b_n^q\right]^{\frac{1}{q}} \\
&= \left[\sum_{m=0}^{\infty} W_\alpha(q,m) a_m^p\right]^{\frac{1}{p}} \times \left[\sum_{n=0}^{\infty} W_\alpha(p,n) b_n^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Since $\sin\left(\frac{\pi}{q}\right) = \sin\left(\frac{\pi}{p}\right)$ and $(p-1)(q-1) = 1$, by (2.4) (for $r = p, q$), we have (3.1). This proves the theorem. ∎

**Theorem 3.2.** *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{2} \le \alpha \le 1$, $a_n \ge 0$, $0 < \sum_{n=0}^{\infty} a_n^p < \infty$, *then*

$$\sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty} \frac{\ln\left(\frac{m+\alpha}{n+\alpha}\right) a_m}{m-n}\right]^p \tag{3.2}$$

$$< \left[\frac{\pi}{\sin(\frac{\pi}{p})}\right]^{2p-2} \sum_{n=0}^{\infty}\left[\left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right)^2 - \frac{2(p-1)\ln(n+1)}{3(2n+1)^{\frac{1}{p}}}\right] a_n^p.$$

*Proof.* By Hölder's inequality and (2.3), we obtain

$$\begin{aligned}
\sum_{m=0}^{\infty} \frac{\ln\left(\frac{m+\alpha}{n+\alpha}\right) a_m}{m-n} &= \sum_{m=0}^{\infty} \frac{\ln\left(\frac{m+\alpha}{n+\alpha}\right)}{m-n}\left[\left(\frac{m+\alpha}{n+\alpha}\right)^{\frac{1}{pq}} a_m\right]\left[\left(\frac{n+\alpha}{m+\alpha}\right)^{\frac{1}{pq}}\right] \\
&\le \left[\sum_{m=0}^{\infty} \frac{\ln\left(\frac{m+\alpha}{n+\alpha}\right)}{m-n}\left(\frac{m+\alpha}{n+\alpha}\right)^{\frac{1}{q}} a_m^p\right]^{\frac{1}{p}} \cdot \left[\sum_{m=0}^{\infty} \frac{\ln\left(\frac{m+\alpha}{n+\alpha}\right)}{m-n}\left(\frac{n+\alpha}{m+\alpha}\right)^{\frac{1}{p}}\right]^{\frac{1}{q}} \\
&= \left[\sum_{m=0}^{\infty} \frac{\ln\left(\frac{m+\alpha}{n+\alpha}\right)}{m-n}\left(\frac{m+\alpha}{n+\alpha}\right)^{\frac{1}{q}} a_m^p\right]^{\frac{1}{p}} \cdot \left[W_\alpha(p,n)\right]^{\frac{1}{q}}.
\end{aligned}$$

Since by (2.4), we have $W_\alpha(p,n) \leq \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^2$, then we find

$$\begin{aligned}\sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty}\frac{\ln\left(\frac{m+\alpha}{n+\alpha}\right)a_m}{m-n}\right]^p &\leq \left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln\left(\frac{m+\alpha}{n+\alpha}\right)}{m-n}\left(\frac{m+\alpha}{n+\alpha}\right)^{\frac{1}{q}}a_m^p\right]\cdot[W_\alpha(p,n)]^{\frac{p}{q}} \\ &\leq \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^{2p-2}\sum_{m=0}^{\infty}\left[\sum_{n=0}^{\infty}\frac{\ln\left(\frac{m+\alpha}{n+\alpha}\right)}{m-n}\left(\frac{m+\alpha}{n+\alpha}\right)^{\frac{1}{q}}\right]a_m^p \\ &= \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^{2p-2}\sum_{m=0}^{\infty}[W_\alpha(q,m)]\,a_m^p.\end{aligned}$$

Hence by (2.4) (for $r = q$), we have (3.2). The theorem is proved. ∎

**Remark 3.1.**

(a) Inequality (3.1) is strengthened version of (1.3), and inequality (3.2) is a strengthened version of (1.4).

(b) For $\alpha = \frac{1}{2}, p = q = 2$, by (3.1) and (3.2), we have the following new strengthened Hilbert's type inequalities:

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln\left(\frac{2m+1}{2n+1}\right)a_m b_n}{m-n} \\ &< \left\{\sum_{n=0}^{\infty}\left[\pi^2-\frac{2\ln(n+1)}{3(n+1)^{\frac{1}{2}}}a_n^2\right]\sum_{n=0}^{\infty}\left[\pi^2-\frac{2\ln(n+1)}{3(2n+1)^{\frac{1}{2}}}b_n^2\right]\right\}^{\frac{1}{2}};\end{aligned} \tag{3.3}$$

$$\sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty}\frac{\ln\left(\frac{2m+1}{2n+1}\right)a_m}{m-n}\right]^p < \pi^2\sum_{n=0}^{\infty}\left[\pi^2-\frac{2\ln(n+1)}{3(2n+1)^{\frac{1}{2}}}\right]a_n^2. \tag{3.4}$$

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