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ON THE FEKETE-SZEGÖ INEQUALITY FOR SOME SUBCLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT. In this present investigation, the authors obtain Fekete-Szegö's inequality for certain normalized analytic functions $f(z)$ defined on the open unit disk for which $1 - \frac{f(z)f''(z)}{f'(z)^2}$ lie in a region starlike with respect to 1 and symmetric with respect to the real axis. Also certain applications of the main result for a class of functions defined by convolution are given. As a special case of this result, Fekete-Szegö's inequality for a class of functions defined through fractional derivatives is also obtained.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all *analytic* functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \Delta := \{z \in \mathbb{C} | \, |z| < 1\}) \tag{1.1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \quad (z \in \Delta)$$

and $C(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), \quad (z \in \Delta),$$

where $\prec$ denotes the subordination between analytic functions. These classes were introduced and studied by Ma and Minda [4]. They have obtained the Fekete-Szegö inequality for the functions in the class $C(\phi)$. Since $f \in C(\phi)$ if and only if $zf'(z) \in S^*(\phi)$, we get the Fekete-Szegö inequality for functions in the class $S^*(\phi)$. Many authors have studied the Fekete-Szegö inequality for different subclasses of analytic functions [1, 2, 3, 7, 8, 10, 11]. For a brief history of Fekete-Szegö problem for the class of starlike, convex and close-to-convex functions, see the recent paper by Srivastava *et al.* [12].

In the present paper, we obtain the Fekete-Szegö inequality for functions in a more general class $M(\phi)$ of functions which we define below. Also we give applications of our results to certain functions defined through convolution (or Hadamard product) and in particular we consider a class $M^{\lambda}(\phi)$ of functions defined by fractional derivatives.

Definition 1.1. Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the unit disk Δ onto a region in the right half plane which is symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in \mathcal{A}$ is in the class $M(\phi)$ if

$$1 - \frac{f(z)f''(z)}{f'(z)^2}. \prec \phi(z)$$

For fixed $g \in \mathcal{A}$, we define the class $M^g(\phi)$ to be the class of functions $f \in \mathcal{A}$ for which $(f * g) \in M(\phi)$.

When $\phi(z) = \frac{1+Az}{1+Bz}$, $(-1 \le B < A \le 1)$, we denote the class $M(\phi)$ by $M[A, B]$.

To prove our main result, we need the following:

Lemma 1.1 ([4]). *If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is an analytic function with positive real part in Δ, then*

$$|c_2 - vc_1^2| \le \begin{cases} -4v + 2 & \text{if } v \le 0, \\ 2 & \text{if } 0 \le v \le 1, \\ 4v - 2 & \text{if } v \ge 1. \end{cases}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p_1(z)$ is $(1 + z)/(1 - z)$ or one of its rotations. If $0 < v < 1$, then the equality holds if and only if $p_1(z)$ is $(1 + z^2)/(1 - z^2)$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z} \quad (0 \le \lambda \le 1)$$

or one of its rotations. If $v = 1$, the equality holds if and only if p_1 is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$.

Also the above upper bound is sharp, and it can be improved as follows when $0 < v < 1$:

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2 \quad (0 < v \leq 1/2)$$

and

$$|c_2 - vc_1^2| + (1-v)|c_1|^2 \leq 2 \quad (1/2 < v \leq 1).$$

2. FEKETE-SZEGÖ PROBLEM

Our main result is the following:

Theorem 2.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If f given by (1.1) belongs to $M_\alpha(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} -\dfrac{B_2}{6} + \dfrac{1-\mu}{4} B_1^2 & \text{if} \quad \mu \leq \sigma_1, \\ \dfrac{B_1}{6} & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \dfrac{B_2}{6} - \dfrac{1-\mu}{4} B_1^2 & \text{if} \quad \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 := 1 - \frac{2}{3B_1^2}\left(B_1 + B_2\right),$$

and

$$\sigma_2 := 1 - \frac{2}{3B_1^2}\left(B_2 - B_1\right).$$

The result is sharp.

Proof. For $f(z) \in M(\phi)$, let

$$p(z) := 1 - \frac{f(z) f''(z)}{f'(z)^2} = 1 + b_1 z + b_2 z^2 + \cdots . \tag{2.1}$$

From (2.1), we obtain

$$-2a_2 = b_1 \quad \text{and} \quad 6\left(a_2^2 - a_3\right) = b_2.$$

Since $\phi(z)$ is univalent and $p \prec \phi$, the function

$$p_1(z) = \frac{1 + \phi^{-1}(p(z))}{1 + \phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \cdots$$

is analytic and has positive real part in Δ. Also we have

$$p(z) = \phi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) \tag{2.2}$$

and from this equation (2.2), we obtain

$$b_1 = \frac{1}{2} B_1 c_1$$

and

$$b_2 = \frac{1}{2} B_1 (c_2 - \frac{1}{2} c_1^2) + \frac{1}{4} B_2 c_1^2.$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{-B_1}{12}\left\{c_2 - vc_1^2\right\} \tag{2.3}$$

where

$$v := \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \frac{3(1-\mu)}{2}B_1\right].$$

Our result now follows by an application of Lemma 1.1. To show that the bounds are sharp, we define the functions K^{ϕ_n} $(n = 2, 3, \ldots)$ by

$$1 - \frac{K^{\phi_n}(z)(K^{\phi_n})''(z)}{(K^{\phi_n})'(z)^2} = \phi(z^{n-1}), \quad K^{\phi_n}(0) = 0 = [K^{\phi_n}]'(0) - 1$$

and the function F_α^λ and G_α^λ $(0 \le \lambda \le 1)$ by

$$1 - \frac{F^\lambda(z)(F^\lambda(z))''(z)}{(F^\lambda(z))'^2} = \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right), \quad F^\lambda(0) = 0 = (F^\lambda)'(0) - 1$$

and

$$1 - \frac{G^\lambda(z)(G^\lambda(z))''(z)}{(G^\lambda(z))'^2} = \phi\left(-\frac{z(z+\lambda)}{1+\lambda z}\right), \quad G^\lambda(0) = 0 = (G^\lambda)'(0).$$

Clearly the functions $K^{\phi n}, F^\lambda, G^\lambda \in M(\phi)$. Also we write $K^\phi := K^{\phi_2}$.

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K^ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, the equality holds if and only if f is K^{ϕ_3} or one of its rotations. If $\mu = \sigma_1$ then the equality holds if and only if f is F^λ or one of its rotations. If $\mu = \sigma_2$ then the equality holds if and only if f is G^λ or one of its rotations. ∎

Remark 2.1. If $\sigma_1 \le \mu \le \sigma_2$, then, in view of Lemma 1.1, Theorem 2.1 can be improved. Let σ_3 be given by

$$\sigma_3 := 1 - \frac{2B_2}{3B_1^2}$$

If $\sigma_1 \le \mu \le \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{2}{3B_1}\left[(B_1 + B_2) - \frac{3B_1}{4}(1-\mu)\right]|a_2|^2 \le \frac{B_1}{6}.$$

If $\sigma_3 \le \mu \le \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{2}{3B_1}\left[\frac{B_1 - B_2}{B_1} + \frac{3B_1}{2}(1-\mu)\right]|a_2|^2 \le \frac{B_1}{6}.$$

Example 2.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If f given by (1.1) belongs to $M(A, B)$, then*

$$|a_3 - \mu a_2^2| \le \begin{cases} -\frac{(B^2 - AB)}{6} + \frac{1-\mu}{4}(A-B)^2 & \text{if} \quad \mu \le \sigma_1, \\ \frac{(A-B)}{6} & \text{if} \quad \sigma_1 \le \mu \le \sigma_2, \\ \frac{(B^2 - AB)}{6} - \frac{1-\mu}{4}(A-B)^2 & \text{if} \quad \mu \ge \sigma_2, \end{cases}$$

where

$$\sigma_1 := 1 - \frac{2}{3(A-B)^2}\left[(B-A)(B-1)\right],$$

and

$$\sigma_2 := 1 - \frac{2}{3(A-B)^2}\left[(B-A)(B+1)\right].$$

The result is sharp.

3. APPLICATIONS TO FUNCTIONS DEFINED BY FRACTIONAL DERIVATIVES

In order to introduce the class $M^\lambda(\phi)$, we need the following:

Definition 3.1 (see [5, 6]; see also [13, 14])**.** Let $f(z)$ be analytic in a simply connected region of the z-plane containing the origin. The *fractional derivative* of f of order λ is defined by

$$D_z^\lambda f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda} d\zeta \quad (0 \leq \lambda < 1)$$

where the multiplicity of $(z-\zeta)^\lambda$ is removed by requiring that $\log(z-\zeta)$ is real for $z-\zeta > 0$.

Using the above Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [5] introduced the operator $\Omega^\lambda : \mathcal{A} \to \mathcal{A}$ defined by

$$(\Omega^\lambda f)(z) = \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z), \quad (\lambda \neq 2, 3, 4, \ldots).$$

The class $M^\lambda(\phi)$ consists of functions $f \in \mathcal{A}$ for which $\Omega^\lambda f \in M(\phi)$. Note that $M^0(\phi) \equiv S^*(\phi)$ and $M^\lambda(\phi)$ is the special case of the class $M^g(\phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n. \tag{3.1}$$

Let $g(z) = z + \sum_{n=2}^\infty g_n z^n$ $(g_n > 0)$. Since $f(z) = z + \sum_{n=2}^\infty a_n z^n \in M^g(\phi)$ if and only if $(f * g) = z + \sum_{n=2}^\infty g_n a_n z^n \in M(\phi)$, we obtain the coefficient estimate for functions in the class $M^g(\phi)$, from the corresponding estimate for functions in the class $M(\phi)$. Applying Theorem 2.1 for the function $(f*g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \cdots$, we get the following Theorem 3.1 after an obvious change of the parameter μ:

Theorem 3.1. *Let the function $\phi(z)$ be given by $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by (1.1) belongs to $M^g(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{g_3}\left[-\frac{B_2}{6} + \frac{1}{4}B_1^2 - \frac{\mu g_3}{4g_2^2}B_1^2\right] & \text{if} \quad \mu \leq \sigma_1, \\ \frac{B_1}{6g_3} & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \frac{1}{g_3}\left[\frac{B_2}{6} - \frac{1}{4}B_1^2 + \frac{\mu g_3}{4g_2^2}B_1^2\right] & \text{if} \quad \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{g_2^2}{g_3}\left[1 - \frac{2}{3B_1^2}[B_1 + B_2]\right],$$

and

$$\sigma_2 := \frac{g_2^2}{g_3}\left[1 - \frac{2}{3B_1^2}[B_2 - B_1]\right].$$

The result is sharp.

Since

$$(\Omega^\lambda f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n,$$

we have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda} \tag{3.2}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}. \tag{3.3}$$

For g_2 and g_3 given by (3.2) and (3.3), Theorem 3.1 reduces to the following:

Theorem 3.2. *Let the function $\phi(z)$ be given by $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by (1.1) belongs to $M_\alpha^\lambda(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(2-\lambda)(3-\lambda)}{36}\gamma & \text{if} \quad \mu \leq \sigma_1 \\ \frac{(2-\lambda)(3-\lambda)}{36}B_1 & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2 \\ -\frac{(2-\lambda)(3-\lambda)}{36}\gamma & \text{if} \quad \mu \geq \sigma_2 \end{cases}$$

where

$$\gamma := \left[-\frac{B_2}{6} + \frac{1}{4}B_1^2 - \frac{3(2-\lambda)}{8(3-\lambda)}B_1^2\mu\right],$$

$$\sigma_1 := \frac{2(3-\lambda)}{3(2-\lambda)}.\left[1 - \frac{2}{3B_1^2}\left[B_1 + B_2\right]\right],$$

and

$$\sigma_2 := \frac{2(3-\lambda)}{3(2-\lambda)}.\left[1 - \frac{2}{3B_1^2}\left[B_2 - B_1\right]\right].$$

The result is sharp.

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