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COMPARISON RESULTS FOR SOLUTIONS OF TIME SCALE MATRIX RICCATI EQUATIONS AND INEQUALITIES

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ABSTRACT. In this paper we derive comparison results for Hermitian solutions of time scale matrix Riccati equations and Riccati inequalities. Such solutions arise from special conjoined bases (X, U) of the corresponding time scale symplectic system via the Riccati quotient $Q = UX^{-1}$. We also discuss properties of a unitary matrix solution $\hat{Q} = (U + iX)\,(U - iX)^{-1}$ of a certain associated Riccati equation.

Key words and phrases: Time scale, Time scale symplectic system, Linear Hamiltonian system, Matrix Riccati equation, Riccati inequality, Conjoined basis.

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1. INTRODUCTION

In this paper we study inequalities for solutions of time scale matrix Riccati equations and inequalities related to time scale symplectic systems. A *time scale* $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$ and time scale differential (sometimes called dynamic) equations incorporate classical differential equations ($\mathbb{T} = \mathbb{R}$), difference equations ($\mathbb{T} = \mathbb{Z}$), q-difference equations ($\mathbb{T} = q^{\mathbb{N}}$), and many other examples. We refer to [3] and [4] for elementary and advanced treatment of the time scale calculus and dynamic equations on time scales.

Let $\mathbb{T}$ be a bounded time scale which is not necessarily connected. If we denote $a := \min \mathbb{T}$ and $b := \max \mathbb{T}$, then we call $\mathbb{T} = [a, b]_{\mathbb{T}}$ the time scale interval. On any time scale we have the forward and backward jump operators $\sigma(t)$ and $\rho(t)$ at t, and the graininess function $\mu(t) := \sigma(t) - t$. For a function f on a time scale we have the notions of the time scale delta-derivative $f^{\Delta}(t)$, the time scale integral $\int_a^b f(t)\,\Delta t$, a piecewise rd-continuous function, and a piecewise rd-continuously delta-differentiable function. See [3], [10], and [1] for details about these notions. We abbreviate the composition $f \circ \sigma$ of a function f with the forward jump operator by f^{σ}.

A *time scale symplectic system* is a linear system (here in the matrix form)

$$(\mathcal{S}) \qquad X^{\Delta} = \mathcal{A}(t)\,X + \mathcal{B}(t)\,U, \quad U^{\Delta} = \mathcal{C}(t)\,X + \mathcal{D}(t)\,U,$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ are complex $n \times n$ matrix functions on $[a, \rho(b)]_{\mathbb{T}}$ which are piecewise rd-continuous, X, U are complex $n \times n$ matrix functions on $[a, b]_{\mathbb{T}}$ which are piecewise rd-continuously delta-differentiable, and the coefficient matrix $\mathcal{S}(t) := \begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t) & \mathcal{D}(t) \end{pmatrix}$ satisfies the identity

$$(1.1) \qquad \mathcal{S}^*(t)\,\mathcal{J} + \mathcal{J}\mathcal{S}(t) + \mu(t)\,\mathcal{S}^*(t)\,\mathcal{J}\mathcal{S}(t) = 0 \quad \text{on } [a, \rho(b)]_{\mathbb{T}}.$$

Here $\mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is a $2n \times 2n$ skew-symmetric matrix and $\mathcal{S}^*$ denotes the complex conjugate of the matrix $\mathcal{S}$.

The subject of this paper are inequalities for solutions of the associated matrix Riccati equation

$$(\mathcal{R}) \qquad R[Q](t) := Q^{\Delta} - [\mathcal{C}(t) + \mathcal{D}(t)\,Q] + Q^{\sigma}[\mathcal{A}(t) + \mathcal{B}(t)\,Q] = 0$$

and the Riccati inequality

$$(\mathcal{RI}) \qquad R[Q](t)\,\big\{I + \mu(t)\,[\mathcal{A}(t) + \mathcal{B}(t)\,Q)]^{-1}\big\} \le 0.$$

In the continuous-time case, i.e. when $\mathbb{T} = [a, b]$ is a real *connected* interval, system ($\mathcal{S}$) reduces to the linear Hamiltonian system

$$X' = A(t)\,X + B(t)\,U, \quad U' = C(t)\,X - A^*(t)\,U,$$

and the Riccati equation ($\mathcal{R}$) and inequality ($\mathcal{RI}$) reduce to the classical matrix Riccati differential equation and inequality

$$R_c[Q](t) := Q' + A^*(t)\,Q + QA(t) + QB(t)\,Q - C(t) = 0 \quad \text{and} \quad R_c[Q](t) \le 0.$$

To see this we take $\mathcal{A}(t) = -\mathcal{D}^*(t) := A(t)$, $\mathcal{B}(t) := B(t)$ and $\mathcal{C}(t) := C(t)$ (matrices $B(t)$ and $C(t)$ are Hermitian), $f^{\Delta}(t) = f'(t)$, $\sigma(t) = t$, and $\mu(t) = 0$. Inequalities for solutions of the continuous-time Riccati equation $R_c[Q](t) = 0$ are established in [12, Chapter 5] and [13, Chapter 4].

In the discrete-time case, i.e. when $\mathbb{T} = \{0, 1, \dots, N+1\}$ is a set of equidistant isolated points, system ($\mathcal{S}$) reduces to the discrete symplectic system

$$X_{k+1} = A_k X_k + B_k U_k, \quad U_{k+1} = C_k X_k + D_k U_k,$$

and the Riccati equation ($\mathcal{R}$) and inequality ($\mathcal{RI}$) reduce to the discrete Riccati equation and inequality

$$R_d[Q]_k := Q_{k+1}(A_k + B_kQ_k) - (C_k + D_kQ_k) = 0 \quad \text{and} \quad R_d[Q]_k(A_k + B_kQ_k)^{-1} \le 0,$$

where the matrices A_k, $\mathcal{B}_k$, C_k, D_k are such that the $2n\times 2n$ matrix $S_k := \left(\begin{smallmatrix} A_k & B_k \\ C_k & D_k \end{smallmatrix}\right)$ is symplectic, i.e. $S_k^*\mathcal{J}S_k = \mathcal{J}$. The relation to the coefficients of the time scale system ($\mathcal{S}$) is given by $\mathcal{A}(k) := A_k - I$, $\mathcal{B}(k) := B_k$, $\mathcal{C}(k) := C_k$, $\mathcal{D}(k) := D_k - I$, $f^\Delta(k) = \Delta f(k) = f(k+1) - f(k)$, $\sigma(k) = k + 1$, and $\mu(k) \equiv 1$. Inequalities for solutions of discrete Riccati equations are established in [2]. The above Riccati inequality for the discrete symplectic case was discovered only recently in [9]. The time scale Riccati inequality ($\mathcal{RI}$) was then obtained in [11].

As it is known, see Proposition 2.1, Hermitian solutions $Q(t)$ of ($\mathcal{R}$) and ($\mathcal{RI}$) arise from conjoined bases (X, U) of ($\mathcal{S}$) with $X(t)$ invertible on $[a, b]_{\mathbb{T}}$ via the Riccati quotient $Q(t) = U(t)\,X^{-1}(t)$. Recall that a solution (X, U) of ($\mathcal{S}$) is a *conjoined basis* if $X^*(t)\,U(t)$ is Hermitian and rank $\begin{pmatrix} X(t) \\ U(t) \end{pmatrix} = n$ for some (and hence for any) $t \in [a, b]_{\mathbb{T}}$.

In this paper we derive inequalities for Hermitian solutions $Q(t)$ and $\underline{Q}(t)$ of two Riccati equations or inequalities of the form ($\mathcal{R}$) and ($\mathcal{RI}$) (even combined, one equation and one inequality) such that $\underline{Q}(t) \ge Q(t)$ for all $t \in [a, b]_{\mathbb{T}}$ whenever $\underline{Q}(a) \ge Q(a)$. Similar results hold for strict or opposite inequalities. Our results generalize to time scales some of the continuous-time ones from [12, Chapter 5]. At the same time we allow solutions of Riccati *inequalities* instead of equations. Finally, we establish time scale results connected to a "matrix method of Atkinson" as it is discussed in [13, Chapter 4] for matrices $\hat{X} = U - iX$ and $\hat{U} = U + iX$, and $\hat{Q} = \hat{U}\hat{X}^{-1}$, where (X, U) is a conjoined basis of ($\mathcal{S}$).

2. TIME SCALE SYMPLECTIC SYSTEMS AND AUXILIARY IDENTITIES

Let us adopt the following notation. When no ambiguity can arise, we will skip the argument (t) in the solutions and coefficients $X(t)$, $U(t)$, $Q(t)$, $\mathcal{A}(t), \dots, \mathcal{S}(t)$, etc. This will considerably simplify and shorten presented calculations.

Identity (1.1) implies that the matrix $I + \mu\mathcal{S}$ is symplectic, hence invertible. This means, using the time scale terminology, that the matrix function $\mathcal{S}$ is regressive on $[a, \rho(b)]_{\mathbb{T}}$. Thus, by [7, Theorem 5.7] or [3, Theorem 5.8], the system ($\mathcal{S}$) possesses unique solutions for any initial time $t_0 \in [a, b]_{\mathbb{T}}$ and arbitrary matrix initial values. A *Wronskian* of two solutions (X, U) and $(\tilde{X}, \tilde{U})$ of ($\mathcal{S}$) is the constant quantity $W := X^*\tilde{U} - U^*\tilde{X}$. Two conjoined bases (X, U) and $(\tilde{X}, \tilde{U})$ of ($\mathcal{S}$) are *normalized* if their Wronskian is the identity matrix, i.e. $W = I$.

The defining property (1.1) is translated in terms of the coefficients by the following equivalent conditions

$$\begin{aligned} &\mathcal{C}^*(I + \mu\mathcal{A}) \text{ and } \mathcal{B}^*(I + \mu\mathcal{D}) \text{ are Hermitian, and } \mathcal{A}^* + \mathcal{D} + \mu\,(\mathcal{A}^*\mathcal{D} - \mathcal{C}^*\mathcal{B}) = 0, \\ &\mathcal{B}\,(I + \mu\mathcal{A}^*) \text{ and } \mathcal{C}\,(I + \mu\mathcal{D}^*) \text{ are Hermitian, and } \mathcal{D} + \mathcal{A}^* + \mu\,(\mathcal{D}\mathcal{A}^* - \mathcal{C}\mathcal{B}^*) = 0. \end{aligned} \tag{2.1}$$

Solutions of ($\mathcal{S}$) satisfy the identities

$$X^\sigma = (I + \mu\mathcal{A})\,X + \mu\mathcal{B}U, \quad U^\sigma = \mu\mathcal{C}X + (I + \mu\mathcal{D})\,U, \tag{2.2}$$

$$X = (I + \mu\mathcal{D}^*)\,X^\sigma - \mu\mathcal{B}^*U^\sigma, \quad U = -\mu\mathcal{C}^*X^\sigma + (I + \mu\mathcal{A}^*)\,U^\sigma. \tag{2.3}$$

They show the relation between the values of X, U and their forward jump X^σ, U^σ.

The connection between a Hermitian solution Q of the Riccati equation and a conjoined basis (X, U) of ($\mathcal{S}$) is described in the following.

Proposition 2.1. *The following statements are equivalent.*

(i) *There exists a conjoined basis* (X, U) *of* ($\mathcal{S}$) *such that* $X(t)$ *is invertible for all* $t \in [a, b]_{\mathbb{T}}$.

(ii) *There exists a Hermitian solution* Q *on* $[a, b]_{\mathbb{T}}$ *of* ($\mathcal{R}$) *such that*

$$I + \mu(\mathcal{A} + \mathcal{B}Q) \text{ is invertible on } [a, \rho(b)]_{\mathbb{T}}. \tag{2.4}$$

Moreover, if any of the conditions (i)-(ii) above holds, then $Q = UX^{-1}$ *on* $[a, b]_{\mathbb{T}}$, *and* $P = \mathcal{P}$ *on* $[a, \rho(b)]_{\mathbb{T}}$, *where*

$$P := X(X^\sigma)^{-1}\mathcal{B} \quad \text{and} \quad \mathcal{P} := [I + \mu\,(\mathcal{A} + \mathcal{B}Q)]^{-1}\mathcal{B}. \tag{2.5}$$

Proof. The details can be found in [6, Theorem 3]. ∎

Remark 2.2. Conditions (i)-(ii) together with $P \geq 0$ or $\mathcal{P} \geq 0$ on $[a, \rho(b)]_{\mathbb{T}}$ are equivalent to the positivity of the quadratic functional associated with system ($\mathcal{S}$). We refer to [6], [5], [8], and [11] for this and other results concerning time scale quadratic functionals.

Together with system ($\mathcal{S}$) we consider another time scale symplectic system

$$X\underline{\ }^\Delta = \underline{\mathcal{A}}(t)\,\underline{X} + \underline{\mathcal{B}}(t)\,\underline{U}, \quad \underline{U}^\Delta = \underline{\mathcal{C}}(t)\,\underline{X} + \underline{\mathcal{D}}(t)\,\underline{U} \tag{$\underline{\mathcal{S}}$}$$

and the corresponding matrix Riccati equation

$$\underline{R}[\underline{Q}](t) := \underline{Q}^\Delta - [\underline{\mathcal{C}}(t) + \underline{\mathcal{D}}(t)\,\underline{Q}] + \underline{Q}^\sigma[\underline{\mathcal{A}}(t) + \underline{\mathcal{B}}(t)\,\underline{Q}] = 0. \tag{$\underline{\mathcal{R}}$}$$

That is, we assume that the coefficients $\underline{\mathcal{A}}$, $\underline{\mathcal{B}}$, $\underline{\mathcal{C}}$, $\underline{\mathcal{D}}$ are complex $n \times n$ matrix functions on $[a, \rho(b)]_{\mathbb{T}}$ which are piecewise rd-continuous, and the coefficient matrix $\underline{\mathcal{S}} := \left(\begin{smallmatrix} \underline{\mathcal{A}} & \underline{\mathcal{B}} \\ \underline{\mathcal{C}} & \underline{\mathcal{D}} \end{smallmatrix}\right)$ satisfies the identity

$$\underline{\mathcal{S}}^*(t)\,\mathcal{J} + \mathcal{J}\underline{\mathcal{S}}(t) + \mu(t)\,\underline{\mathcal{S}}^*(t)\,\mathcal{J}\underline{\mathcal{S}}(t) = 0 \quad \text{on } [a, \rho(b)]_{\mathbb{T}}.$$

Next we establish our first auxiliary result. In the sequel we shall abbreviate $(f^\sigma)^*$ by $f^{\sigma *}$.

Proposition 2.3. *Let* (X, U) *be a conjoined basis of* ($\mathcal{S}$) *with* $X(t)$ *invertible on* $[a, b]_{\mathbb{T}}$. *Then for any solution* $(\underline{X}, \underline{U})$ *of* ($\underline{\mathcal{S}}$) *we have*

$$(\underline{X}^*\underline{U} - \underline{X}^*Q\underline{X})^\Delta = \underline{X}^{\sigma *}\begin{pmatrix} I & Q^\sigma \end{pmatrix}\mathcal{J}\,(\underline{\mathcal{S}} - \mathcal{S})\begin{pmatrix} I \\ Q \end{pmatrix}\underline{X} + V^{\sigma *}\underline{\mathcal{B}}\,V,$$

where $Q := UX^{-1}$ *is a solution of* ($\mathcal{R}$) *and* $V := \underline{U} - Q\underline{X}$ *on* $[a, b]_{\mathbb{T}}$. *Moreover, if* $(\underline{X}, \underline{U})$ *is a conjoined basis of* ($\underline{\mathcal{S}}$) *with* $\underline{X}(t)$ *invertible on* $[a, b]_{\mathbb{T}}$, *then with* $\underline{Q} := \underline{U}\,\underline{X}^{-1}$ *we have*

$$V^{\sigma *}\underline{\mathcal{B}}\,V = \underline{X}^{\sigma *}(\underline{Q}^\sigma - Q^\sigma)\,\underline{\mathcal{B}}\,(\underline{Q} - Q)\,\underline{X}.$$

Proof. For the first part we have, by using the time scale product rule $(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta$,

$$\begin{aligned}
(\underline{X}^*\underline{U} - \underline{X}^*Q\underline{X})^\Delta &= (\underline{X}^\Delta)^*(\underline{U} - Q\underline{X}) + \underline{X}^{\sigma *}[\underline{U}^\Delta - (Q^\Delta\underline{X} + Q^\sigma\underline{X}^\Delta)] \\
&\overset{(\underline{\mathcal{S}}),(\mathcal{R})}{=} (\underline{\mathcal{A}X} + \underline{\mathcal{B}U})^*(\underline{U} - Q\underline{X}) + \underline{X}^{\sigma *}(\underline{\mathcal{C}X} + \underline{\mathcal{D}U}) \\
&\quad - \underline{X}^{\sigma *}[\mathcal{C} + \mathcal{D}Q - Q^\sigma(\mathcal{A} + \mathcal{B}Q)]\underline{X} - \underline{X}^{\sigma *}Q^\sigma(\underline{\mathcal{A}X} + \underline{\mathcal{B}U}) \\
&\overset{(2.3)}{=} \underline{X}^{\sigma *}[(I + \mu\underline{\mathcal{D}})\,\underline{\mathcal{A}}^* - \mu\underline{\mathcal{C}\mathcal{B}}^*]\,(\underline{U} - Q\underline{X}) + \underline{X}^{\sigma *}(\underline{\mathcal{C}} - \mathcal{C})\,\underline{X} \\
&\quad + \underline{U}^{\sigma *}[(I + \mu\underline{\mathcal{A}})\,\underline{\mathcal{B}}^* - \mu\underline{\mathcal{B}\mathcal{A}}^*]\,(\underline{U} - Q\underline{X}) - \underline{X}^{\sigma *}\mathcal{D}Q\underline{X} \\
&\quad - \underline{X}^{\sigma *}Q^\sigma(\underline{\mathcal{A}} - \mathcal{A} - \mathcal{B}Q)\,\underline{X} + \underline{X}^{\sigma *}(\underline{\mathcal{D}} - Q^\sigma\underline{\mathcal{B}})\,\underline{U} \\
&\overset{(2.1)}{=} \underline{X}^{\sigma *}\begin{pmatrix} I & Q^\sigma \end{pmatrix}\mathcal{J}\,(\underline{\mathcal{S}} - \mathcal{S})\begin{pmatrix} I \\ Q \end{pmatrix}\underline{X} + V^{\sigma *}\underline{\mathcal{B}}\,V.
\end{aligned}$$

Finally, if $\underline{X}$ is invertible on $[a, b]_{\mathbb{T}}$, then $V = (\underline{Q} - Q)\,\underline{X}$, so that the proof is complete. ∎

Remark 2.4. Let (X, U), Q, and V be as in Proposition 2.3. Then

$$V = \underline{U} - (X^*)^{-1} U^* \underline{X} = (X^*)^{-1}(X^* \underline{U} - U^* \underline{X}) = (X^*)^{-1} W,$$

where $W := X^* \underline{U} - U^* \underline{X}$ is a Wronskian-type matrix. Since (X, U) and $(\underline{X}, \underline{U})$ are in general solutions of two different systems, we cannot call this matrix the "true" Wronskian. If W is constant on $[a, b]_{\mathbb{T}}$, then we have $V^\sigma = (X^{\sigma *})^{-1} W$ and

$$V^{\sigma *} \underline{\mathcal{B}}\, V = W^* (X^\sigma)^{-1} \underline{\mathcal{B}} (X^*)^{-1} W = V^* X (X^\sigma)^{-1} \underline{\mathcal{B}}\, V.$$

When (X, U) and $(\underline{X}, \underline{U}) = (\tilde{X}, \tilde{U})$ are solutions of the *same* system, we obtain the following.

Proposition 2.5. *Let (X, U) be a conjoined basis of ($\mathcal{S}$) with $X(t)$ invertible on $[a, b]_{\mathbb{T}}$. Then for any other solution $(\tilde{X}, \tilde{U})$ of ($\mathcal{S}$) we have*

$$(\tilde{X}^* \tilde{U} - \tilde{X}^* Q \tilde{X})^\Delta = W^* (X^\sigma)^{-1} \mathcal{B} (X^*)^{-1} W = V^* P V, \tag{2.6}$$

$$(X^{-1} \tilde{X})^\Delta = (X^\sigma)^{-1} \mathcal{B} (X^*)^{-1} W = X^{-1} P V, \tag{2.7}$$

where $Q := U X^{-1}$ is a solution of ($\mathcal{R}$), W is the (constant) Wronskian of (X, U) and $(\tilde{X}, \tilde{U})$, $V := \tilde{U} - Q \tilde{X} = (X^*)^{-1} W$, and P is defined in (2.5)*.*

Proof. Equalities in (2.6) follow from Proposition 2.3 and Remark 2.4. Furthermore, by the time scale product rule and $(X^{-1})^\Delta = -(X^\sigma)^{-1} X^\Delta X^{-1}$, calculations

$$\begin{aligned} (X^{-1} \tilde{X})^\Delta &= -(X^\sigma)^{-1} X^\Delta X^{-1} \tilde{X} + (X^\sigma)^{-1} \tilde{X}^\Delta \\ &\overset{(\mathcal{S})}{=} (X^\sigma)^{-1} \mathcal{B} (\tilde{U} - Q \tilde{X}) = (X^\sigma)^{-1} \mathcal{B} V = X^{-1} P V. \end{aligned}$$

show the identities in (2.7). ∎

Corollary 2.6. *Suppose that (X, U) and $(\tilde{X}, \tilde{U})$ are normalized conjoined bases of ($\mathcal{S}$) such that $X(t)$ is invertible on $[a, b]_{\mathbb{T}}$. Then*

$$(\tilde{X}^* \tilde{U} - \tilde{X}^* Q \tilde{X})^\Delta = (X^\sigma)^{-1} \mathcal{B} (X^*)^{-1} = X^{-1} P (X^*)^{-1} = (X^{-1} \tilde{X})^\Delta,$$

where $Q := U X^{-1}$ and P is defined in (2.5)*.*

3. SOLUTIONS OF RICCATI EQUATIONS

In this section we establish the following comparison result for Hermitian solutions of two time scale Riccati matrix equations.

Theorem 3.1. *Assume that Q and $\underline{Q}$ are Hermitian solutions of the time scale Riccati equations ($\mathcal{R}$) and ($\underline{\mathcal{R}}$), respectively, on $[a, \rho(b)]_{\mathbb{T}}$ such that*

$$I + \mu(\mathcal{A} + \mathcal{B} Q), \ I + \mu(\underline{\mathcal{A}} + \underline{\mathcal{B}}\, \underline{Q}) \text{ are invertible on } [a, \rho(b)]_{\mathbb{T}} \tag{3.1}$$

and satisfying on $[a, \rho(b)]_{\mathbb{T}}$ the inequality

$$K := \left\{ \begin{pmatrix} I & Q^\sigma \end{pmatrix} \mathcal{J} \left(\underline{\mathcal{S}} - \mathcal{S} \right) \begin{pmatrix} I \\ Q \end{pmatrix} + \left(\underline{Q}^\sigma - Q^\sigma \right) \underline{\mathcal{B}} \left(\underline{Q} - Q \right) \right\} [I + \mu(\underline{\mathcal{A}} + \underline{\mathcal{B}}\, \underline{Q})]^{-1} \geq 0. \tag{3.2}$$

If $\underline{Q}(a) \geq Q(a)$, then $\underline{Q}(t) \geq Q(t)$ for all $t \in [a, b]_{\mathbb{T}}$.

Proof. From Proposition 2.1 we have that $Q = UX^{-1}$ and $\underline{Q} = \underline{U}\,\underline{X}^{-1}$ on $[a,b]_{\mathbb{T}}$, where (X,U) and $(\underline{X},\underline{U})$ are conjoined bases of ($\mathcal{S}$) and ($\underline{\mathcal{S}}$), respectively, such that $X(t)$ and $\underline{X}(t)$ are invertible on $[a,b]_{\mathbb{T}}$. By using (2.2) for the solution $(\underline{X},\underline{U})$ and system ($\underline{\mathcal{S}}$) we get the identity $\underline{X}(\underline{X}^{\sigma})^{-1} = [I + \mu(\underline{\mathcal{A}} + \underline{\mathcal{B}}\,\underline{Q})]^{-1}$, and then Proposition 2.3 yields

$$\begin{aligned}\left\{\underline{X}^*(\underline{Q} - Q)\,\underline{X}\right\}^{\Delta} &= (\underline{X}^*\underline{U} - \underline{X}^*Q\underline{X})^{\Delta}\\ &= \underline{X}^{\sigma *}\left\{\begin{pmatrix} I & Q^{\sigma}\end{pmatrix}\mathcal{J}\,(\underline{\mathcal{S}} - \mathcal{S})\begin{pmatrix} I\\ Q\end{pmatrix} + (\underline{Q}^{\sigma} - Q^{\sigma})\,\underline{\mathcal{B}}\,(\underline{Q} - Q)\right\}\underline{X}\\ &= \underline{X}^{\sigma *}K\underline{X}^{\sigma} \geq 0.\end{aligned}$$

The assumption $\underline{Q}(a) \geq Q(a)$ implies that $\left\{\underline{X}^*(\underline{Q} - Q)\,\underline{X}\right\}(a) \geq 0$. Hence, $\left\{\underline{X}^*(\underline{Q} - Q)\,\underline{X}\right\}(t) \geq 0$ for all $t \in [a,b]_{\mathbb{T}}$. The invertibility of $\underline{X}$ on $[a,b]_{\mathbb{T}}$ now yields the conclusion. ∎

Remark 3.2. From the above proof we can obtain various modifications of Theorem 3.1 in the following sense. The three inequalities "$\geq$" in Theorem 3.1 can be all replaced by the opposite inequality "$\leq$". Similarly, if $K \geq (>) 0$ and $\underline{Q}(a) > (\geq) Q(a)$, then $\underline{Q}(t) > Q(t)$ for all $t \in (a,b]_{\mathbb{T}}$. On the other hand, if $K \leq (<) 0$ and $\underline{Q}(a) < (\leq) Q(a)$, then $\underline{Q}(t) < Q(t)$ for all $t \in (a,b]_{\mathbb{T}}$.

Remark 3.3. Assumption (3.1) is always satisfied on a (sufficiently small) neighborhood of every right-dense or left-dense point t_0, because for such points we have $\mu(t) \to 0$ as $t \to t_0$.

When Q and $\underline{Q} = \tilde{Q}$ are solutions of the *same* Riccati equation, we obtain the following.

Corollary 3.4. *Assume that Q and $\tilde{Q}$ are Hermitian solutions of the time scale Riccati equation ($\mathcal{R}$) such that*

$$I + \mu(\mathcal{A} + \mathcal{B}Q),\ I + \mu(\mathcal{A} + \mathcal{B}\tilde{Q}) \text{ are invertible on } [a,\rho(b)]_{\mathbb{T}} \tag{3.3}$$

and satisfying on $[a,\rho(b)]_{\mathbb{T}}$ the inequality

$$(\tilde{Q}^{\sigma} - Q^{\sigma})\,\mathcal{B}(\tilde{Q} - Q)\,[I + \mu(\mathcal{A} + \mathcal{B}\tilde{Q})]^{-1} \geq 0.$$

If $\tilde{Q}(a) \geq Q(a)$, then $\tilde{Q}(t) \geq Q(t)$ for all $t \in [a,b]_{\mathbb{T}}$.

Example 3.1. (i) Let $\mathcal{C} \equiv 0$, $I + \mu\mathcal{A}$ be invertible, and $\mathcal{D} = -(I + \mu\mathcal{A}^*)^{-1}\mathcal{A}^*$ on $[a,\rho(b)]_{\mathbb{T}}$, and $Q \equiv 0$ on $[a,b]_{\mathbb{T}}$. Then, by Corollary 3.4, for any Hermitian solution $\tilde{Q}$ of ($\mathcal{R}$) with $I + \mu(\mathcal{A} + \mathcal{B}\tilde{Q})$ invertible and $\tilde{Q}^{\sigma}\mathcal{B}\,\tilde{Q}[I + \mu(\mathcal{A} + \mathcal{B}\tilde{Q})]^{-1} \geq 0$ on $[a,\rho(b)]_{\mathbb{T}}$, and with $\tilde{Q}(a) \geq 0$ we have $\tilde{Q}(t) \geq 0$ for all $t \in [a,b]_{\mathbb{T}}$.

(ii) Assume that $\mathcal{A} \equiv 0$ in the above example. This implies that $\mathcal{D} \equiv 0$ and $\mathcal{B}$ is Hermitian. Then for any Hermitian solution $\tilde{Q}$ of the time scale Riccati equation $\tilde{Q}^{\Delta} + \tilde{Q}^{\sigma}\mathcal{B}(t)\,\tilde{Q} = 0$ with $I + \mu\mathcal{B}\tilde{Q}$ invertible and $\tilde{Q}^{\sigma}\mathcal{B}\,\tilde{Q}(I + \mu\mathcal{B}\tilde{Q})^{-1} \geq 0$ on $[a,\rho(b)]_{\mathbb{T}}$, and with $\tilde{Q}(a) \geq 0$ we have $\tilde{Q}(t) \geq 0$ for all $t \in [a,b]_{\mathbb{T}}$.

Remark 3.5. In the continuous-time case, i.e. when $\mu(t) \equiv 0$ and $\sigma(t) = t$ on the connected interval $[a,\rho(b)]_{\mathbb{T}} = [a,b]$, condition (3.2) reads as

$$\begin{pmatrix} I & Q\end{pmatrix}\mathcal{J}\,(\underline{\mathcal{S}} - \mathcal{S})\begin{pmatrix} I\\ Q\end{pmatrix} + (\underline{Q} - Q)\,\underline{\mathcal{B}}\,(\underline{Q} - Q) \geq 0. \tag{3.4}$$

In this case $\mathcal{B}$ and $\mathcal{C}$ are Hermitian and $\mathcal{D} = -\mathcal{A}^*$ on $[a,b]$. In particular, inequalities

$$\mathcal{J}(\underline{\mathcal{S}} - \mathcal{S}) \geq 0 \quad \text{and} \quad \underline{\mathcal{B}} \geq 0 \quad \text{on } [a,b]$$

imply condition (3.4), so that we obtain the statement of [12, Lemma 5.1.1] as a special case of Theorem 3.1.

4. SOLUTIONS OF RICCATI INEQUALITIES

In this section we generalize comparison results from the previous section to solutions $Q(t)$ and $\underline{Q}(t)$ of two time scale Riccati *inequalities*.

Proposition 4.1. *The following statements are equivalent.*

(i) *There exists a solution (X, U) of the system*

$$X^{\Delta} = \mathcal{A}(t)\,X + \mathcal{B}(t)\,U, \quad N(t) := X^{\sigma *}[U^{\Delta} - \mathcal{C}(t)\,X - \mathcal{D}(t)\,U] \leq 0, \tag{4.1}$$

such that X^*U is Hermitian and X is invertible on $[a,b]_{\mathbb{T}}$.

(ii) *There exists a Hermitian solution Q on $[a,b]_{\mathbb{T}}$ of ($\mathcal{RI}$) satisfying condition* (2.4).

Moreover, if any of the conditions (i)-(ii) above holds, then $Q = UX^{-1}$ on $[a,b]_{\mathbb{T}}$.

Proof. It is similar to the proof of Proposition 2.1. Alternatively, consult [11, Section 7]. ∎

Remark 4.2. Let us denote by $F(t)$ the $n \times n$ Hermitian matrix representing the left-hand side of the Riccati inequality ($\mathcal{RI}$), that is

$$F := R[Q]\,[I + \mu(\mathcal{A} + \mathcal{B}Q)]^{-1}.$$

Since the connection between solutions (X, U) and Q in Proposition 4.1 is given by the Riccati quotient $Q = UX^{-1}$, it follows by a simple calculation that on $[a, \rho(b)]_{\mathbb{T}}$ we have

$$F = (X^{\sigma *})^{-1} N (X^{\sigma})^{-1}, \quad N = X^{\sigma *} R[Q]\,X = X^{\sigma *} F X^{\sigma},$$

where the Hermitian matrix N is defined in (4.1).

Remark 4.3. Let (X, U) be a solution of system (4.1) from Proposition 4.1. We observed in the proof of [11, Theorem 7.1] that (X, U) solves the following time scale symplectic system

$$X^{\Delta} = \mathcal{A}X + \mathcal{B}U, \quad U^{\Delta} = [\mathcal{C} + F(I + \mu\mathcal{A})]\,X + (\mathcal{D} + \mu F\mathcal{B})\,U. \tag{$\mathcal{SI}$}$$

That is, the coefficient matrix $\mathcal{S} + \left(\begin{smallmatrix} 0 & 0 \\ F(I+\mu\mathcal{A}) & \mu F\mathcal{B} \end{smallmatrix}\right)$ of the above system satisfies identity (1.1) and (X, U) is a conjoined basis of ($\mathcal{SI}$) with X invertible on $[a,b]_{\mathbb{T}}$. Therefore, we can replace system ($\mathcal{S}$) and Riccati equation ($\mathcal{R}$) by system ($\mathcal{SI}$) and Riccati inequality ($\mathcal{RI}$), and obtain the following generalization of Theorem 3.1.

Let $\underline{\mathcal{A}}$, $\underline{\mathcal{B}}$, $\underline{\mathcal{C}}$, $\underline{\mathcal{D}}$ be $n \times n$ matrix functions as in Section 3. Consider the time scale matrix Riccati inequality

$$\underline{F} := \underline{R}[\underline{Q}]\,[I + \mu(\underline{\mathcal{A}} + \underline{\mathcal{B}}\underline{Q})]^{-1} \leq 0 \tag{$\underline{\mathcal{RI}}$}$$

and the corresponding time scale symplectic system

$$\underline{X}^{\Delta} = \underline{\mathcal{A}X} + \underline{\mathcal{B}U}, \quad \underline{U}^{\Delta} = [\underline{\mathcal{C}} + \underline{F}(I + \mu\underline{\mathcal{A}})]\,\underline{X} + (\underline{\mathcal{D}} + \mu\underline{F\mathcal{B}})\,\underline{U}. \tag{$\underline{\mathcal{SI}}$}$$

Theorem 4.4. *Assume that Q and $\underline{Q}$ are Hermitian solutions of the time scale Riccati inequalities ($\mathcal{RI}$) and ($\underline{\mathcal{RI}}$), respectively, on $[a, \rho(b)]_{\mathbb{T}}$ such that condition* (3.1) *holds and satisfying on $[a, \rho(b)]_{\mathbb{T}}$ the inequality*

$$\begin{aligned} \Bigg\{ \begin{pmatrix} I & Q^{\sigma} \end{pmatrix} \mathcal{J} \left[\underline{\mathcal{S}} - \mathcal{S} + \begin{pmatrix} 0 & 0 \\ \underline{F}(I + \mu\underline{\mathcal{A}}) - F(I + \mu\mathcal{A}) & \mu(\underline{F\mathcal{B}} - F\mathcal{B}) \end{pmatrix} \right] \begin{pmatrix} I \\ Q \end{pmatrix} & \\ + (\underline{Q}^{\sigma} - Q^{\sigma})\,\underline{\mathcal{B}}\,(\underline{Q} - Q) \Bigg\}\, [I + \mu(\underline{\mathcal{A}} + \underline{\mathcal{B}}\,\underline{Q})]^{-1} \geq 0. & \end{aligned} \tag{4.2}$$

If $\underline{Q}(a) \geq Q(a)$, then $\underline{Q}(t) \geq Q(t)$ for all $t \in [a,b]_{\mathbb{T}}$.

Proof. By Proposition 4.1 and Remark 4.3 we have $Q = UX^{-1}$ and $\underline{Q} = \underline{U}\,\underline{X}^{-1}$, where (X, U) and $(\underline{X}, \underline{U})$ are conjoined bases of $(\mathcal{SI})$ and $(\underline{\mathcal{SI}})$, respectively, with X and $\underline{X}$ invertible on $[a, b]_{\mathbb{T}}$. Thus, the result follows from Theorem 3.1, where we replace the matrices $\mathcal{S}$ and $\underline{\mathcal{S}}$ by the coefficient matrices $\mathcal{S} + \left(\begin{smallmatrix} 0 & 0 \\ F(I+\mu\mathcal{A}) & \mu F\mathcal{B} \end{smallmatrix}\right)$ and $\underline{\mathcal{S}} + \left(\begin{smallmatrix} 0 & 0 \\ \underline{F}(I+\mu\underline{\mathcal{A}}) & \mu\underline{F}\underline{\mathcal{B}} \end{smallmatrix}\right)$. ∎

Remark 4.5. (i) We can see that Theorem 3.1 is a special case of Theorem 4.4 in which we take $F = \underline{F} = 0$. Hence, Theorem 4.4 allows to compare solutions of two Riccati inequalities or equations (or combined, one inequality and one equation).

(ii) Similar statements as in Remark 3.2 now hold also for Theorem 4.4.

When Q and $\underline{Q} = \tilde{Q}$ are solutions of the *same* Riccati inequality, we obtain the following.

Corollary 4.6. *Assume that Q and $\tilde{Q}$ are Hermitian solutions of the time scale Riccati inequality $(\mathcal{RI})$ such that condition (3.3) holds and satisfying on $[a, \rho(b)]_{\mathbb{T}}$ the inequality*

$$\left\{ (\underline{F} - F)\,[I + \mu(\mathcal{A} + \mathcal{B}Q)] + (\tilde{Q}^\sigma - Q^\sigma)\,\mathcal{B}(\tilde{Q} - Q) \right\} [I + \mu(\mathcal{A} + \mathcal{B}\tilde{Q})]^{-1} \geq 0.$$

If $\tilde{Q}(a) \geq Q(a)$, then $\tilde{Q}(t) \geq Q(t)$ for all $t \in [a, b]_{\mathbb{T}}$.

Example 4.1. Let $\mathcal{A} \equiv 0$, $\mathcal{B}$ and $\mathcal{C}$ be Hermitian, $\mathcal{D} = \mu\mathcal{C}\mathcal{B}$, and $\mathcal{C} \geq 0$ on $[a, \rho(b)]_{\mathbb{T}}$. Then for $Q \equiv 0$ we have $F = -\mathcal{C} \leq 0$, i.e. $Q \equiv 0$ is a solution of the Riccati inequality $(\mathcal{RI})$ and satisfies condition (2.4). Take $\underline{F} \equiv 0$. Then, by Theorem 4.4, for any Hermitian solution $\underline{Q}$ of the Riccati equation $(\underline{\mathcal{R}})$ with $I + \mu(\underline{\mathcal{A}} + \underline{\mathcal{B}}\underline{Q})$ invertible and $(\underline{\mathcal{C}} + \underline{Q}^\sigma \underline{\mathcal{B}}\underline{Q})\,[I + \mu(\underline{\mathcal{A}} + \underline{\mathcal{B}}\,\underline{Q})]^{-1} \geq 0$ on $[a, \rho(b)]_{\mathbb{T}}$, and with $\underline{Q}(a) \geq 0$ we have $\underline{Q}(t) \geq 0$ for all $t \in [a, b]_{\mathbb{T}}$.

Remark 4.7. In the continuous-time case, inequality (4.2) reduces to

$$(I \quad Q)\,\mathcal{J}\left[\underline{\mathcal{S}} - \mathcal{S} + \begin{pmatrix} 0 & 0 \\ \underline{F} - F & 0 \end{pmatrix}\right] \begin{pmatrix} I \\ Q \end{pmatrix} + (\underline{Q} - Q)\,\underline{\mathcal{B}}\,(\underline{Q} - Q) \geq 0. \tag{4.3}$$

Hence, similarly as in Remark 3.5, conditions

$$\mathcal{J}\left[\underline{\mathcal{S}} - \mathcal{S} + \begin{pmatrix} 0 & 0 \\ \underline{F} - F & 0 \end{pmatrix}\right] \geq 0 \quad \text{and} \quad \underline{\mathcal{B}} \geq 0 \quad \text{on } [a, b]$$

imply condition (4.3).

5. MATRIX METHOD OF ATKINSON

In this section we derive time scale results connected to a "matrix method of Atkinson" as it is presented in [13, Chapter 4]. Let (X, U) be a conjoined basis of $(\mathcal{S})$, X not necessarily invertible, and consider the following complex $n \times n$ matrix functions

$$\hat{X}(t) := U(t) - iX(t), \quad \hat{U}(t) := U(t) + iX(t)$$

on $[a, b]_{\mathbb{T}}$, where i is the imaginary unit, i.e. $i^2 = -1$. Since the defining properties of a conjoined basis imply that $X^*U - U^*X = 0$ and $X^*X + U^*U > 0$ on $[a, b]_{\mathbb{T}}$, it follows that

$$\hat{X}^*\hat{X} = \hat{U}^*\hat{U} = X^*X + U^*U > 0, \tag{5.1}$$

i.e. both $\hat{X}$ and $\hat{U}$ are invertible on $[a, b]_{\mathbb{T}}$. For $t \in [a, b]_{\mathbb{T}}$ define the Riccati quotient

$$\hat{Q}(t) := \hat{U}(t)\,\hat{X}^{-1}(t). \tag{5.2}$$

Condition (5.1) implies that $\hat{Q}$ is unitary and, consequently, all eigenvalues of $\hat{Q}$ have absolute value 1.

Define on $[a,\rho(b)]_{\mathbb{T}}$ another coefficient matrices

$$\hat{\mathcal{A}} := [\mathcal{A}+\mathcal{D}-i\,(\mathcal{B}-\mathcal{C})]/2, \qquad \hat{\mathcal{C}} := [\mathcal{D}-\mathcal{A}+i\,(\mathcal{B}+\mathcal{C})]/2,$$
$$\hat{\mathcal{B}} := [\mathcal{D}-\mathcal{A}-i\,(\mathcal{B}+\mathcal{C})]/2, \qquad \hat{\mathcal{D}} := [\mathcal{A}+\mathcal{D}+i\,(\mathcal{B}-\mathcal{C})]/2.$$

Then (and only then) a direct calculation shows that the pair $(\hat{X},\hat{U})$ solves on $[a,\rho(b)]_{\mathbb{T}}$ the linear system

$$(\hat{\mathcal{S}}) \qquad \hat{X}^{\Delta} = \hat{\mathcal{A}}(t)\,\hat{X} + \hat{\mathcal{B}}(t)\,\hat{U}, \quad \hat{U}^{\Delta} = \hat{\mathcal{C}}(t)\,\hat{X} + \hat{\mathcal{D}}(t)\,\hat{U}.$$

Theorem 5.1. *Let $\hat{Q}(t)$ be defined by* (5.2).

(i) *The matrix $\hat{Q}(t)$ has an eigenvalue $\lambda = 1$ if and only if the matrix $X(t)$ is singular.*

(ii) *The matrix $\hat{Q}(t)$ satisfies on $[a,\rho(b)]_{\mathbb{T}}$ the first order time scale linear equations*

$$\hat{Q}^{\Delta} = i\,\hat{Q}^{\sigma} M_1(t) = i\,M_2(t)\,\hat{Q},$$

where M_1 and M_2 are $n\times n$ matrices defined by

$$M_1 := 2\,(\hat{U}^{\sigma *})^{-1}\begin{pmatrix} U^{\sigma *} & X^{\sigma *}\end{pmatrix} \mathcal{S} \begin{pmatrix} X \\ U \end{pmatrix} \hat{X}^{-1},$$
$$M_2 := 2\,(\hat{X}^{\sigma *})^{-1}\begin{pmatrix} U^{\sigma *} & X^{\sigma *}\end{pmatrix} \mathcal{S} \begin{pmatrix} X \\ U \end{pmatrix} \hat{U}^{-1}.$$

(iii) *The matrix $\hat{Q}(t)$ satisfies on $[a,\rho(b)]_{\mathbb{T}}$ the matrix Riccati equation*

$$(\hat{\mathcal{R}}) \qquad \hat{Q}^{\Delta} - [\hat{\mathcal{C}}(t) + \hat{\mathcal{D}}(t)\,\hat{Q}] + \hat{Q}^{\sigma}[\hat{\mathcal{A}}(t) + \hat{\mathcal{B}}(t)\,\hat{Q}] = 0.$$

Proof. (i) If there exists a vector $c \in \operatorname{Ker} X(t)$, $c \neq 0$, then $\xi := \hat{X}(t)\,c$ is a nonzero eigenvector of $\hat{Q}(t)$ corresponding to the eigenvalue $\lambda = 1$, because $(\hat{Q}-I)\,\xi = (\hat{U}-\hat{X})\,\hat{X}^{-1}\xi = 2iXc = 0$. Conversely, if $\lambda = 1$ is an eigenvalue of $\hat{Q}(t)$ with an eigenvector $\xi \neq 0$, then put $c := \hat{X}^{-1}(t)\,\xi$. The above calculation shows that $2iXc = (\hat{Q}-I)\,\xi = 0$, i.e. $X(t)$ is singular.

(ii) By the time scale product rule, the definition of $\hat{X}$, $\hat{U}$, and the fact that $\hat{Q}$ is Hermitian, we have

$$\begin{aligned}
\hat{Q}^{\Delta} &= (U+iX)^{\Delta}\hat{X}^{-1} - \hat{Q}^{\sigma}(U-iX)^{\Delta}\hat{X}^{-1} \\
&= \big\{(I-\hat{Q}^{\sigma})\,U^{\Delta} + i\,(I+\hat{Q}^{\sigma})\,X^{\Delta}\big\}\hat{X}^{-1} \\
&= (\hat{X}^{\sigma *})^{-1}\big\{(\hat{X}^{\sigma *}-\hat{U}^{\sigma *})\,(\mathcal{C}X+\mathcal{D}U) + i\,(\hat{X}^{\sigma *}+\hat{U}^{\sigma *})\,(\mathcal{A}X+\mathcal{B}U)\big\}\hat{X}^{-1} \\
&= 2i\,(\hat{X}^{\sigma *})^{-1}\big\{X^{\sigma *}(\mathcal{C}X+\mathcal{D}U) + U^{\sigma *}(\mathcal{A}X+\mathcal{B}U)\big\}\hat{X}^{-1} \\
&= i\,\hat{Q}^{\sigma *}M_1 = i\,\hat{Q}^{\sigma}M_1 = i\,M_2\,\hat{Q}.
\end{aligned}$$

In the last line of the above computation we used the invertibility of $\hat{X}$ and $\hat{U}$.

(iii) This follows from the definition of $\hat{Q}$ in (5.2) and from the form of system $(\hat{\mathcal{S}})$. ∎

Remark 5.2. As it is observed in [13, pg. 165], the matrix $\hat{Q} = \hat{U}\hat{X}^{-1}$ is related to the Cayley transform of the Hermitian solution $Q_1 := UX^{-1}$ of $(\mathcal{R})$. More precisely, if X is invertible on $[a,b]_{\mathbb{T}}$, then $\hat{Q} = (-Q_1-iI)\,(-Q_1+iI)^{-1}$, i.e. $\hat{Q}$ is the Cayley transform of $-Q_1$. On the other hand, if U is invertible on $[a,b]_{\mathbb{T}}$, then with $Q_2 := XU^{-1}$ we have $\hat{Q} = -(Q_2-iI)\,(Q_2+iI)^{-1}$, i.e. $-\hat{Q}$ is the Cayley transform of Q_2.

Finally, we note that the coefficient matrix of system $(\hat{\mathcal{S}})$ does not satisfy identity (1.1) defining a time scale symplectic system. This can be expected since already in the continuous-time case, i.e. when $\mu(t) \equiv 0$ on $[a,b]$, this matrix is not Hamiltonian ($\hat{\mathcal{B}}$ and $\hat{\mathcal{C}}$ are not Hermitian), see [13, pg. 165].

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