The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 3, Issue 2, Article 1, pp. 1-10, 2006

UNIQUENESS OF MEROMORPHIC FUNCTIONS THAT SHARE THREE VALUES

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Received 2 February, 2005; accepted 14 September, 2005; published 7 July, 2006.
Communicated by: K. Noor

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ABSTRACT. In the paper dealing with the uniqueness problem of meromorphic functions we prove five theorems one of which will improve a result given by Lahiri [5] and the remaining will supplement some previous results.

Key words and phrases: Weighted sharing, meromorphic function, uniqueness.

2000 *Mathematics Subject Classification.* 30D35.

ISSN (electronic): 1449-5910

The author is grateful to Prof. Indrajit Lahiri for supplying him the paper [5] from which the author has found the problem of this paper.

1. INTRODUCTION DEFINITIONS AND RESULTS

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities). The notation $S(r, f)$ denotes any quantity satisfying $S(r, f) = 0(T(r, f))$ as $r \longrightarrow \infty$, outside any set of finite linear measure.

We use I to denote any set of infinite linear measure of $0 < r < \infty$.
G. Broch [1] proved the following theorem.

Theorem 1.1. [1, 5, 10] *Let f and g share $0, 1, \infty$ CM. If*

$$\limsup_{\substack{r\longrightarrow\infty \\ r \in I}} \frac{2\overline{N}(r, 0; f) + 2\overline{N}(r, \infty; f) - m(r, 1; g)}{T(r, f)} < 1 \tag{1.1}$$

then $f \equiv g$ or $f.g \equiv 1$.

N. Terglane [9] proved the following theorem.

Theorem 1.2. [5, 9, 10] *Let f and g share $1, \infty$ CM and 0 IM. If*

$$N(r, 1; f) - \overline{N}(r, 1; f) = S(r, f)$$

and

$$\limsup_{\substack{r\longrightarrow\infty \\ r \in I}} \frac{3\overline{N}(r, 0; f) + 2\overline{N}(r, \infty; f) - m(r, 1; g)}{T(r, f)} < 1$$

then $f \equiv g$ or $f.g \equiv 1$.

E. Mues and M. Reinders [8] proved the following result.

Theorem 1.3. [5, 8] *Let f and g share $0, \infty$ IM and 1 CM. If*

$$\limsup_{\substack{r\longrightarrow\infty \\ r \in I}} \frac{3\overline{N}(r, 0; f) + 3\overline{N}(r, \infty; f)}{T(r, f)} < 1$$

then $f \equiv g$ or $f.g \equiv 1$.

H. X. Yi improved the above results and proved the following two theorems.

Theorem 1.4. [5, 10] *Let f and g share $1, \infty$ CM and 0 IM. If*

$$\limsup_{\substack{r\longrightarrow\infty \\ r \in I}} \frac{3\overline{N}(r, 0; f) + 2\overline{N}(r, \infty; f) - m(r, 1; g)}{T(r, f)} < 1 \tag{1.2}$$

then $f \equiv g$ or $f.g \equiv 1$.

Theorem 1.5. [5, 10] *Let f and g share $0, \infty$ IM and 1 CM. If*

$$\limsup_{\substack{r\longrightarrow\infty \\ r\in I}} \frac{3\overline{N}(r,0;f)+3\overline{N}(r,\infty;f)-m(r,1;g)}{T(r,f)} < 1 \tag{1.3}$$

then $f \equiv g$ or $f.g \equiv 1$.

To state the next results we have to introduce the notion of gradation of sharing known as weighted sharing.

Definition 1.1. [2, 3] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

With the notion of weighted sharing of values improving Theorem 1.4 and Theorem 1.5 Lahiri [5] proved the following two theorems.

Theorem 1.6. [5] *Let f and g share $(0,0), (1,2), (\infty,\infty)$. If condition (1.2) holds then either $f \equiv g$ or $f.g \equiv 1$.*

Theorem 1.7. [5] *Let f and g share $(0,0), (1,2), (\infty,0)$. If condition (1.3) holds then either $f \equiv g$ or $f.g \equiv 1$.*

Though the standard definitions and notations are available in [2], we explain some notations which are used in the paper.

Definition 1.2. [4] We denote by $N(r,a;f|=1)$ the counting function of simple a points of f.

Definition 1.3. [3, 4] If s is a positive integer, we denote by $\overline{N}(r,a;f|\geq s)$ the reduced counting function of those a-points of f whose multiplicities are not less than s.

Definition 1.4. [11, 12] Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of f and g where $p > q$, by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of f and g where $p = q = 1$ and by $\overline{N}_E^{(2}(r,1;f)$ the counting function of those 1-points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,1;g), N_E^{1)}(r,1;g), \overline{N}_E^{(2}(r,1;g)$.

Definition 1.5. [12] Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_{f>2}\,(r,1;g)$ the reduced counting function of those 1-points of f and g such that $p > q = 2$. $\overline{N}_{g>2}\,(r,1;f)$ is defined analogously.

Definition 1.6. Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_{f>1}(r,1;g)(\overline{N}_{g>1}(r,1;f))$ the reduced counting function of those 1-points of f and g such that $p > q = 1 (q > p = 1)$.

Definition 1.7. Let f and g be two nonconstant meromorphic functions such that f and g share the value $(1,2)$. Let z_0 be a 1-point of f with multplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_E^{(3}(r,1;g)$ the counting function of those 1-points of f and g where $p=q\geq 3$, each point in this counting function is counted only once. In the same way we can define $\overline{N}_E^{(3}(r,1;g)$

Definition 1.8. [3, 5] Let f,g share a value IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)$, and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

Definition 1.9. [4] Let $a,b\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f|\,g=b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.

Definition 1.10. [4] Let $a,b\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f|\,g\neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.

Now one may ask :
Is it possible in any way to replace the condition (1.3) in Theorem 1.7 by a weaker one so that the conclusion of the theorem remain same?
In this paper we will provide an answer to the question. However the author does not know whether the condition (1.2) in Theorem 1.6 can be further relaxed.
In [5] Lahiri raised a problem of further relaxation of the sharing $(1,2)$ in Theorems 1.6 and 1.7.

Inspired by this problem the present author also investigate the situations when the two functions share the value 1 with weight one or zero. We now state the following five theorems which are our main results. The first theorem is an improvement of Theorem 1.7.

Theorem 1.8. *Let f and g share $(0,0),(1,2),(\infty;0)$. If*

$$\limsup_{\substack{r\longrightarrow\infty\\ r\in I}}\frac{3\overline{N}(r,0;f)+3\overline{N}(r,\infty;f)-\overline{N}_E^{(3}(r,1;f)-\overline{N}_L(r,1;g)-m(r,1;g)}{T(r,f)}<1 \tag{1.4}$$

then $f\equiv g$ or $f.g\equiv 1$.

Theorem 1.9. *Let f and g share $(0,0),(1,1),(\infty;\infty)$. If*

$$\limsup_{\substack{r\longrightarrow\infty\\ r\in I}}\frac{3\overline{N}(r,0;f)+2\overline{N}(r,\infty;f)+\overline{N}_{f>2}(r,1;g)-m(r,1;g)}{T(r,f)}<1$$

then $f\equiv g$ or $f.g\equiv 1$.

Theorem 1.10. *Let f and g share $(0,0),(1,0),(\infty;\infty)$. If*

$$\limsup_{\substack{r\longrightarrow\infty\\ r\in I}}\frac{3\overline{N}(r,0;f)+2\overline{N}(r,\infty;f)+N_{\otimes}(r,1;f,g)-m(r,1;g)}{T(r,f)}<1$$

then $f\equiv g$ or $f.g\equiv 1$, where $N_{\otimes}(r,1;f,g)=\overline{N}_L(r,1;f)+\overline{N}_{f>1}(r,1;g)+\overline{N}_{g>1}(r,1;f)$.

Theorem 1.11. *Let f and g share $(0,0),(1,1),(\infty;0)$. If*

$$\limsup_{\substack{r\longrightarrow\infty\\ r\in I}}\frac{3\overline{N}(r,0;f)+3\overline{N}(r,\infty;f)+\overline{N}_{f>2}(r,1;g)-m(r,1;g)}{T(r,f)}<1 \tag{1.5}$$

then $f \equiv g$ *or* $f.g \equiv 1$.

Theorem 1.12. *Let* f *and* g *share* $(0,0),(1,0),(\infty;0)$. *If*

$$\limsup_{\substack{r\longrightarrow\infty \\ r\in I}} \frac{3\overline{N}(r,0;f)+3\overline{N}(r,\infty;f)+N_{\otimes}(r,1;f,g)-m(r,1;g)}{T(r,f)} < 1 \tag{1.6}$$

then $f \equiv g$ *or* $f.g \equiv 1$, *where* $N_{\otimes}(r,1;f,g)=\overline{N}_L(r,1;f)+\overline{N}_{f>1}(r,1;g)+\overline{N}_{g>1}(r,1;f)$.

Example 1.1. *Let* $f=(1-e^z)^3$, $g=\frac{3(e^z-1)}{e^{2z}}$. *Clearly* f,g *share* $(0,0),(\infty,\infty)$ *and* $(1,\infty)$. *Here* $\overline{N}_L(r,1;f)=0$, $\overline{N}_{f>1}(r,1;g)=0$, $\overline{N}_{g>1}(r,1;f)=0$. *Also* $T(r,f)=3T(r,e^z)+O(1)$, $T(r,g)=2T(r,e^z)+O(1)$ *and* $\overline{N}(r,0;f)\sim T(r,e^z)$, $\overline{N}(r,\infty;f)=0$, $N(r,1;g)\sim 2T(r,e^z)$ *but niether* $f\equiv g$ *nor* $fg\equiv 1$. *So the conditions in Theorem 1.9 and Theorem 1.10 are sharp.*

Example 1.2. *Let* $f=\frac{1}{(1-e^z)^3}$, $g=\frac{e^{2z}}{3(e^z-1)}$. *Clearly* f,g *share* $(0,\infty),(\infty,0)$ *and* $(1,\infty)$. *Here* $\overline{N}_E^{(3}(r,1;f)=0$, $\overline{N}_L(r,1;g)=0$. *Again* $T(r,f)=3T(r,e^z)+O(1)$, $T(r,g)=2T(r,e^z)+O(1)$, $\overline{N}(r,0;f)=0$, $\overline{N}(r,\infty;f)\sim T(r,e^z)$, $N(r,1;g)\sim 2T(r,e^z)$ *but niether* $f\equiv g$ *nor* $fg\equiv 1$. *So the condition (1.4) in Theorem 1.8 is sharp. Also* $\overline{N}_L(r,1;f)=0$, $\overline{N}_{f>1}(r,1;g)=0$, $\overline{N}_{g>1}(r,1;f)=0$. *but niether* $f\equiv g$ *nor* $fg\equiv 1$. *So the conditions (1.5) in Theorem 1.11 and (1.6) in Theorem 1.12 are also sharp.*

2. LEMMAS

In this section we present some lemmas which will be needed in the sequel. Henceforth we shall denote by H the following function

$$H=\left(\frac{f''}{f'}-\frac{2f'}{f-1}\right)-\left(\frac{g''}{g'}-\frac{2g'}{g-1}\right).$$

Lemma 2.1. [4] *Let* f, g *share* $(0,0).(1,0),(\infty,0)$ *then*
(i) $T(r,f)\le 3T(r,g)+S(r,f)$.
(ii) $T(r,g)\le 3T(r,f)+S(r,g)$.

Lemma 2.2. [11, 12] *If* f, g *share (1,0) and* $H\not\equiv 0$ *then*

$$N_E^{1)}(r,1;f)\le N(r,H)+S(r,f)+S(r,g).$$

Lemma 2.3. [7] *The following holds*

$$N(r,0;f'|f\neq 0)\le \overline{N}(r,\infty;f)+\overline{N}(r,0;f)+S(r,f).$$

Lemma 2.4. *Let* f *and* g *be two nonconstant meromorphic functions sharing* $(1,0)$. *Then*

$$\begin{aligned}(i)\qquad & \overline{N}_L(r,1;f)+2\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)-\overline{N}_{f>1}(r,1;g)-\overline{N}_{g>1}(r,1;f)\\ \le\ & N(r,1;g)-\overline{N}(r,1;g).\end{aligned}$$

$$\begin{aligned}(ii)\qquad & \overline{N}_L(r,1;g)+2\overline{N}_L(r,1;f)+\overline{N}_E^{(2}(r,1;g)-\overline{N}_{g>1}(r,1;f)-\overline{N}_{f>1}(r,1;g)\\ \le\ & N(r,1;f)-\overline{N}(r,1;f).\end{aligned}$$

Proof. We prove (i) only because (ii) can be proved similarly. Let z_0 be a 1- point of f of multiplicity p a 1-point of g of multiplicity q. We denote by $N_1(r)$, $N_2(r)$ and $N_3(r)$ the counting functions of those 1-points of f and g when $1\le q<p$, $2\le q=p$ and $p<q$ respectively where in the first counting function each point is counted $q-1$ times and in the remaining two counting functions each point is counted $q-2$ times.

Since f, g share $(1,0)$, we note that a simple 1 point of g is either a simple 1 point of f or a 1 point of f with multiplicity ≥ 2. So we can write

$$N(r,1;g)-\overline{N}(r,1;g) = \overline{N}^{(2}_{E}(r,1;f)+\overline{N}_{L}(r,1;g) + N_1(r)+N_2(r)+N_3(r). \tag{2.1}$$

Also we note that

$$N_1(r)\geq \overline{N}_{L}(r,1;f)-\overline{N}_{f>1}(r,1;g), \tag{2.2}$$

$$N_2(r)\geq \overline{N}^{(2}_{E}(r,1;f)-\overline{N}(r,1;f,g|=2), \tag{2.3}$$

$$N_3(r)\geq \overline{N}_{L}(r,1;g)-\overline{N}_{g>1}(r,1;f), \tag{2.4}$$

where by $\overline{N}(r,1;f,g|=2)$ we mean the reduced counting functions of 1-points of f and g with multiplicities two for each one.

Using (2.2)-(2.4) in (2.1) we deduce that

$$\begin{aligned} N(r,1;g)-\overline{N}(r,1;g) \geq\ & \overline{N}_{L}(r,1;f)+2\overline{N}_{L}(r,1;g) \\ & +2\overline{N}^{(2}_{E}(r,1;f)-\overline{N}(r,1;f,g|=2) \\ & -\overline{N}_{f>1}(r,1;g)-\overline{N}_{g>1}(r,1;f). \end{aligned} \tag{2.5}$$

Now (i) follows from (2.5). This proves the lemma. ∎

Lemma 2.5. [12] *If f, g share $(1,1)$ Then*

$$\begin{aligned} (i) \quad & 2\overline{N}_{L}(r,1;f)+2\overline{N}_{L}(r,1;g)+\overline{N}^{(2}_{E}(r,1;f)-\overline{N}_{f>2}(r,1;g) \\ \leq\ & N(r,1;g)-\overline{N}(r,1;g). \end{aligned}$$

$$\begin{aligned} (ii) \quad & 2\overline{N}_{L}(r,1;f)+2\overline{N}_{L}(r,1;g)+\overline{N}^{(2}_{E}(r,1;g)-\overline{N}_{g>2}(r,1;f) \\ \leq\ & N(r,1;f)-\overline{N}(r,1;f). \end{aligned}$$

Lemma 2.6. *Let f and g be two nonconstant meromorphic functions sharing $(1,2)$. Then*

$$\begin{aligned} (i) \quad & 2\overline{N}_{L}(r,1;f)+3\overline{N}_{L}(r,1;g)+2\overline{N}^{(3}_{E}(r,1;f)+\overline{N}(r,1;f|=2) \\ \leq\ & N(r,1;g)-\overline{N}(r,1;g). \end{aligned}$$

$$\begin{aligned} (ii) \quad & 2\overline{N}_{L}(r,1;g)+3\overline{N}_{L}(r,1;f)+2\overline{N}^{(3}_{E}(r,1;g)-\overline{N}(r,1;g|=2) \\ \leq\ & N(r,1;f)-\overline{N}(r,1;f). \end{aligned}$$

Proof. We prove (i) only because (ii) can be proved similarly. Let z_0 be a 1- point of f of multiplicity p, a 1-point of g of multiplicity q. We denote by $N^2_1(r)$, $N^2_2(r)$ and $N^2_3(r)$ the counting functions of those 1-points of f and g when $3\leq q<p$, $3\leq q=p$ and $3\leq p<q$ respectively each point in these counting functions is counted $q-2$ times.

Since f, g share $(1,2)$, we note that

$$N(r,1;g)-\overline{N}(r,1;g) = \overline{N}^{(3}_{E}(r,1;f)+\overline{N}_{L}(r,1;f)+\overline{N}_{L}(r,1;g) +\overline{N}(r,1;f|=2)+N^2_1(r)+N^2_2(r)+N^2_3(r). \tag{2.6}$$

Also we note that

$$N^2_1(r)\geq \overline{N}_{L}(r,1;f), \tag{2.7}$$

$$N^2_2(r)\geq \overline{N}^{(3}_{E}(r,1;f), \tag{2.8}$$

$$N_3^2(r) \geq 2\overline{N}_L(r,1;g), \tag{2.9}$$

Using (2.7)-(2.9) in (2.6) we deduce that

$$N(r,1;g) - \overline{N}(r,1;g) \geq 2\overline{N}_L(r,1;f) + 3\overline{N}_L(r,1;g) + 2\overline{N}_E^{(3}(r,1;f) + \overline{N}(r,1;f| = 2).$$

This proves the lemma. ∎

Lemma 2.7. [5] *Let f, g share $(0,0)$,$(1,0)$,$(\infty,0)$) and $H \not\equiv 0$. Then*

$$\begin{aligned} N(r,H) &\leq \overline{N}_*(r,0;f,g) + \overline{N}_*(r,\infty;f,g) + \overline{N}_*(r,1;f,g) \\ &\quad + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g'), \end{aligned}$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not the zeros of $f(f-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined.

Lemma 2.8. *Let f, g share $(1,2)$. Then*

$$\begin{aligned} &\overline{N}(r,0;f) + \overline{N}(r,\infty;f) - \overline{N}_E^{(3}(r,1;f) - \overline{N}_L(r,1;g) \\ \geq\ &\frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) + \frac{1}{2}N_0(r,0;f') + S(r,f), \end{aligned}$$

where $N_0(r,0;f')$ is the counting function of those zeros of f' which are not the zeros of $f(f-1)$.

Proof. Using Lemma 2.3 we get

$$\begin{aligned} \overline{N}_E^{(3}(r,1;f) + \overline{N}_L(r,1;g) &= \overline{N}_E^{(3}(r,1;g) + \overline{N}_L(r,1;g) \\ &\leq \overline{N}(r,1;g| \geq 3) \\ &= \overline{N}(r,1;f| \geq 3) \\ &\leq \frac{1}{2}N(r,0;f' \big| f = 1) \\ &\leq \frac{1}{2}N(r,0;f' \big| f \neq 0) - \frac{1}{2}N_0(r,0;f') \\ &\leq \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) - \frac{1}{2}N_0(r,0;f') + S(r,f). \end{aligned}$$

So

$$\begin{aligned} &\overline{N}(r,0;f) + \overline{N}(r,\infty;f) - \overline{N}_E^{(3}(r,1;f) - \overline{N}_L(r,1;g) \\ \geq\ &\frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) + \frac{1}{2}N_0(r,0;f') + S(r,f), \end{aligned}$$

This proves the lemma. ∎

Lemma 2.9. *Let f, g share $(0,0)$,$(1,0)$,(∞,k), $0 \leq k \leq \infty$ and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r,f) &\leq 3\overline{N}(r,0;f) + 2\overline{N}(r,\infty;f) + \overline{N}(r,\infty;f| \geq k+1) + \overline{N}_L(r,1;f) \\ &\quad + \overline{N}_{f>1}(r,1;g) + \overline{N}_{g>1}(r,1;f) - m(r,1;g) + S(r,f). \end{aligned}$$

Proof. By the second fundamental theorem we get

$$\begin{aligned} T(r,f) + T(r,g) &\leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;g) \\ &\quad + \overline{N}(r,\infty;g) + \overline{N}(r,1;f) + \overline{N}(r,1;g) \\ &\quad - N_0(r,0;f') - N_0(r,0;g') + S(r,f) + S(r,g). \end{aligned} \tag{2.10}$$

Since f and g share $(0,0)$ and (∞,k), $\overline{N}_*(r,\infty;f,g) \leq \overline{N}(r,0;f)$ and $\overline{N}(r,\infty;f,g) \leq \overline{N}(r,\infty;f|\geq k+1)$. By Lemmas 2.1, 2.2, 2.4 and 2.7 we get

$$\begin{aligned}
(2.11)\qquad & \overline{N}(r,1;f)+\overline{N}(r,1;g)\\
= & N_E^{1)}(r,1;f)+\overline{N}_L(r,1;f)+\overline{N}_L(r,1;g)\\
& +\overline{N}_E^{(2}(r,1;f)+\overline{N}(r,1;g)\\
\leq & N_E^{1)}(r,1;f)+N(r,1;g)+\overline{N}_{f>1}(r,1;g)\\
& +\overline{N}_{g>1}(r,1;f)-\overline{N}_L(r,1;g)\\
\leq & \overline{N}(r,0;f)+\overline{N}(r,\infty;f|\geq k+1)+\overline{N}_*(r,1;f,g)\\
& +T(r,g)-m(r,1;g)+O(1)+\overline{N}_{f>1}(r,1;g)\\
& +\overline{N}_{g>1}(r,1;f)-\overline{N}_L(r,1;g)+\overline{N}_0(r,0;f^{'})\\
& +\overline{N}_0(r,0;g^{'})+S(r,f)+S(r,g)\\
\leq & \overline{N}(r,0;f)+\overline{N}(r,\infty;f|\geq k+1)+T(r,g)\\
& -m(r,1;g)+\overline{N}_L(r,1;f)+\overline{N}_{f>1}(r,1;g)\\
& +\overline{N}_{g>1}(r,1;f)+\overline{N}_0(r,0;f^{'})+\overline{N}_0(r,0;g^{'})+S(r,f)
\end{aligned}$$

Using (2.11) in (2.10) and noting that $\overline{N}(r,0;f)=\overline{N}(r,0;g)$ and $\overline{N}(r,\infty;f)=\overline{N}(r,\infty;g)$ we obtain the conclusion of the lemma. This proves the lemma. ∎

Lemma 2.10. *Let f, g share $(0,0)$,$(1,1)$,(∞,k), $0\leq k\leq\infty$ and $H\not\equiv 0$. Then*

$$\begin{aligned}
T(r,f) \leq\ & 3\overline{N}(r,0;f)+2\overline{N}(r,\infty;f)+\overline{N}(r,\infty;f|\geq k+1)+\overline{N}_{f>2}(r,1;g)\\
& -m(r,1;g)+S(r,f).
\end{aligned}$$

Proof. We omit the proof since using Lemmas 2.1, 2.2, 2.5 and 2.7 the proof of the lemma can be carried out in the line of Lemma 2.9. ∎

Lemma 2.11. *Let f, g share $(0,0)$,$(1,2)$,$(\infty,0)$ and $H\not\equiv 0$. Then*

$$\begin{aligned}
T(r,f) \leq\ & 3\overline{N}(r,0;f)+3\overline{N}(r,\infty;f)-\overline{N}_E^{(3}(r,1;f)\\
& -\overline{N}_L(r,1;g)-m(r,1;g)+S(r,f).
\end{aligned}$$

Proof. By the second fundamental theorem we get

$$\begin{aligned}
(2.12)\qquad T(r,f)+T(r,g) \leq\ & \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)\\
& +\overline{N}(r,\infty;g)+\overline{N}(r,1;f)+\overline{N}(r,1;g)\\
& -N_0(r,0;f^{'})-N_0(r,0;g^{'})+S(r,f)+S(r,g).
\end{aligned}$$

Since f, g share $(1,2)$ implies $N_E^{1)}(r,1;f) = N(r,1;f\big| = 1)$, by Lemmas 2.1, 2.2, 2.6 and 2.7 we see that

$$
\begin{aligned}
(2.13)\qquad & \overline{N}(r,1;f) + \overline{N}(r,1;g) \\
= \;& N(r,1;f\big| = 1) + \overline{N}(r,1;f\big| = 2) + \overline{N}_E^{(3}(r,1;f) \\
& +\overline{N}_L(r,1;f) + \overline{N}_L(r,1;g) + \overline{N}(r,1;g) \\
\le \;& N(r,1;f\big| = 1) + \overline{N}(r,1;f\big| = 2) + \overline{N}_E^{(3}(r,1;f) \\
& +\overline{N}_L(r,1;f) + \overline{N}_L(r,1;g) + N(r,1;g) - 2\overline{N}_L(r,1;f) \\
& -3\overline{N}_L(r,1;g) - 2\overline{N}_E^{(3}(r,1;f) - \overline{N}(r,1;f\big| = 2) \\
\le \;& \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}_*(r,1;f,g) + T(r,g) \\
& -m(r,1;g) + O(1) - \overline{N}_L(r,1;f) - 2\overline{N}_L(r,1;g) \\
& -\overline{N}_E^{(3}(r,1;f) + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g') \\
& +S(r,f) + S(r,g) \\
\le \;& \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + T(r,g) - m(r,1;g) \\
& -\overline{N}_E^{(3}(r,1;f) - \overline{N}_L(r,1;g) + \overline{N}_0(r,0;f') \\
& +\overline{N}_0(r,0;g') + S(r,f)
\end{aligned}
$$

From (2.12) and (2.13) the lemma follows. This proves the lemma. ∎

Lemma 2.12. [10] *If f, g share $(0,0)$,$(1,0)$,$(\infty,0)$ and $H \equiv 0$. Then f,g share $(0,\infty)$,$(1,\infty)$, (∞,∞).*

3. Proofs of the theorems

Proof of Theorem 1.8. Suppose $H \not\equiv 0$. Then from Lemma 2.11 and condition (1.4) we get a contradiction. So $H \equiv 0$. Hence by Lemma 2.12 f and g share $(0,\infty)$, $(1,\infty)$, (∞,∞). Now Lemma 2.8 and condition (1.4) implies condition (1.1) of Theorem 1.1. So by Theorem 1.1 the theorem follows. This proves the theorem. ∎

Proof of Theorem 1.11. Since f, g share $(\infty;0)$ $\overline{N}(r,\infty;f\big| \ge k+1) = \overline{N}(r,\infty;f)$. Suppose $H \not\equiv 0$. Then from Lemma 2.10 and condition (1.5) we get a contradiction. So $H \equiv 0$. Now the theorem follows from Lemma 2.12 and Theorem 1.1. This proves the theorem. ∎

Proof of Theorem 1.12. Since f, g share $(\infty;0)$ $\overline{N}(r,\infty;f\big| \ge k+1) = \overline{N}(r,\infty;f)$. Suppose $H \not\equiv 0$. Then from Lemma 2.9 and condition (1.6) we obtain a contradiction. So $H \equiv 0$ and the theorem follows from Lemma 2.12 and Theorem 1.1. This completes the proof of the theorem. ∎

Proof of Theorem 1.9. Suppose $H \not\equiv 0$. Since f, g share $(\infty;\infty)$ we obtain from Lemma 2.10 for $k = \infty$

$$T(r,f) \le 3\overline{N}(r,0;f) + 2\overline{N}(r,\infty;f) + \overline{N}_{f>2}(r,1;g) - m(r,1;g) + S(r,f)$$

which leads to a contradiction. So $H \equiv 0$. Now the theorem follows from Lemma 2.12 and Theorem 1.1. This proves the theorem. ∎

Proof of Theorem 1.10. Using Lemma 2.9 for $k = \infty$ and proceeding in the same way as in the proof of Theorem 1.9 we can prove the theorem. This proves the theorem. ∎

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