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AN EASY AND EFFICIENT WAY FOR SOLVING A CLASS OF SINGULAR TWO POINT BOUNDARY VALUE PROBLEMS

MUHAMMED I. SYAM, MUHAMMED N. ANWAR AND BASEM S. ATTILI

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MATHEMATICAL SCIENCES DEPARTMENT, UNITED ARAB EMIRATES UNIVERSITY, P. O. BOX 17551, AL-AIN, UNITED ARAB EMIRATES,
b.attili@uaeu.ac.ae

ABSTRACT. We will consider an efficient and easy way for solving a certain class of singular two point boundary value problems. We will employ the least squares method which proved to be efficient for this type of problems. Enough examples that were considered by others will be solved with comparison with the results presented there.

Key words and phrases: Two Point BVP's, Singular points, Least Squares.

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1. INTRODUCTION

Singular two point boundary value problems for ordinary differential equations arise very frequently in several areas of science and engineering. For example; the analysis of heat conduction through a solid with heat generation leads to a boundary value problem that involve the solution of an ordinary differential equation of the form

$$\frac{d}{dx}\left(p(x)\frac{dT}{dx}\right)+q(x)T=f(x,T)$$

where the dependent variable represents the temperature and the right hand function represents the heat generation. Furthermore, the well-known Thomas-Fermi model in atomic physics that describes the charge concentration $y(x)$ of electrons in an ion is governed by a special type of singular ordinary differential equation of the form

$$\sqrt{x}y''(x)=\sqrt{y(x)}.$$

Very often singularities are encountered at one or more points in the interval over which the problem is defined. We mention here two examples to illustrate our point.

(1) When separation of variables is attempted on the heat equation in a solid sphere or the electrostatic potential in the sphere, the singular equation
$$-\frac{1}{\sin(\varphi)}\left[\Phi'(\varphi)\sin(\varphi)\right]'+\lambda\Phi(\varphi)=0,\ \varphi\in[0,\pi]$$
appears. The source of the singularity here is the vanishing of the function p at the endpoints.

(2) The equation
$$-\left(\left(1-x^2\right)u'\right)'=f(x),\ x\in[-1,1]$$
represents the steady state temperature distribution in a bar extending from -1 to 1 if the thermal conductivity is $(1-x^2)$. The same type of singularity occurs here also.

Other areas of engineering where such singular problems occur is the analysis of behavior of magnetoelectroelastic material, which has the ability of converting the magnetic, electric, and mechanical; energies from one to another. The mathematical modeling of characteristics of such material Chandrasekhar[4] is a very active area of research and investigations. Therefore, the solution of such model represents a challenge, which involves solving a particular singular boundary value problem of a form similar to the above equations. Another area of application is the modeling and analysis of the dynamics of compressible perfect fluids such as plasmas and dilute gases Davis[9]. Many other examples and applications involving singular boundary value problems are cited in the work of Weinitschke[32] and Schneider[28].

The numerical treatment of such singular boundary value problems has always been far from trivial, because the existence of the singularity in the underlying equation. Several authors have been extensively involved in the solution of such class of problems and numerous innovative methods and approach have enriched the scientific literature; each with its particular merits and advantages.

The starting of some serious work on the numerical methods for the previously mentioned problem was in 1965, when Parter[24] and in his study of numerical methods for generalized axially symmetric potentials in a rectangle was led to a singular second order BVP. He considered finite difference methods to approximate the solution of such singular problem. Others that considered finite differences are Jamet[17], Doedel and Reddien[10], Chawla, McKee and Shaw[5] and Chawla, Subramanian and Sathi[6], Cohen and Jones[8] and Gustafsson[16]. Different approaches and order of convergence were achieved depending on the type of the singularity in addition to stability and convergence analysis. Rayleigh-Ritz-Galerkin method was considered by Ciarlet, Natterer and Varga[7] and Jespersen[18] and Eriksson and Thome'e[13]. Other methods like collocation was considered by Reddien[26]. Invariant embedding was considered by Attili[2], Scott[29], Elder[11], Kadalbajoo, and Raman[19] and Nelson[23]. Recently, general existence and uniqueness results for solutions of the singular problem were given in Fink, Gatica, Hrenandez and Waltman[14] and Baxley[3] while for the eigenvalue problem was considered by Nassif[22]. Initial value methods through shooting was employed by Attili, Elgindi and Elgebeily[1] produced a convergence rate of at least 2 when Runge-Kutta of order 4 is used. This is due to the singularity. A more accurate iterative shooting method is given by Elgebeily and Attili[12]. Cubic splines with the series solution as explained in the invariant embedding case above was used successfully by Kadalbajoo and Aggarwal[20].

We will employ a traditional easy to use least squares method which is one of the weighted residual methods. It proved to be efficient in computing such singularities. What makes it attractive is its easiness and accurate results it yields. It compares and competes very well with other methods mentioned above in terms of accuracy and amount of work. We will present the method in the nest section. Enough examples will be solved in the final section that illustrates the point.

2. LEAST SQUARES METHOD

We will present here an overview of weighted residual methods since the least squares method is one special case, see Finlayson[15] and Lapidus and Pinder[21]. It is well known that these are some approximation methods to solve arbitrary linear and nonlinear differential equations. Both, ordinary and partial differential equations, are considered. We will start with the following one-dimensional differential equation

$$Lu + f = 0, \quad x \in [a, b], \tag{2.1}$$

where $u(x)$ is the unknown function and $f(x)$ is a given function and L denotes a linear differential operator which specifies the actual form of the differential equation (2.1). For example $L = \frac{d}{dx}$ or $L = \frac{d^2}{dx^2}$. The boundary conditions can be given in simple form by

$$u(a) = u_a, \quad u(b) = u_b. \tag{2.2}$$

Multiplying (2.1) by an arbitrary weight function $v(x)$ and integrating over the interval $[a, b]$ one obtains

$$\int_a^b v\,(Lu + f)\,dx = 0 \tag{2.3}$$

It is clear that (2.1) and (2.3) are equivalent, because $v(x)$ is an arbitrary function. This means we seek a numerical solution to (2.3) subject to (2.2) in the form

$$\hat{u} = \alpha_1\varphi_1 + \alpha_2\varphi_2 + \ .... \ + \alpha_n\varphi_n \tag{2.4}$$

Here $\varphi_1,\ \varphi_2,\ \dots,\ \varphi_n$ are basis functions of x that form a linearly independent and complete set and $\alpha_1,\ \alpha_2,\ \dots,\ \alpha_n$ are unknown coefficients that need to be determined. In vector form (2.4) becomes

$$\hat{u} = \Psi a \tag{2.5}$$

where

$$a = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ \alpha_n \end{pmatrix}; \quad \Psi = (\varphi_1,\ \varphi_2,\ \dots,\ \varphi_n)$$

Substituting $\hat{u}(x)$ given by (2.4) in place of $u(x)$ in (2.3), we are supposed to obtain

$$\int_a^b v\,(L\hat{u} + f)\,dx = 0, \tag{2.6}$$

but in fact replacing $u(x)$ by its approximation $\hat{u}(x)$ in (2.1), it will not be satisfied exactly; that is,

$$Lu + f = e, \tag{2.7}$$

where $e(x)$ is a measure for the error called the residual. This leads to

$$\int_a^b v(x)e(x)dx = 0, \tag{2.8}$$

Obviously, the residual, $e(x)$ depends on the unknown parameters given by vector a; that is, depends on $\alpha_1,\ \alpha_2,\ \dots,\ \alpha_n$. Therefore the coefficients $\alpha_1,\ \alpha_2,\ \dots,\ \alpha_n$ must be determined so that (2.8) is satisfied. Generally

$$v(x) = b_1 v_1(x) + b_2 v_2(x) + \ \dots\ + b_n v_n(x)$$

where $v_1(x),\ v_2(x),\ \dots, v_n(x)$ are known functions of x known as basis and $b_1,\ b_2,\ \dots,\ b_n$ are certain parameters. In vector format it is written as

$$v(x) = bV$$

where $V = (v_1(x),\ v_2(x),\ \dots, v_n(x))$ and $b = (b_1,\ b_2,\ \dots,\ b_n)$. Substituting in (2.8) leads to

$$b^T \int_a^b V^T(x)e(x)dx = 0. \tag{2.9}$$

Since b^T is arbitrary, this means

$$\int_a^b V^T(x)e(x)dx = 0, \tag{2.10}$$

or

$$\int_a^b v_1(x)e(x)dx = 0,\ \int_a^b v_2(x)e(x)dx = 0,\ \dots,\ \int_a^b v_n(x)e(x)dx = 0. \tag{2.11}$$

Now, we have n equations in n unknowns; that is, the coefficients $\alpha_1,\ \alpha_2,\ \dots,\ \alpha_n$. Using (2.5) and (2.7) yields

$$e(x) = L(\Psi a) + f = L(\Psi).a + f$$

and the condition (2.10) can be rewritten as

$$\left(\int_a^b V^T(x)L(\Psi)dx\right).a = -\int_a^b V^T(x)f(x)dx.$$

In matrix form

$$Ka = h, \tag{2.12}$$

where

$$K = \int_a^b V^T(x)L(\Psi)dx, \quad h = -\int_a^b V^T(x)f(x)dx.$$

A linear system of n equations in n coefficients $\alpha_1,\ \alpha_2,\ \dots,\ \alpha_n$ to be determined.

The choice of $v_i(x)$ and hence $V(x)$ determines the method. In the least square method, which is under consideration here, the functions $v_i(x)$ are defined as

$$v_i(x) = \frac{\partial e}{\partial \alpha_i}; \quad i = 1,\ \dots,\ n.$$

Accordingly (2.11) in this case becomes

$$\int_a^b \frac{\partial e}{\partial \alpha_i} e(x)dx. = 0; \quad i = 1,\ \dots,\ n.$$

This is due to the fact that to determine the parameters $\alpha_1,\ \alpha_2,\ \dots,\ \alpha_n$, it will involve minimizing

$$E(\alpha_1,\ \alpha_2,\ \dots,\ \alpha_n) = \int_a^b e^2(x)dx.$$

over the interval $[a, b]$. A sufficient condition is

$$\frac{\partial E(\alpha_1,\ \alpha_2,\ \dots,\ \alpha_n)}{\partial \alpha_i} = 2\int_a^b e(x)\frac{\partial e}{\partial \alpha_i}dx. = 0;\ i = 1,\ \dots,\ n.$$

In explicit form the linear system (2.11) has the form

$$\begin{bmatrix} \int_a^b L(\varphi_1)L(\varphi_1)dx. & \cdots & \int_a^b L(\varphi_1)L(\varphi_n)dx \\ \int_a^b L(\varphi_2)L(\varphi_1)dx & \cdots & \int_a^b L(\varphi_2)L(\varphi_n)dx \\ \vdots & & \vdots \\ \int_a^b L(\varphi_n)L(\varphi_1)dx & & \int_a^b L(\varphi_n)L(\varphi_n)dx \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \\ \alpha_n \end{bmatrix} = -\begin{bmatrix} \int_a^b L(\varphi_1)fdx. \\ \int_a^b L(\varphi_2)fdx. \\ \\ \int_a^b L(\varphi_n)fdx. \end{bmatrix},$$

with K symmetric in this case, for more see see Finlayson[15]. Note that if $\varphi_1,\ \varphi_2,\ \dots,\ \varphi_n$, the basis functions are mutually orthogonal (if the sequence is not orthogonal we will accept that we can always reduce it to an orthogonal one) and complete set, the matrix K in the system above will reduce to a diagonal system making the calculations easier.

3. NUMERICAL EXPERIMENTS AND RESULTS

To demonstrate the simplicity and accuracy of the proposed method we will consider several examples cited by several authors in the literature. Consider the following examples.

Example 3.1. *We consider the singular initial value problem governed by a Lane-Emden type equation given by:*

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} - 2(x^2+3)y = 0,\ 0 < x < 1$$

$$y(0) = 1,\ \ y'(0) = 0$$

whose exact solution is $y(x) = e^{x^2}$. Such equation appears in many applications among them the governing mathematical models of various problems in physics, astrophysics, and thermal behavior of gas clouds [5], [9], and [27]. We have used two polynomial representations namely:

$$P_N(x) = \sum_{i=0}^{N} a_i x^i,\ \ N = 5,\ 9.$$

Implementing the proposed method, we arrive at

$$P_5(x) = 1 + 0.942595x^2 + 0.516105x^3 - 0.700402x^4 + 0.960801x^5$$

and

$$\begin{aligned} P_9(x) \ &= \ 1 + 0.999932x^2 + 0.00237407x^3 + 0.476123x^4 + 0.110602x^5 \\ &\quad -0.109141x^6 + 0.3886159x^7 - 0.258623x^8 + 0.108399x^9 \end{aligned}$$

Table 3.1 gives the absolute errors in both polynomials at selected values of x while Figure 3 represents the graphs of the exact solution with $P_5(x)$.

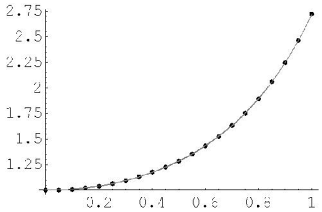

Figure 1: The graphs of y(x) and P5(x)

	Least Squares Technique			Least Squares Technique	
x_i	$\lvert y(x_i) - P_5(x_i)\rvert$	$\lvert y(x_i) - P_9(x_i)\rvert$	0.3	$1.26(-6)$	$3.84(-10)$
0	0.0	0.0	0.4	$2.24(-6)$	$3.66(-10)$
0.02	$3.28(-6)$	$2.11(-10)$	0.5	$2.38(-6)$	$4.49(-10)$
0.04	$1.20(-6)$	$1.78(-10)$	0.6	$1.42(-6)$	$8.15(-10)$
0.06	$1.11(-6)$	$3.21(-10)$	0.7	$1.05(-6)$	$3.95(-10)$
0.08	$1.24(-6)$	$3.00(-10)$	0.8	$1.02(-6)$	$2.12(-10)$
0.1	$1.68(-6)$	$1.69(-10)$	0.9	$3.55(-6)$	$5.49(-10)$
0.2	$2.08(-6)$	$5.73(-10)$	1.0	$8.17(-6)$	$5.92(-10)$

$Table(3.1)$

Results reveal the efficiency and computational accuracy of the proposed method. The above problem was considered by Wazwaz[30, 31]. The solutions he obtained were based on series solution using Adomian decomposition method. The solutions obtained have similar form to Taylor series expansion about $x = 0$. This means there will be some convergence problems when the point is far from zero. For example, the absolute errors at $x = 1$ using the polynomials of degree five and nine generated by the Adomian Decomposition method are 0.218282 and 0.175226, respectively.

Example 3.2. *We consider the singular initial value problem governed by the inhomogenous Lane-Emden type equation namely:*

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y = 6 + 12x + x^2 + x^3, \quad 0 < x < 1$$

$$y(0) = 0, \ \ y'(0) = 0$$

whose exact solution is$y(x) = x^2 + x^3$. Again this problem was studied by Wazwaz [31] where its solution was obtained using Adomian decomposition method leading to a power series solution that converges to the exact solution upon adopting the noise terms phenomenon. We implemented the least squares method using the fourth order polynomial expression of the form

$$P_4(x) = a_2x^2 + a_3x^3 + a_4x^4$$

which resulted in obtaining the exact solution

$$P_4(x) = x^2 + x^3$$

Note that we solve only a system of three equations in three unknowns.

Example 3.3. *Consider the two point boundary value problem governed by the Bessel equation of the form*

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0 \quad , \qquad 0 < x < 1$$

$$y'(0) = 0, y(1) = 1$$

whose exact solution is $y(x) = \frac{J_0(x)}{J_0(1)}$ with $J_0(x)$ is the Bessel function of first kind of order zero. This example was studied in [25] using a cubic spline approach and subtracting the singularity by resorting to a power series solution in a neighborhood of the singular point. The cubic spline technique will be costly to apply compared to the least squares approach to yield the two polynomial representations

$$P_N(x) = \sum_{i=0}^{N} a_i x^i, \ N = 4, 7.$$

With

$$P_4(x) = 1.306839 - 0.327112x^2 + 0.001956x^3 + 0.018317x^4$$

and

$$\begin{aligned} P_7(x) &= 1.306851 - 0.326713x^2 + 0.0000027237616x^3 - 0.020404x^4 \\ &+0.0000389x^5 - 0.0.00061807x^6 + 0.0000334x^7. \end{aligned}$$

Table 3.2 gives the resulting values of absolute errors attained for both cases above at specific values of x . Here $ECS(x)$ means the absolute error using the cubic spline technique [25] with step size $h = \frac{1}{40}$. Moreover, Figure 3 represents the graphs of the exact solution with $P_4(x)$.

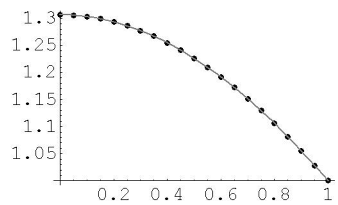

Figure 2: The graphs of the exact solution with P4(x).

x_i	$\lvert y(x_i) - P_4(x_i)\rvert$	$\lvert y(x_i) - P_7(x_i)\rvert$	$ECS(x)$
0	$1.32(-6)$	$3.68(-14)$	$1.16(-4)$
0.025	$1.21(-6)$	$2.80(-14)$	$1.16(-4)$
0.05	$1.12(-6)$	$6.14(-14)$	$1.16(-4)$
0.075	$1.11(-6)$	$2.34(-14)$	$1.16(-4)$
0.1	$1.54(-6)$	$3.02(-14)$	$1.16(-4)$
0.2	$1.68(-6)$	$7.62(-14)$	$1.15(-4)$
0.3	$1.29(-5)$	$7.66(-14)$	$1.16(-4)$
0.4	$3.32(-5)$	$4.63(-14)$	$1.13(-4)$
0.5	$9.03(-6)$	$1.13(-14)$	$1.10(-4)$
0.6	$1.96(-5)$	$3.17(-14)$	$1.07(-4)$
0.7	$2.37(-6)$	$7.66(-14)$	$1.02(-4)$
0.8	$1.91(-5)$	$5.12(-14)$	$9.60(-5)$
0.9	$7.95(-6)$	$3.72(-15)$	$8.20(-5)$
1	0.0	$1.21(-16)$	$7.10(-5)$

$Table(3.2)$

Results show that the proposed method gives considerable small errors in addition to its straight forward implementation

Example 3.4. *We consider the nonlinear homogeneous singular two-point boundary value problem governed by:*

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + xy^5 = 0 \quad , \qquad 0 < x < 1$$

$$y(0) = 1,\ y(1) = \frac{\sqrt{3}}{2}$$

whose exact solution is$y(x) = \frac{\sqrt{3}}{\sqrt{3+x^2}}$. Implementing the proposed method using the fourth order polynomial expression of the form:

$$P_4(x) = \frac{\sqrt{3}}{2} + (x-1)\sum_{i=0}^{3} a_i x^i$$

We arrive at:

$$P_4(x) = 1 + 0.0047629x - 0.19389x^2 + 0.0541207x^3 + 0.001040x^4.$$

Table 3.3 gives the absolute errors attained at specific values of x .

x_i	$\lvert y(x_i) - P_4(x_i)\rvert$	x_i	$\lvert y(x_i) - P_4(x_i)\rvert$
0	0.0	0.6	$4.00(-5)$
0.1	$1.40(-5)$	0.7	$2.20(-5)$
0.2	$5.50(-5)$	0.8	$3.10(-5)$
0.3	$9.90(-5)$	0.9	$6.00(-6)$
0.4	$3.70(-5)$	1	$2.11(-14)$
0.5	$4.80(-5)$		

$Table(3.3)$

The same problem was considered by Attili, Elgindi and Elgebeily[1], the errors they obtained at $x = 1.0$ using Runge-Kutta 4th order method were $2.298807-3$, $3.531472-4$, $8.470276-5$, $2.172234-5$, $5.610755-6$ and $1.391789-6$ at $\frac{1}{2^N};\ N = 2, 3,\ \ldots, 7$ respectively. Results we obtained here for such nonlinear problem confirm the efficiency of the proposed method and compares well with the results of others.

Example 3.5. *Consider the singular two-point boundary value problem with oscillatory coefficients; namely:*

$$\sin(\frac{\pi x}{2})\frac{d^2y}{dx^2} + \frac{\pi}{2}\cos(\frac{\pi x}{2})\frac{dy}{dx} + \frac{\pi^2}{2}\sin(\frac{\pi x}{2})y = 0 \ \ ,0 < x < 1$$

$$y'(0) = 0,\ \ y(1) = 0$$

whose exact solution is $y(x) = \cos(\frac{\pi x}{2})$. We implemented the proposed method using the fifth and ninth order polynomial expressions of the form:

$$P_N(x) = (x-1)\sum_{i=0}^{N} a_i x^i,\ N = 4, 8$$

Implementing the proposed method; we arrive at:

$$\begin{aligned} P_4(x) \ &= \ (x-1)(-1 - 0.998237x + 0.221297x^2 \\ &\quad +0.268343x^3 - 0.062113x^4) \end{aligned}$$

and

$$\begin{aligned} P_9(x) &= (x-1)(-1-0.999999x+0.2336999x^2+0.233707x^3-0.020001x^4 \\ &\quad -0.019876x^5+0.0007334x^6+0.00106060x^7+0.000119x^8) \end{aligned}$$

Table 3.4 gives the absolute errors at specific values of x and Figure 3 represents the graphs of the exact solution against $P_4(x)$.

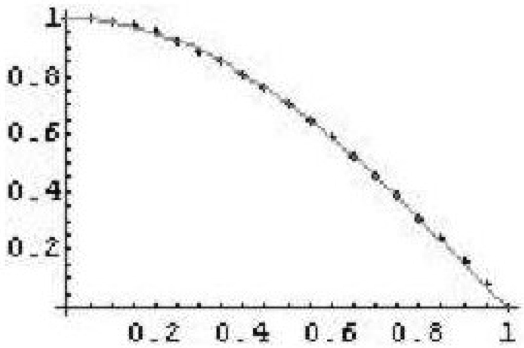

Figure 3: The graphs of the exact solution with P4(x).

x_i	$\lvert y(x_i)-P_4(x_i)\rvert$	$\lvert y(x_i)-P_8(x_i)\rvert$	x_i	$\lvert y(x_i)-P_4(x_i)\rvert$	$\lvert y(x_i)-P_8(x_i)\rvert$
0	0.0	0.0	0.4	3.40(−7)	2.12(−14)
0.025	2.20(−7)	3.81(−14)	0.5	4.00(−7)	2.39(−14)
0.05	1.70(−7)	1.19(−14)	0.6	4.00(−7)	2.34(−14)
0.075	5.40(−7)	4.06(−14)	0.7	2.90(−7)	1.22(−14)
0.1	7.40(−7)	2.62(−14)	0.8	1.50(−7)	9.94(−15)
0.2	5.70(−7)	3.29(−14)	0.9	5.21(−8)	6.04(−15)
0.3	3.70(−7)	3.26(−14)	1	0.0	2.01(−16)

$Table(3.4)$

The same example was considered by Elgebeily and Attili[12] using iterative shooting and the errors given their using $h=\frac{1}{2^{16}}$ were $1.901388E-5$ and $1.085963E-10$ at $x=0.5$ and $x=1.0$ respectively. Our results compare very well with these results.

4. CONCLUSIONS

In this paper an approximate method based on polynomial presentation of the solution of a class of singular second order two-point boundary, and initial value problems is presented. The method has been demonstrated to be characterized by its simplicity, efficiency, and accuracy. It has been implemented to obtain the solutions of a number of problems. The problems considered are selected due to their importance in many applications in Engineering and Physics. Furthermore, for the sake of comparison the above problems are chosen because their solutions are either known in analytical closed form, or are numerically obtained by other authors using different approximate approaches.

Numerical results obtained were accurate. The amount of work compares well with other proposed methods. The proposed method has been tested for solving problems involving singular linear, nonlinear, homogeneous, and inhomogenous equations leading to results of excellent agreements with exact solutions and achieving smaller absolute errors than other methods available in the literature.

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