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ON ZEROS OF DIAGONALLY QUASICONVEX MULTIFUNCTIONS

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ABSTRACT. In this paper, we extended the notion of diagonally quasiconvexity for multifunctions and established several existence results for zeros of diagonally quasiconvex multifunctions. As applications we obtain the results of fixed points, coincidence points and best approximations for multifunctions. Using our result we also prove the existence of solutions to the variational-like inequality problem and generalized vector equilibrium problem. The results of this paper generalize some known results in the literature.

Key words and phrases: Fixed point theorems, Coincidence theorems, Multifinctions, Best approximations, Variational inequalities, KKM-map.

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1. INTRODUCTION

Using the methods of the KKM theory, see for example [12, 17], in this paper we prove certain results on the existence of zeros of multifunctions. As corollaries some results of Q. H. Ansari [1], Q. H. Ansari and J. -C. Yao [3], E. Tarafdar [15], F. E. Browder [5], W. Takahashi [16] and Ky Fan [8, 9] are obtained.

We shall use the following notation and results. Let A be a nonempty subset of a topological vector space X. We denote by 2^A the family of all subsets of A. If A is a non-empty subset of a topological vector space X, we shall denote by $int_X(A)$ and $co(A)$ the interior of A in X and the convex hull of A, respectively. Let X and Y be two topological vector spaces. Let $F : X \to 2^Y$ be a multifunction. The inverse of F defined by

$$x \in F^{-1}(y) \text{ if and only if } y \in F(x).$$

Let C be a convex nonempty subset of X. A multifunction $H(x, y) : C \times C \to 2^Y$ is said to be diagonally quasiconvex in y if, for any finite subset $\{y_1, \ldots, y_n\} \subset C$ and any $y_0 \in co\{y_1, \ldots, y_n\}$, we have

$$\bigcap_{i=1}^{n} H(y_0, y_i) \subseteq H(y_0, y_0). \tag{1.1}$$

A multifunction $H(x, y) : C \times C \to 2^Y$ is said to be diagonally quasiconcave in y if, for any finite subset $\{y_1, \ldots, y_n\} \subset C$ and any $y_0 \in co\{y_1, \ldots, y_n\}$, we have

$$H(y_0, y_0) \subseteq \bigcup_{i=1}^{n} H(y_0, y_i). \tag{1.2}$$

Remark 1.1. 1. Let $H : C \times C \to 2^{\mathbb{R}}$ be a multifunction, such that

$$H(x, y) = (\phi(x, y), +\infty), \text{ for all } x, y \in C,$$

where $\phi : C \times C \to \mathbb{R}$ is a single-valued function. Then the condition (1.1) reduces to

$$\phi(y_0, y_0) \leq \max_{1 \leq i \leq n} \phi(y_0, y_i),$$

and condition (1.2) to

$$\min_{1 \leq i \leq n} \phi(y_0, y_i) \leq \phi(y_0, y_0),$$

and we have generalization of diagonally quasiconvex and diagonally quasiconcave functions, see for example [2, 18].

2. Let $H : C \times C \to 2^{\mathbb{R}}$ be a multifunction, such that

$$H(x, y) = (||f(y) - x||, +\infty), \text{ for all } x, y \in C,$$

where $f : C \to X$ is a single-valued function. Then the condition (1.1) reduces to

$$||f(y_0) - y_0|| \leq \max_{1 \leq i \leq n} ||f(y_i) - y_0||,$$

and we obtain generalization of

(i) almost affine mapping,

$$||f(\lambda y_1 + (1 - \lambda)y_2) - u|| \leq \lambda||f(y_1) - u|| + (1 - \lambda)||f(y_2) - u||,$$

(ii) and almost quasiconvex mapping,

$$||f(\lambda y_1 + (1 - \lambda)y_2) - u|| \leq \max\{||f(y_1) - u||, ||f(y_2) - u||\},$$

see for example [7, 11, 12, 13, 14].

3. Let $H : C \times C \to 2^{\mathbb{Y}}$ be a multifunction, such that
$$H(x, y) = \{z : z - g(x, y) \in K\}, \text{ for all } x, y \in C,$$
where Y is a topological vector space with a closed and convex cone K and $g(x, y) : C \times C \to Y$ is a single-valued function. Then the condition (1.1) reduces to, for all $z \in Y$ the set $\{y : g(x, y) \leq_K z\}$ is convex and we have generalization of K-quasiconvex function, see for example [6].

A multifunction $G : X \to 2^Y$ is called a KKM-map if, for every finite subset $\{x_1, \dots, x_n\}$ of X, $co\{x_1, \dots, x_n\} \subset \bigcup\limits_{i=1}^{n} G(x_i)$.

The following version of Fan-KKM type theorem, see for example [12], will be used to prove the main result of this paper.

Theorem 1.1. *Let C be a nonempty convex set in a topological vector space X. For each $y \in C$ let $G(y)$ be a nonempty closed subset of C and let $G : C \to 2^C$ be a KKM-map. If there is a nonempty subset B_0 of C such that the intersection $\bigcap\limits_{y \in B_0} G(y)$ is compact and B_0 is contained in a compact convex subset of C then $\bigcap\limits_{y \in C} G(y) \neq \emptyset$.*

2. EXISTENCE RESULTS

Theorem 2.1. *Let X and Y be topological vector spaces, C a nonempty convex subset of X, and $H : C \times C \to 2^Y$ a multifunction such that*

(a) *$H(x, y)$ is a diagonally quasiconvex multifunction in y and $H(x, y)$ is nonempty for all $x, y \in C$,*
(b) $C = \bigcup\limits_{y \in C} int_C\{x \in C : 0 \in H(x, y)\}$,
(c) *there exists a non-empty subset B_0 of C such that intersection*
$$\bigcap_{y \in B_0} C \setminus int_C\{x \in C : 0 \in H(x, y)\}$$
is compact and B_0 is contained in a compact convex subset of C.

Then there exists $y_0 \in C$ such that $0 \in H(y_0, y_0)$.

Proof. Assume that
$$0 \notin H(y, y) \text{ for all } y \in C.$$
Define $G : C \to 2^C$ by
$$G(y) = C \setminus int_C\{x \in C : 0 \in H(x, y)\}.$$
Suppose that
$$G(y) = \emptyset \text{ for some } y \in C.$$
Then
$$y \in \{x \in C : 0 \in H(x, y)\}$$
and thus
$$0 \in H(y, y),$$
a contradiction of our assumption. Then $G(y)$ is nonempty and closed in C. From assumption (c) we have, $\bigcap\limits_{y \in B_0} G(y)$ compact and B_0 is contained in a compact convex subset of C. From

assumption (b) we have,

$$\bigcap_{y\in C} G(y) = \left(\bigcup_{y\in C} int_C\{x \in C : 0 \in H(x,y)\}\right)^C = \emptyset,$$

and G cannot be a KKM-map. Therefore, exists $\{y_1,\dots,y_n\} \subset C$ and some

$$y_0 \in co\{y_1,\dots,y_n\} \text{ such that } y_0 \notin \bigcup_{i=1}^{n} G(y_i).$$

So,

$$0 \in H(y_0, y_i) \text{ for all } i = 1,\dots,n.$$

From assumption (a) we have,

$$0 \in H(y_0, y_0).$$

∎

Corollary 2.2. *Let X and Y be topological vector spaces, C a nonempty compact convex subset of X, and $H : C \times C \to 2^Y$ a multifunction such that*

(a) *$H(x,y)$ is a diagonally quasiconvex multifunction in y and $H(x,y)$ is a nonempty for all $x, y \in C$,*
(b) $C = \bigcup_{y\in C} int_C\{x \in C : 0 \in H(x,y)\}$.

Then there exists $y_0 \in C$ such that $0 \in H(y_0, y_0)$.

Corollary 2.3. *Let X and Y be topological vector spaces, C a nonempty convex subset of X, and let $H, L : C \times C \to 2^Y$ be two multifunctions such that*

(a) *for all $x, y \in C$, $H(x,y) \subseteq L(x,y)$, $L(x,y)$ is a diagonally quasiconvex multifunction in y and and $H(x,y)$ is a nonempty for all $x, y \in C$,*
(b) $C = \bigcup_{y\in C} int_C\{x \in C : 0 \in H(x,y)\}$,
(c) *there exists a non-empty subset B_0 of C such that intersection*

$$\bigcap_{y\in B_0} C \setminus int_C\{x \in C : 0 \in H(x,y)\}$$

is compact and B_0 is contained in a compact convex subset of C.

Then there exists $y_0 \in C$ such that $0 \in L(y_0, y_0)$.

Corollary 2.4. *Let C be a nonempty convex subset of a topological vector space X, and let $g : C \times C \to \mathbb{R}$ be a function such that*

(a) *$g(x,y)$ is a diagonally quasiconvex function in y,*
(b) $C = \bigcup_{y\in C} int_C\{x \in C : g(x,y) < 0\}$,
(c) *there exists a non-empty subset B_0 of C such that intersection*

$$\bigcap_{y\in B_0} C \setminus int_C\{x \in C : g(x,y) < 0\}$$

is compact and B_0 is contained in a compact convex subset of C.

Then there exists $y_0 \in C$ such that $g(y_0, y_0) < 0$.

Corollary 2.5. *Let C be a nonempty convex subset of a topological vector space X, and let $f, g : C \times C \to C$ be two multifunctions such that*

(a) *for all* $x, y \in C$, $f(x,y) \leq g(x,y)$ *and* $f(x,y)$ *is a diagonally quasiconvex function in* y,

(b) $C = \bigcup_{y \in C} int_C\{x \in C : g(x,y) < 0\}$,

(c) *there exists a non-empty subset* B_0 *of* C *such that intersection*

$$\bigcap_{y \in B_0} C \setminus int_C\{x \in C : g(x,y) < 0\}$$

is compact and B_0 *is contained in a compact convex subset of* C.

Then there exists $y_0 \in C$ *such that* $f(y_0, y_0) < 0$.

3. APPLICATIONS

As the first application of our results we have the following results of F. E. Browder, Q. H. Ansari and E. Tarafdar.

Theorem 3.1. [5] *Let* C *be a nonempty compact convex subset of a topological vector space* X*, and let* $S : C \to 2^C$ *be a multifunction such that*

(i) *for each* $x \in C$, $S(x)$ *is convex and* $S(x)$ *is nonempty,*

(ii) *for each* $y \in C$, $S^{-1}(y)$ *is open in* C.

Then there exists $x_0 \in C$ *such that* $x_0 \in S(x_0)$.

Proof. Taking $H(x,y) = y - S(x)$ in Corollary 2.2. ∎

Theorem 3.2. [1] *Let* C *be a nonempty convex subset of a topological vector space* X*, and let* $S, T : C \to 2^C$ *be two multifunctions. Assume that*

(i) *for each* $x \in C$, $co(S(x)) \subseteq T(x)$ *and* $S(x)$ *is nonempty,*

(ii) $C = \bigcup_{y \in C} int_C S^{-1}(y)$,

(iii) *there exists a nonempty subset* B_0 *of* C *such that intersection* $\bigcap_{y \in B_0} C \setminus int_C S^{-1}(y)$ *is compact and* B_0 *is contained in a compact convex subset of* C.

Then there exists $x_0 \in C$ *such that* $x_0 \in T(x_0)$.

Proof. Taking $H(x,y) = y - S(x)$ and $L(x,y) = y - co(S(x))$ in Corollary 2.3. ∎

Theorem 3.3. [16] *Let* C *be a nonempty convex subset of a topological vector space* X*, and let* $T : C \to 2^C$ *be a multifunction such that*

(i) *for each* $x \in C$, $T(x)$ *is convex and nonempty,*

(ii) *for each* $y \in C$, $T^{-1}(y)$ *contains an open set* O_y *which may be empty,*

(iii) $\cup_{y \in C} O_y = C$,

(iv) *there exists a nonempty set* X_0 *contained in a compact convex subset* X_1 *of* C *such that* $D = \cap_{x \in X_0} O_x^C$ *is either empty or compact.*

Then there exists $x_0 \in C$ *such that* $x_0 \in T(x_0)$.

Proof. Taking $H(x,y) = y - T(x)$ in Theorem 2.1. ∎

As an application of our theorem on zeros of multifunctions we have the best approximations theorems.

Theorem 3.4. *(Best approximations theorem) Let* C *be a nonempty compact convex subset of a normed space* X*, and let* $f, g : C \to X$ *be two continuous functions. Assume that for any finite subset* $\{y_1, \dots, y_n\} \subset C$ *and any* $y \in co\{y_1, \dots, y_n\}$,

$$||g(y) - f(y)|| \leq \max_{1 \leq i \leq n} ||g(y_i) - f(y)||,$$

that there exists $x_0 \in C$ *such that* $||g(x_0) - f(x_0)|| = \inf_{x \in C} ||g(x) - f(x_0)||$.

Proof. Assume that for each $x \in C$, $||g(x) - f(x)|| > \inf_{u \in C} ||u - f(x)||$. Taking

$$H(x, y) = (||g(y) - f(x)|| - ||g(x) - f(x)||, +\infty), x, y \in C,$$

in Corollary 2.2. From assumption

$$||g(y) - f(y)|| \leq \max_{1 \leq i \leq n} ||g(y_i) - f(y)||,$$

it is easily shown that H is a diagonally convex. If H satisfies condition

$$C = \bigcup_{y \in C} int_C\{x \in C : 0 \in H(x, y)\},$$

then there exists $y_0 \in C$ such that $0 \in H(y_0, y_0)$ that is,

$$||g(y_0) - f(y_0)|| < ||g(y_0) - f(y_0)||,$$

and we have a contradiction. Hence,

$$C \neq \bigcup_{y \in C} int_C\{x \in C : 0 \in H(x, y)\},$$

and exists $x_0 \in C$ such that $0 \notin H(x_0, y)$ for each $y \in C$, that is,

$$||g(x_0) - f(x_0)|| \leq ||g(y) - f(x_0)||.$$

∎

In view of Remark 1.1, we have as corollary the following result of J. B. Prolla .

Theorem 3.5. [11] *Let* C *be a nonempty compact convex subset of a normed space* X*, and* $g : C \to C$ *a continuous, almost affine, onto mapping. Then, for each continuous mapping* $f : C \to X$ *there exists an* $x_0 \in C$ *such that* $||g(x_0) - f(x_0)|| = \inf_{u \in C} ||u - f(x_0)||$.

Remark 3.1. If $g(x) = x, x \in C$, Theorem 3.4 reduces to well-known best approximations theorem of Ky Fan [9].

Theorem 3.6. *(Simultaneous approximations theorem,* [10]*) Let* X *be a normed space,* C *a nonempty convex compact subset of* X*,* $G_i : C \times C \to 2^C,\ i = 1, \dots, n$*, continuous mappings with compact and convex values and* $\alpha_i \geq 1$*. If* $x \mapsto G_i(x, \cdot),\ i = 1, \dots, n$ *are convex, that is, for all* $x_1, x_2, y \in C$ *and* $\lambda \in [0, 1]$*,*

$$\lambda G_i(x_1, y) + (1 - \lambda) G_i(x_2, y) \subseteq G_i(\lambda x_1 + (1 - \lambda) x_2, y),\ i = 1, \dots, n\ ,$$

then there exists $x_0 \in C$ *such that*

$$\sum_{i=1}^{n} ||G_i(x_0, x_0)||^{\alpha_i} = \inf_{u \in C} \sum_{i=1}^{n} ||G_i(u, x_0)||^{\alpha_i}.$$

Proof. Taking

$$H(x, y) = (\sum_{i=1}^{n} ||G_i(y, x)||^{\alpha_i} - \sum_{i=1}^{n} ||G_i(x, x)||^{\alpha_i}, +\infty)$$

in Corollary 2.2 and it is similar to the proof of Theorem 3.4. ∎

Theorem 3.7. *(Theorem 3.60,* [12]*) Let* C *be a nonempty, convex subset of a topological vector space. Let* $A \subset C \times C$ *and* $B \subset C \times C$ *satisfy*

(i) $A \subset B$,

(ii) *For each* $y \in C$ *the section* $\{x \in C|(x,y) \in A\}$ *is open in* C,

(iii) *For each* $x \in C$ *the section* $\{y \in C|(x,y) \in A\}$ *is nonempty, and the section* $\{y \in C|(x,y) \in B\}$ *is convex,*

(iv) C *has a nonempty compact convex subset* B_0 *such that the set* $\{x \in C|(x,y) \notin A$ *for all* $y \in B_0\}$ *is compact.*

Then there exists $x_0 \in C$ *such that* $(x_0, x_0) \in B$.

Proof. Taking $H(x,y) = (x,y) - A$ and $L(x,y) = (x,y) - B$ in Corollary 2.3. ∎

Remark 3.2. If $A = B$, Theorem 3.7 reduces to result of Ky Fan [8].

Theorem 3.8. *(Theorem 3.46,* [12]*) Let* C *be a nonempty, convex subset of topological vector space,* Y *a normed space. Let* $f : C \times C \to Y$ *be a continuous function, affine in* y*, that is, for all* $x, y_1, y_2 \in C$ *and* $\lambda \in [0,1]$,

$$f(x, \lambda y_1 + (1-\lambda)y_2) = \lambda f(x, y_1) + (1-\lambda) f(x, y_2).$$

Assume the following conditions hold:

(i) C *has a nonempty compact convex subset* B_0 *such that the set*

$$D = \{x \in C : ||f(x,y)|| \geq ||f(x,x)|| \text{ for all } y \in B_0\}$$

is compact,

(ii) *for any* $x \in C, f(x,y) = 0$ *has at least one solution* y *in* C.

Then there exists a point $y_0 \in C$ *such that* $f(y_0, y_0) = 0$.

Proof. We using corollary 2.4. Let $g(x,y) = ||f(x,y)|| - ||f(x,x)||$ for all $x, y \in C$. Since function f affine in y we have that function g is diagonally quasiconvex in y. From assumption (i) we have that there exists a nonempty subset B_0 of C such that intersection

$$\bigcap_{y \in B_0} C \setminus int_C\{x \in C : g(x,y) < 0\}$$

is compact and B_0 is contained in a compact convex subset of C. If there exists $x \in C$ such that $g(x,x) < 0$ then

$$||f(x,x)|| < ||f(x,x)||$$

and we have a contradiction. Hence, because f is continuous, we have

$$C \neq \bigcup_{y \in C} \{x \in C : ||f(x,y)|| < ||f(x,x)||\},$$

and so there exists $x \in C$ such that

$$||f(x,y)|| \geq ||f(x,x)|| \text{ for all } y \in C.$$

From assumption (ii) we have $f(x,y) = 0$ for some $y \in C$, hence $f(x,x) = 0$. ∎

We prove the existence of solutions to the variational-like inequality problem (VLIP) of Q. H. Ansari and J. -C. Yao, find $\overline{x} \in C$ such that

$$\langle T(\overline{x}), \eta(y, \overline{x}) \rangle \geq 0, \text{ for all } y \in C,$$

using our result of zeros of multifunctions. We will use the following definition.

Definition 3.1. [3] Let X be a locally convex topological space with topological dual $X^\star$ and C a nonempty subset of X. For a given bifunction $\eta : C \times C \to X$, an operator $T : C \to X^\star$ is called

(i) $\eta-$pseudomonotone if,

$$\langle T(x), \eta(y,x)\rangle \geq 0 \text{ implies } \langle T(y), \eta(y,x)\rangle \geq 0, \text{ for all } x, y \in C,$$

or equivalently

$$\langle T(y), \eta(y,x)\rangle < 0 \text{ implies } \langle T(x), \eta(y,x)\rangle < 0, \text{ for all } x, y \in C;$$

(ii) $\eta-$pseudodissipative if,

$$\langle T(y), \eta(y,x)\rangle \geq 0 \text{ implies } \langle T(x), \eta(y,x)\rangle \geq 0, \text{ for all } x, y \in C,$$

or equivalently

$$\langle T(x), \eta(y,x)\rangle < 0 \text{ implies } \langle T(y), \eta(y,x)\rangle < 0, \text{ for all } x, y \in C.$$

Theorem 3.9. [3] *Let C be a nonempty and convex subset of a locally convex topological vector space X and let $\eta : C \times C \to X$ be a bifunction such that $\eta(x,x) = 0$, for all $x \in C$. Assume that*

(i) *$T : C \to X^\star$ is $\eta-$pseudomonotone and $\eta-$pseudodissipative;*
(ii) *for each fixed $y \in C$, the map $x \mapsto \langle T(y), \eta(y,x)\rangle$ is upper semicontinuous on C;*
(iii) *for each fixed $x \in C$, the map $y \mapsto \langle T(x), \eta(y,x)\rangle$ is quasi-convex;*
(iv) *there exists a nonempty, compact and convex subset B of C and a nonempty and compact subset D of C such that for each $x \in C \backslash D$, there exists $\widetilde{y} \in B$ such that $\langle T(x), \eta(\widetilde{y}, x)\rangle < 0$. Then the VLIP has a solution, that is, exists $\overline{x} \in C$ such that*

$$\langle T(\overline{x}), \eta(y, \overline{x})\rangle \geq 0, \text{ for all } y \in C.$$

Proof. Assume that the VLIP has no solution. Then for each $x \in C$, exists $y \in C$ such that $\langle T(x), \eta(y,x)\rangle < 0$ and we have

$$C = \bigcup_{y \in C} \{x \in C : \langle T(x), \eta(y,x)\rangle < 0\}.$$

Taking

$$H(x,y) = \{y - z : \langle T(x), \eta(z,x)\rangle < 0\}, \ x, y \in C,$$

in Theorem 2.1. From assumption (iii) we obtain that H is diagonally quasiconvex in y. From assumptions (i) we have

$$\{x \in C : \langle T(x), \eta(y,x)\rangle < 0\} = \{x \in C : \langle T(y), \eta(y,x)\rangle < 0\}$$

and by (ii) we have that

$$int_C\{x \in C : \langle T(x), \eta(y,x)\rangle < 0\} = \{x \in C : \langle T(x), \eta(y,x)\rangle < 0\}.$$

Since there is

$$C = \bigcup_{y \in C} \{x \in C : \langle T(x), \eta(y,x)\rangle < 0\}$$

we have

$$C = \bigcup_{y \in C} int_C\{x \in C : \langle T(x), \eta(y,x)\rangle < 0\}$$

and

$$C = \bigcup_{y \in C} int_C\{x \in C : 0 \in H(x,y)\}.$$

From assumption (iv) there exists a nonempty, compact and convex subset B of C and a nonempty and compact subset D of C such that

$$C \setminus D \subseteq \bigcup_{y \in B} \{x \in C : \langle T(x), \eta(y,x)\rangle < 0\},$$

and

$$C \setminus D \subseteq \bigcup_{y \in B} int_C\{x \in C : \langle T(x), \eta(y,x)\rangle < 0\}.$$

So,

$$D \supseteq \bigcap_{y \in B} C \setminus int_C\{x \in C : \langle T(x), \eta(y,x)\rangle < 0\},$$

and exists a nonempty subset $B_0 = B$ of C such that intersection

$$\bigcap_{y \in B_0} C \setminus int_C\{x \in C : 0 \in H(x,y)\}$$

is compact and B_0 is contained in a compact convex subset of C. So, exists $y_0 \in C$ such that $0 \in H(y_0, y_0)$, that is,

$$\langle T(y_0), \eta(y_0, y_0)\rangle < 0.$$

Since, $\eta(y_0, y_0) = 0$, we have a contradiction. ∎

The following result of W. Takahashi has the applications of minimax inequalities. The proof of this result is given as an application of the Corollary 2.3 .

Theorem 3.10. [15] *Let C be a nonempty convex subset of a topological vector space X. Let $r \in \mathbb{R}$ and $f : C \times C \to (-\infty, \infty], g : C \times C \to (-\infty, \infty]$ such that*

- (i) *$g(x,y) \le f(x,y)$ for all $x, y \in C$,*
- (ii) *f is quasiconcave in its first variable,*
- (iii) *g is lower semicontinuous in its second variable,*
- (iv) *C has a nonempty subset B_0 such that $\{y \in C : g(x,y) \le r$ for all $x \in B_0\}$ is compact and B_0 is contained in a compact convex subset of C.*

Then either exists a point $z \in C$ such that $g(x,z) \le r$ for all $x \in C$ or there exists a point $y_0 \in C$ such that $f(y_0, y_0) > r$.

Proof. Let

$$H(x,y) = (r - g(y,x), +\infty) \text{ and } L(x,y) = (r - f(y,x), +\infty) \text{ for all } x, y \in C.$$

From assumptions (i) and (ii) we have $H(x,y) \subseteq L(x,y)$ for all $x, y \in C$ and $L(x,y)$ is a diagonally quasiconvex multifunction in y.
From assumption (iv) we have that there exists a non-empty subset B_0 of C such that intersection

$$\bigcap_{y \in B_0} C \setminus int_C\{x \in C : 0 \in H(x,y)\}$$

is compact and B_0 is contained in a compact convex subset of C. If $f(x,x) \le r$ for all $x \in C$ we have $0 \notin L(x,x)$ for all $x \in C$ and obtain

$$C \ne \bigcup_{y \in C} int_C\{x \in C : 0 \in H(x,y)\},$$

i.e.

$$C \ne \bigcup_{y \in C} int_C\{x \in C : g(y,x) > r\}.$$

From assumption (iii) we have

$$int_C\{x \in C : g(y,x) > r\} = \{x \in C : g(y,x) > r\},$$

and then exists a point $z \in C$ such that $g(x,z) \le r$ for all $x \in C$. ∎

Finally, we establish an existence result for generalized vector equilibrium problem (GVEP) which is to find $x_0 \in K$ such that

$$F(x_0, y) \nsubseteq -intC(x_0), \text{ for all } y \in K,$$

where $C(x)$ is a closed convex cone and $F : K \times K \to 2^Y$ a multifunction, by using result of the existence zeros of multifunctions.

Theorem 3.11. *Let X and Y be topological vector spaces, K a nonempty convex subset of X, and $C : K \to 2^Y$ is a multifunction such that for each $x \in K$, $C(x)$ is a closed convex cone in Y with $intC(x) \neq \emptyset$. Let $F, G : K \times K \to 2^Y$ be two multifunctions such that*

(a) *for all $x, y \in K$, $F(x, y) \subseteq -intC(x)$, implies $G(x, y) \subseteq -intC(x)$,*
(b) *for each $x \in K$, $G(x, x) \nsubseteq -intC(x)$ and $G(x, y)$ is a diagonally quasiconcave multifunction in y,*
(c) $K = \bigcup\limits_{y \in K} int_K\{x \in K : F(x, y) \subseteq -intC(x)\}$,
(d) *there exists a non-empty subset D_0 contained in a nonempty compact convex subset of D_1 such that for each $x \in K \setminus D_1$, there exists $y \in D_0$ with $F(x, y) \subseteq -intC(x)$.*

Then there exists $x_0 \in C$ such that $F(x_0, y) \nsubseteq -intC(x_0)$, for all $y \in K$.

Proof. Assume that the GVEP has no solution. Then for each $x \in K$, exists $y \in K$ such that

$$F(x, y) \subseteq -intC(x).$$

Let for all $x, y \in K$,

$$L(x, y) = \{y - z : G(x, z) \subseteq -intC(x)\}$$

and

$$H(x, y) = \{y - z : F(x, z) \subseteq -intC(x)\}.$$

Now, by Corollary2.3 we obtain there exists a point $\overline{x} \in K$, such that,

$$0 \in L(\overline{x}, \overline{x}),$$

and

$$G(\overline{x}, \overline{x}) \subseteq -intC(\overline{x}),$$

which is a contradiction of assumption (b). Hence, the solution set of GVEP is nonempty. ∎

Example 3.1. *A multifunction $F : K \times K \to 2^Y$ is called C_x-quasiconvex-like multifunction in y, see for example* [4, 6], *if for all $x, y_1, y_2 \in K$, and $\lambda \in [0, 1]$, we have either*

$$F(x, \lambda y_1 + (1 - \lambda)y_2) \subseteq F(x, y_1) - C(x)$$

or

$$F(x, \lambda y_1 + (1 - \lambda)y_2) \subseteq F(x, y_2) - C(x).$$

Let $K = [-2, 2],\ C(x) = \{0\}$, for all $x \in K$. We define $F : K \times K \to 2^{\mathbb{R}^2}$ by

$$F(x, y) = \begin{cases} \mathbb{R} \setminus \{-1\}, & (x, y) \in K \times [-2, -1], \\ \mathbb{R} \setminus \{0\}, & (x, y) \in K \times (-1, 1), \\ \mathbb{R} \setminus \{1\}, & (x, y) \in K \times [1, 2]. \end{cases}$$

We have,

$$F(y_0, y_0) \nsubseteq F(y_0, y_1) - C(y_0)$$

and

$$F(y_0, y_0) \nsubseteq F(y_0, y_2) - C(y_0),$$

where

$$y_0 = \lambda y_1 + (1 - \lambda)y_2, \lambda = \frac{1}{2}, y_1 = -1, y_2 = 1.$$

So, F is not C_x-quasiconvex-like multifunction in y.
On the other hand, we have that F is diagonally quasiconcave in y.

Remark 3.3. (i) The assumption (c) in Theorem 3.11 can be replaced by the following condition

for each $y \in K,\ F(\cdot, y)$ is upper semicontinuous multifunction on $K.$

Namely, if $F(\cdot, y)$ is upper semicontinuous multifunction for each $y \in K$, then we obtain

$$int_K\{x \in K : F(x, y) \subseteq -intC(x)\} = \{x \in K : F(x, y) \subseteq -intC(x)\},$$

for each $y \in K$, and since is,

$$K = \bigcup_{y \in K} \{x \in K : F(x, y) \subseteq -intC(x)\},$$

we have

$$K = \bigcup_{y \in K} int_K\{x \in K : F(x, y) \subseteq -intC(x)\}.$$

(ii) The assumption (b) can be replaced by

for each $x \in K, G(x, x) \nsubseteq -intC(x)$

and $G(x, y)$ is a C_x-quasiconvex-like multifunction in $y.$

In this case, Theorem 3.11 improves Theorem 2.1 in [4].

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