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ON STABILITY OF A PEXIDERIZED EQUATION ON AMENABLE ABELIAN GROUPS

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ABSTRACT. The aim of the paper is to prove the stability of the Pexiderized equation

$$f(x) = g(y + x) - h(y - x),$$

for any amenable abelian group.

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1. INTRODUCTION

The problem of stability of a given functional equation was first raised by S. M. Ulam [19] in 1940. In 1941, Hyers proved the following theorem that we state it in the language of abelian semigroups: Let $(\mathcal{G},+)$ be an abelian semigroup and $f:\mathcal{G}\to\mathbb{R}$ be a function satisfying $|f(x+y)-f(x)-f(y)|\leq\varepsilon$ for some $\varepsilon>0$ and for all $x,y\in\mathcal{G}$. Then there exists an additive function $T:\mathcal{G}\to\mathbb{R}$ such that $|T(x)-f(x)|\leq\varepsilon$ for all $x\in\mathcal{G}$.

In 1978, Th. M. Rassias [17] extended the theorem of Hyers by considering an unbounded Cauchy difference. Another generalization of Hyers' result was given by J. M. Rassias in a series of interesting papers [10, 11, 12, 14, 15]. These results have provided a lot of influence in the development of what we now call *Cauchy–Ulam stability* of functional equations. Since then the topic of stability of functional equations was extensively studied and extended in several ways by many mathematicians. The reader is referred to [1, 2, 3, 6, 8, 9, 13] and references therein for a comprehensive account on stability of functional equations.

One of most useful equations is the Pexiderized equation $f(x)=g(y+x)-h(y-x)$ where f,g,h are complex functions on an abelian (additive) group $\mathcal{G}$. In the case that $\mathcal{G}$ is uniquely 2-divisible (i.e. an abelian group in which the map $\varphi:\mathcal{G}\to\mathcal{G},\varphi(x)=2x$ is bijective) and $g=h=k$ and $f=2k$ we obtain the additive type equation $2k(x)=k(y+x)-k(y-x);x,y\in\mathcal{G}$ whose solutions are clearly those of the Jensen type equation $k(\frac{y-x}{2})=\frac{1}{2}(k(y)-k(x)),x,y\in\mathcal{G}$. The stability of this equation was studied in [16].

Our aim in the present paper is to prove the stability of the Pexiderized equation on an amenable abelian group. Recall that a group $\mathcal{G}$ is called amenable if there exists an invariant mean μ on $\mathcal{G}$, i.e. a positive linear functional μ on the space $l^\infty(\mathcal{G})$ of all bounded complex functions on $\mathcal{G}$ such that $\mu(1)=1$ and μ is right invariant in the sense that $\mu(f_x)=\mu(f);f\in l^\infty(\mathcal{G}),x\in\mathcal{G}$ in which $f_x(t):=f(tx)\ (t\in\mathcal{G})$. The reader is referred to [4] for more details on invariant means.

The analogue problem for the equation $f(x+y)+g(x-y)=h(x)+k(y)$ has been studied in [7] but in a different approach.

2. MAIN RESULTS

In this section, using ideas from [3, 18], we establish the HyersŰ-Ulam stability problem for the functional equation $f(x)=g(y+x)-h(y-x)$, where f,g,h are complex functions on an amenable abelian group $\mathcal{G}$.

Theorem 2.1. *Suppose $(\mathcal{G},+)$ is an amenable abelian group and $f,g,h:\mathcal{G}\to\mathbb{C}$ are mappings for which there exists $\varepsilon>0$ such that*

$$|f(x)-g(y+x)+h(y-x)|\leq\varepsilon, \tag{2.1}$$

for all $x,y\in\mathcal{G}$. Then there exists a unique additive mapping $T:\mathcal{G}\to\mathbb{C}$ such that

$$|f(x)-f(0)-T(x)|\leq 4\varepsilon,$$

for all $x\in\mathcal{G}$.

Proof. Let us define $F(x):=f(x)-f(0),G(x):=g(x)-g(0),H(x):=h(x)-h(0)$. It follows from (2.1) that

$$\begin{aligned}|F(x)-G(y+x)+H(y-x)| &\leq |f(x)-g(y+x)+h(y-x)|\\ &\quad+|f(0)-g(0)+h(0)|\\ &\leq 2\varepsilon. \qquad (2.2)\end{aligned}$$

Set $x = 0$ in (2.2) to obtain

$$|H(y) - G(y)| \leq 2\varepsilon.$$

Hence

$$\begin{aligned} |F(x) - (H(y+x) - H(y-x))| &\leq ||F(x) - G(y+x) + H(y-x)| \\ &\quad +|G(y+x) - H(y+x)| \\ &\leq 4\varepsilon. \end{aligned} \tag{2.3}$$

For each fixed element $x \in \mathcal{G}$, the function $(H_x - H_{-x})(y) = H(y+x) - H(y-x)$ is bounded since, by (2.3), $|H(y+x)-H(y-x)| \leq |F(x)-(H(y+x)-H(y-x))|+|F(x)| \leq 4\varepsilon+|F(x)|$. Let μ_y be a right invariant mean on the space $l^\infty(\mathcal{G})$ (the suffix y indicate that μ_y acts on functions of the variable y). Define $T(x) := \mu_y(H_x - H_{-x})$. Using the commutativity of $\mathcal{G}$ we have

$$\begin{aligned} T(x+z) &= \mu_y(H_{x+z} - H_{-z-x}) \\ &= \mu_y(H_{x+z} - H_{z-x}) + \mu_y(H_{z-x} - H_{-z-x}) \\ &= \mu_y((H_x - H_{-x})_z) + \mu_y((H_z - H_{-z})_{-x}) \\ &= \mu_y(H_x - H_{-x}) + \mu_y(H_z - H_{-z}) \\ &= T(x) + T(z). \end{aligned}$$

Hence T is additive. Moreover,

$$\begin{aligned} |T(x) + f(0) - f(x)| &= |T(x) - F(x)| = |\mu_y(H_x - H_{-x} - F(x))| \\ &\leq \sup_{y\in\mathcal{G}} |H(y+x) - H(y-x) - F(x)| \\ &\leq 4\varepsilon. \end{aligned}$$

If T' is another additive mapping fulfilling $|F(x) - T'(x)| \leq 4\varepsilon$ for all $x \in \mathcal{G}$, then

$$\begin{aligned} |T(x) - T'(x)| &= \frac{1}{n}|T(nx) - T'(nx)| \\ &\leq \frac{1}{n}(|T(nx) - F(nx)| + |F(nx) - T'(nx)|) \\ &\leq \frac{8\varepsilon}{n}, \end{aligned}$$

by the additivity of T and T'. Letting n tend to ∞, we get $T(x) = T'(x); x \in \mathcal{G}$. This proves the uniqueness assertion. ∎

Remark 2.1. There is a very useful tool in the study of stability of functional equations that is Hyers' type sequence [5].

Setting $y = x$ in (2.3), we get

$$|F(x) - H(2x)| \leq 4\varepsilon$$

and therefore

$$|T(x) - H(2x)| \leq |T(x) - F(x)| + |F(x) - H(2x)| \leq 8\varepsilon \tag{2.4}$$

Using induction on n we infer from (2.4) that

$$|T(x) - 2^{-n+1}H(2^n x)| \leq 2^{-n+4}\varepsilon$$

for all $x \in \mathcal{G}$. Hence we obtain the Hyers sequence $\{2^{-n+1}H(2^n x)\}$ with the limit $T(x) = \lim_{n\to\infty} 2^{-n+1}H(2^n x)$. Thus we can define the required additive mappings by using H.

Corollary 2.2. *The equation* $2k(x) = k(y+x) - k(y-x); x, y \in \mathcal{G}$ *for a mapping* $k : \mathcal{G} \to \mathbb{C}$ *is stable for any finite abelian group* $\mathcal{G}$.

Corollary 2.3. *The equation* $2k(x) = k(y+x) - k(y-x); x, y \in \mathcal{G}$ *is stable on any amenable abelian group.*

Proof. Apply Theorem 2.1 with $g = h = k$ and $f = 2k$. Then the mapping T obviously satisfies $2T(x) = T(y+x) - T(y-x); x, y \in \mathcal{G}$ and $|k(x) - T(x)| \le |k(x) - k(0) - T(x)| + |k(0)| \le 4\varepsilon + \frac{1}{2}\varepsilon = \frac{9}{2}\varepsilon$ for all $x \in \mathcal{G}$. ∎

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