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q-NORMS ARE REALLY NORMS

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ABSTRACT. Replacing the triangle inequality, in the definition of a norm, by $\|x+y\|^q \leq 2^{q-1}\left(\|x\|^q+\|y\|^q\right)$, we introduce the notion of a q-norm. We establish that every q-norm is a norm in the usual sense, and that the converse is true as well.

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1. INTRODUCTION

The parallelogram law states that $\|x+y\|^2+\|x-y\|^2 = 2(\|x\|^2+\|y\|^2)$ holds for all vectors x and y in a Hilbert space. This law implies that the so-called parallelogram inequality $\|x+y\|^2 \leq 2(\|x\|^2+\|y\|^2)$ trivially holds. S. Saitoh [2] noted the inequality $\|x+y\|^2 \leq 2(\|x\|^2+\|y\|^2)$ may be more suitable than the usual triangle inequality. He used this inequality to the setting of a natural sum Hilbert space for two arbitrary Hilbert spaces.

Obviously the classical triangle inequality in an arbitrary normed space implies the above inequality. This motivates us to introduce an apparently extension of the triangle inequality. More precisely, we introduce the notion of a q-norm, by replacing, in the definition of a norm, the triangle inequality by $\|x+y\|^q \leq 2^{q-1}(\|x\|^q+\|y\|^q)$, where $q \geq 1$. We establish that every q-norm is a norm in the usual sense, and that the converse is true as well. The reader is referred to [1] for undefined terms and notations.

2. MAIN RESULTS

We start our work with the following definition.

Definition 2.1. Let $\mathcal{X}$ be a real or complex linear space and $q \in [1,\infty)$. A mapping $\|\cdot\| : \mathcal{X} \to [0,\infty)$ is called a q-norm on $\mathcal{X}$ if it satisfies the following conditions:

(1) $\|x\| = 0 \Leftrightarrow x = 0$,
(2) $\|\lambda x\| = \|\lambda\|\|x\|$ for all $x \in \mathcal{X}$ and all scalar λ,
(3) $\|x+y\|^q \leq 2^{q-1}(\|x\|^q+\|y\|^q)$ for all $x, y \in \mathcal{X}$.

We first prove a rather trivial result.

Proposition 2.1. *Every norm in the usual sense is a q-norm.*

Proof. One can easily verify that the function $f(t) = \frac{1+t^q}{2} - (\frac{1+t}{2})^q$ has a nonnegative derivative and so it is monotonically increasing on $[0,\infty)$. It follows that $(\frac{1+\frac{\|y\|}{\|x\|}}{2})^q \leq \frac{1+(\frac{\|y\|}{\|x\|})^q}{2}$ whenever $\|x\| \leq \|y\|$. Therefore $\|\frac{x+y}{2}\|^q \leq (\frac{\|x\|+\|y\|}{2})^q \leq \frac{\|x\|^q+\|y\|^q}{2}$ for all $x, y \in \mathcal{X}$. It follows that $\|.\|$ is a q-norm. ∎

Now we state the following lemma which is interesting on its own right.

Lemma 2.2. *Let $\mathcal{X}$ be a real or complex linear space. Let $\|\cdot\| : \mathcal{X} \to [0,\infty)$ be a mapping satisfying (1) and (2) in the definition of a q-norm. Then $\|\cdot\|$ is a norm if and only if the set $B = \{x \mid \|x\| \leq 1\}$ is convex.*

Proof. If $\|\cdot\|$ is a norm, then B is clearly a convex set. Conversely, let B be convex and $x, y \in \mathcal{X}$. We can assume that $x \neq 0, y \neq 0$. Putting $x' = \frac{x}{\|x\|}$ and $y' = \frac{y}{\|y\|}$ we have $x', y' \in B$.

Now $\lambda x' + (1-\lambda)y' \in B$ for all $0 \leq \lambda \leq 1$. In particular, for $\lambda = \frac{\|x\|}{\|x\|+\|y\|}$ we obtain

$$\|\frac{x}{\|x\|+\|y\|} + \frac{y}{\|x\|+\|y\|}\| = \|\lambda x' + (1-\lambda)y'\| \leq 1.$$

So that $\|x+y\| \leq \|x\| + \|y\|$. ∎

We are just ready to prove our main result.

Theorem 2.3. *Every q-norm is a norm in the usual sense.*

Proof. We shall show that $B = \{x : \|x\| \leq 1\}$ is convex. Let $x, y \in B$. Then we have

$$\|x+y\|^q \leq 2^{q-1}(\|x\|^q+\|y\|^q) \leq 2^{q-1}(1+1) = 2^q,$$

whence $\|\frac{x+y}{2}\|^2 \leq 1$, so $\frac{1}{2}x + \left(1 - \frac{1}{2}\right) y \in B$. Thus if $A = \left\{\frac{k}{2^n} \mid n = 1, 2, \ldots ; k = 0, 1, \ldots, n\right\}$, then for each $\lambda \in A$ we have $\lambda x + (1 - \lambda) y \in B$.

Let $0 \leq \lambda \leq 1$ and $z = \lambda x + (1 - \lambda) y$. Since A is dense in $[0, 1]$, there exists a decreasing sequence $\{r_n\}$ in A such that $\lim_n r_n = \lambda$. Put $\beta_n = \frac{1-r_n}{1-\lambda}$. Obviously $0 \leq \beta_n \leq 1$, $\lim_n \beta_n = 1$ and $\frac{r_n+\beta_n-1}{r_n} \leq 1$. Since $\frac{r_n+\beta_n-1}{r_n}x \in B$ and $r_n \in A$ we conclude that

$$\beta_n z = \lambda\beta_n x + (1 - \lambda)\beta_n y = r_n \frac{r_n + \beta_n - 1}{r_n} x + (1 - r_n) y \in B.$$

Thus $\beta_n\|z\| = \|\beta_n z\| \leq 1$ for all n. Tending n to infinity we get $\|z\| \leq 1$, i.e. $z \in B$. ∎

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