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MERIT FUNCTIONS AND ERROR BOUNDS FOR MIXED QUASIVARIATIONAL INEQUALITIES

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ABSTRACT. It is well known that the mixed quasivariational inequalities are equivalent to the fixed point problems. We use this equivalent alternative formulation to construct some merit functions for mixed quasivariational inequalities and obtain error bounds under some conditions. Since mixed quasivariational inequalities include the classical variational inequalities and the complementarity problems as special cases, our results continue to hold for these problems.

Key words and phrases: Mixed quasivariational inequalities, merit functions, error bounds, residue vector.

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1. INTRODUCTION

Variational inequality theory, introduced by Stampacchia [20] in the early 1960's, has witnessed an explosive growth in theoretical advances, algorithmic developments and applications across all disciplines of pure and applied sciences. Variational inequalities have been generalized and extended in various directions using innovative techniques. A useful and significant generalization of variational inequalities is called the mixed quasi variational inequality involving the nonlinear bifunction which enables us to study the free-moving, unilateral and equilibrium problems arising in elasticity, fluid flow through porous media, finance, economics, transportation, circuit and structural analysis in a unified framework, see [1]-[12]. As a result of interaction among different branches of mathematical and engineering sciences, there exist now a variety of techniques including the projection method and its variant forms, auxiliary principle, resolvent equations, to suggest and analyze various iterative algorithms for solving variational inequalities and related optimization problems. It is well known that the projection method and its variant forms cannot be extended for mixed quasi variational inequalities due to the presence of the bifunction. However, if the bifunction is a proper, convex and lower semicontinuous function with respect to the first argument, then it has been shown [11] that the mixed quasi variational inequalities are equivalent to the fixed-point problem and the resolvent equations. This alternative equivalent formulation has been used to suggest and analyze some iterative methods for solving mixed quasivariational inequalities. In recent years, much attention has been given to reformulate the variational inequality as an optimization problem. A function which can constitute an equivalent optimization problem is called a merit (gap) function. Merit functions turn out to be very useful in designing new globally convergent algorithms and in analyzing the rate of convergence of some iterative methods. Various merit (gap) functions for variational inequalities and complementarity problems have been suggested and proposed by many authors, see [4, 5, 16, 17, 18, 19] and the references therein. Error bounds are functions which provide a measure of the distance between a solution set and an arbitrary point. Therefore, error bounds play an important role in the analysis of global or local convergence analysis of algorithms for solving variational inequalities. To the best of our knowledge, very few merit functions have been considered for mixed quasivariational inequalities.

In this paper, we construct some merit functions for the mixed quasivariational inequalities using the equivalence between the fixed-point and the mixed quasivariational inequalities coupled with the auxiliary principle technique. Proofs of our results is simple and straightforward as compared with other methods. As special cases, we obtain a number of known and new results for variational inequalities.

2. FORMULATIONS AND BASIC FACTS

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let K be a closed and convex set in H and $T : H \longrightarrow H$ be a nonlinear operator. Let $\varphi(.,u) : H \times H \longrightarrow R \cup \{+\infty\}$ be a function such that for all $u \in H, \varphi(.'u)$ is not identically ∞.

A *mixed quasivariational inequality* consists in finding $u \in H$, such that

$$\langle Tu, v-u\rangle + \varphi(v,u) \geq \varphi(u,u), \quad \forall v \in H. \tag{2.1}$$

It is well known [2]-[12] that a large class of obstacle, unilateral, contact, free, moving, and equilibrium problems arising in economics, finance, physical, mathematical, engineering and applied sciences can be studied in the unifying and general framework of (2.1). For example, the mixed quasivariational inequality (2.1) characterizes the Signorini problem with non-local friction. If S is an open bounded domain in R^n with regular boundary ∂S, representing the interior of an elastic body subject to external forces and if a part of the boundary may come into contact with a rigid foundation, then (2.1) is simply a statement of the virtual work for an elastic body restrained by friction forces, assuming that a non-local law of friction holds. The strain energy of the body corresponding to an admissible displacement v is $\langle Tv, v\rangle$. Thus $\langle Tu, v-u\rangle$ is the work produced by the stresses through strains caused by the virtual displacement $v-u$. The friction forces are represented by the bifunction $\varphi(.,.)$. Similar problems arise in the study of the fluid flow through porous media. For the physical and mathematical formulation of the mixed quasivariational inequalities of type (2.1); see [2, 3, 8, 9].

For $\varphi(v,u) = \varphi(v), \forall u \in H$, (2.1) shrinks to finding $u \in H$, such that

$$\langle Tu, v-u\rangle + \varphi(v) \geq \varphi(u), \quad \forall v \in H, \tag{2.2}$$

which is called the mixed variational inequality or variational inequality of the second kind; see [2]-[13]. If Tu is a gradient and φ is convex, then (2.2) corresponds to a free optimization problem.

If the function $\varphi(.)$ is the indicator function of a closed and convex set K in H, that is

$$\varphi(u) = \begin{cases} 0, & \text{if} \quad u \in K, \\ +\infty, & \text{otherwise}, \end{cases}$$

then (2.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle \geq 0, \quad \forall v \in K. \tag{2.3}$$

which is known as the classic variational inequality introduced and studied by Stampacchia [20] in 1964. For the state of the art in this theory; see [1]-[20].

We also need the following well-known concepts and results.

Definition 2.1. The operator $T : H \longrightarrow H$ is said to be
(a) *strongly monotone,* iff, there exists a constant $\alpha > 0$, such that

$$\langle Tu - Tv, u-v\rangle \quad \geq \quad \alpha\|u-v\|^2. \quad \forall u, v \in H.$$

(b) *Lipschitz continuous,* iff, there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leq \|u-v\|, \quad \forall \quad u, v \in H.$$

(c) *hemicontinuous,* if $\forall u, v \in H$, the mapping $t \longrightarrow \langle T(u + t(v-u)), v-u\rangle$ is continuous $\forall t \in [0,1]$.

In particular, from (a) and (b), it follows that $\alpha \leq \beta$.

Definition 2.2. The bifunction $\varphi(.,.)$ is said to be *skew-symmetric,* iff

$$\varphi(u,v) + \varphi(v,u) \leq \varphi(u,u) + \varphi(v,v), \quad \forall u, v \in H.$$

Clearly, if the skew-symmetric bifunction $\varphi(.,.)$ is linear in both arguments, then

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) = \varphi(u-v, u-v) \geq 0, \quad \forall u, v \in H.$$

Definition 2.3. A function $M : H \longrightarrow R \cup \{+\infty\}$ is called a merit (gap) function for the mixed quasivariational inequalities (2.1), if and only if,
(i). $M(u) \geq 0, \quad \forall v \in H.$
(ii). $M(\overline{u}) = 0$, iff, $\overline{u} \in H$ solves (2.1).

Definition 2.4. Let A be a maximal monotone set-valued operator. Then the resolvent operator associated with A is defined as

$$J_A(u) = (I + \rho A)^{-1}(u), \quad \forall u \in H,$$

where $\rho > 0$ is a constant and I is the identity operator. It is well known that the operator J_A is a single-valued and Lipschitz map on H.

Remark 2.1. It is well known that the subdifferential $\partial\varphi(.,u)$ of a proper convex and lower-semicontinuous function on H for each $u \in H$ is a maximal monotone operator, so its resolvent is defined by

$$J_{\varphi(u)} = (I + \rho\partial\varphi(.,u))^{-1} \equiv (I + \rho\partial\varphi(u))^{-1}, \tag{2.4}$$

where $\partial\varphi(u) \equiv \partial\varphi(.,u)$, unless otherwise specified.

The resolvent operator $J_{\varphi(u)}$ has the following characterization.

Lemma 2.1. *Let $\varphi(.,u)$ be a proper convex lower-semicontinuous function on H for each $u \in H$. For a given $u \in H$, $z \in H$ satisfies the inequality*

$$\langle u - z, v - u\rangle + \rho\varphi(v,u) \geq \rho\varphi(u,u), \quad \forall v \in H, \tag{2.5}$$

if and only if

$$u = J_{\varphi(u)}z,$$

where $J_{\varphi(u)}$ is the resolvent operator and $\rho > 0$ is a constant.

Lemma 2.2. *Let the operator T be monotone and hemicontinuous. If the bifunction $\varphi(.,.)$ is convex in the first argument, then problem (2.1) is equivalent to finding $u \in H$ such that*

$$\langle Tv, v - u\rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in H. \tag{2.6}$$

Proof. Let $u \in H$ be a solution of (2.1). Then

$$\langle Tu, v - u\rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in H,$$

which implies, using the monotonicity of T,

$$\langle Tv, v - u\rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in H.$$

Conversely let $u \in H$ be such that (2.6) hold. For $t \in [0,1], u, v \in H$, let $v_t = u + t(v-u) \in H$. Taking $v = v_t$ in (2.6), we have

$$\begin{aligned} 0 &\leq t\langle Tv_t, v - u\rangle + \varphi(v_t,u) - \varphi(u,u) \\ &\leq t\langle Tv_t, v - u\rangle + t\{\varphi(v,u) - \varphi(u,u)\}, \end{aligned}$$

since $\varphi(.,.)$ is convex with respect to the first argument. Dividing the above inequality by t and letting $t \longrightarrow 0$, we have

$$\langle Tu, v - u\rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in H,$$

the required (2.1). ∎

Remark 2.2. Inequality of type (2.6) is called the *dual mixed quasi variational inequality.* From Lemma 2.2, it is clear that the solution sets of both problems (2.1) and (2.6) are equivalent. Lemma 2.2 plays an important part in the approximation of the variational inequalities. Lemma 2.2 can be viewed as a natural generalization of a Minty's Lemma.

We now study those conditions under which the mixed quasivariational inequality (2.1) has a unique solution, which is the main motivation for our next result.

Theorem 2.3. *Let T be a strongly monotone with constant $\alpha > 0$ and Lipschitz continuous operator with constant $\beta > 0$. If the bifunction $\varphi(.,.)$ is skew-symmetric and $0 < \rho < \frac{2\alpha}{\beta^2}$, then the mixed quasivariational inequality (2.1) has a unique solution.*

Proof.

(a). Uniqueness. Let $u_1 \neq u_2 \in H$ be two solutions of (2.1). Then, we have

$$\langle Tu_1, v - u_1\rangle + \varphi(v, u_1) - \varphi(u_1, u_1) \geq 0, \quad \forall v \in H, \tag{2.7}$$

$$\langle Tu_2, v - u_2\rangle + \varphi(v, u_2) - \varphi(u_2, u_2) \geq 0, \tag{2.8}$$

Taking $v = u_2$ in (2.7) and $v = u_1$ in (2.8), adding the resultant and using the skew-symmetry of the bifunction $\varphi(.,.)$, we have

$$\begin{aligned}\langle Tu_1 - Tu_2, u_1 - u_2\rangle &\leq \varphi(u_1, u_2) - \varphi(u_1, u_1) - \varphi(u_2, u_2) + \varphi(u_2, u_1) \\ &\leq 0.\end{aligned}$$

Since T is strongly monotone, there exists a constant $\alpha > 0$, such that

$$\alpha\|u_1 - u_2\|^2 \leq \langle Tu_1 - Tu_2, u_1 - u_2\rangle \leq 0,$$

which implies that $u_1 = u_2$, the uniqueness of the solution of (2.1).

(b). Existence. We now use the auxiliary principle technique to prove the existence of a solution of (2.1). For a given $u \in H$, we consider the problem of finding a $w \in H$ such that

$$\langle w, v - w\rangle + \rho\varphi(v, w) \geq \rho\varphi(w, w) + \langle u, v - w\rangle - \rho\langle Tu, v - w\rangle, \quad \forall v \in H, \tag{2.9}$$

where $\rho > 0$ is a constant. Inequality of type (2.9) is called the auxiliary variational inequality associated with the problem (1). It is clear that the relation (2.9) defines a mapping $u \longrightarrow w$. It is enough to show that the mapping $u \longrightarrow w$ defined by the relation (2.9) has a fixed point belonging to H satisfying the mixed quasivariational inequality (2.1). Let w_1, w_2 be two solutions of (2.9) related to $u_1, u_2 \in H$ respectively. It is sufficient to show that for a well chosen $\rho > 0$,

$$\|w_1 - w_2\| \leq \theta\|u_1 - u_2\|,$$

with $0 < \theta < 1$, where θ is independent of u_1 and u_2. Taking $v = w_2$(respectively w_1) in (2.9) related to u_1 (respectively u_2), adding the resultant and using the skew-symmetry of the bifunction $\varphi(.,.)$, we have

$$\langle w_1 - w_2, w_1 - w_2\rangle \leq \langle u_1 - u_2 - \rho(Tu_1 - Tu_2), w_1 - w_2\rangle,$$

from which, we have

$$\begin{aligned}\|w_1 - w_2\|^2 &\leq \|u_1 - u_2 - \rho(Tu_1 - Tu_2)\|^2 \\ &\leq \|u_1 - u_2\|^2 - 2\rho\langle u_1 - u_2, Tu_1 - Tu_2\rangle + \rho^2\|Tu_1 - Tu_2\|^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u_1 - u_2\|^2,\end{aligned}$$

since T is both strongly monotone and Lipschitz continuous operator with constants $\alpha > 0$ and $\beta > 0$ respectively. Thus

$$\|w_1 - w_2\| \leq \theta\|u_1 - u_2\|,$$

where $\theta = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} < 1$ for $0 < \rho < \frac{2\alpha}{\beta^2}$ showing that the mapping defined by (2.9) has a fixed point belonging to H, which is the solution of (2.1), the required result.

∎

We note that if the operator T is linear, symmetric, positive and the bifunction $\varphi(.,.)$ is convex in the first argument, then the solution of the auxiliary mixed quasi variational inequality (2.9) is equivalent to finding the minimum of the function $I[w]$, where

$$I[w] = \frac{1}{2}\langle w - u, w - u\rangle + \rho\langle Tu, w - u\rangle + \rho\varphi(u, w) - \rho\varphi(u, u), \tag{2.10}$$

which is a differentiable function associated with the inequality (2.9). This auxiliary functional can be used to construct a gap (merit) function whose stationary points solve the variational inequality (2.1). In fact, one can easily show that the mixed quasi variational inequality (2.1) is equivalent to the optimization problem. This approach is used to suggest and analyze some descent-type iterative methods for solving mixed quasi variational inequalities.

We also need the following condition .

Assumption 1. $\forall u, v, w \in H$, *the operator* $J_{\varphi(u)}$ *satisfies the condition*

$$\|J_{\varphi(u)}w - J_{\varphi(v)}w\| \leq \nu\|u - v\|, \tag{2.11}$$

where $\nu > 0$ *is a constant.*

For the applications and the examples of Assumption 1, see [11, 12, 13, 15].

3. MAIN RESULTS

In this section, we consider three merit functions for the mixed quasivariational inequalities (2.1) and obtain error bounds for the solution of the mixed quasivariational inequalities (2.1).

From now onward, it is assumed that the function $\varphi(.,.)$ is proper convex and lower-semicontinuous on H for all $u \in H$, unless otherwise specified.

We need the following result, which can be proved by using Lemma 2.1.

Lemma 3.1. *The mixed quasi variational inequality (2.1) has a solution* $u \in H$ *if and only if* $u \in H$ *satisfies the relation*

$$u = J_{\varphi(u)}[u - \rho Tu], \tag{3.1}$$

where $\rho > 0$ *is a constant.*

Lemma 3.1 implies that problems (2.1) and (3.1) are equivalent. This alternative equivalent formulation plays an important part in suggesting and analyzing several iterative methods for solving variational inequalities. This fixed-point formulation has been used to suggest and analyze several iterative methods for solving the mixed quasivariational inequalities (2.1).

We now consider the residue vector

$$R_\rho(u) \equiv R(u) := u - J^{\rho}_{\varphi(u)}[u - \rho Tu] \equiv u - J_{\varphi(u)}[u - \rho Tu]. \tag{3.2}$$

It is clear from Lemma 3.1 that (2.1) has a solution $u \in H$, iff, $u \in H$ is a root of the equation

$$R(u) = 0. \tag{3.3}$$

It is known that the normal residue vector $R(u)$ defined by the relation (3.2) is merit function for the mixed quasivariational inequality (2.1). We use this merit function to derive the error bound for the solution of (2.1).

Theorem 3.2. *Let the function* $\varphi(., u)$ *be proper convex and lower-semicontinuous on* H *for all* $u \in H$ *and skew-symmetric. Let* $\overline{u} \in H$ *be a solution of (2.1) and let Assumption 1 hold. If the operator* T *is both strongly monotone and Lipschitz continuous with constants* $\alpha > 0$ *and and* $\beta > 0$ *respectively, then*

$$k_1\|R(u)\| \leq \|u - \overline{u}\| \leq k_2\|R(u)\|, \quad \forall u \in H, \tag{3.4}$$

where k_1, k_2 *are generic constants.*

Proof. Let $\overline{u} \in H$ be solution of (2.1). Then

$$\langle T\overline{u}, v - \overline{u}\rangle + \varphi(v, \overline{u}) - \varphi(\overline{u}, \overline{u}) \geq 0, \quad \forall v \in H. \tag{3.5}$$

Taking $v = J_{\varphi(u)}[u - \rho Tu]$ in (3.5), we have

$$\langle T\overline{u}, J_{\varphi(u)}[u - \rho Tu] - \overline{u}\rangle + \varphi(J_{\varphi(u)}[u - \rho Tu], \overline{u}) - \varphi(\overline{u}, \overline{u}) \geq 0. \tag{3.6}$$

Letting $u = J_{\varphi(u)}[u - \rho Tu], \quad z = u - \rho Tu$ and $v = \overline{u}$ in (2.5), we have

$$\begin{aligned} \langle \rho Tu + J_{\varphi(u)}[u - \rho Tu] - u, \quad & \overline{u} - J_{\varphi(u)}[u - \rho Tu]\rangle + \rho\varphi(\overline{u}, J_{\varphi(u)}[u - \rho Tu]) \\ & -\rho\varphi(J_{\varphi(u)}[u - \rho Tu], J_{\varphi(u)}[u - \rho Tu]) \geq 0. \end{aligned} \tag{3.7}$$

Adding (3.6), (3.7) and using the skew-symmetry of the bifunction $\varphi(.,.)$, we obtain

$$\langle T\overline{u} - Tu + (1/\rho)(u - J_{\varphi(u)}[u - \rho Tu]), J_{\varphi(u)}[u - \rho Tu] - \overline{u}\rangle \geq 0. \tag{3.8}$$

Since T is a strongly monotone, there exists a constant $\alpha > 0$, such that

$$\begin{aligned} \alpha\|\overline{u} - u\|^2 &\leq \langle T\overline{u} - Tu, \overline{u} - u\rangle \\ &= \langle T\overline{u} - Tu, \overline{u} - J_{\varphi(u)}[u - \rho Tu]\rangle \\ &\quad + \langle T\overline{u} - Tu, J_{\varphi(u)}[u - \rho Tu] - u\rangle \\ &\leq (1/\rho)\langle u - J_{\varphi(u)}[u - \rho Tu], J_{\varphi(u)}[u - \rho Tu] - u + u - \overline{u}\rangle \\ &\quad + \langle T\overline{u} - Tu, J_{\varphi(u)}[u - \rho Tu] - u\rangle \\ &\leq -(1/\rho)\|R(u)\|^2 + (1/\rho)\|R(u)\|\|u - \overline{u}\| \\ &\quad + \|T\overline{u} - Tu\|\|R(u)\| \\ &\leq (1/\rho)(1 + \beta)\|R(u)\|\|\overline{u} - u\| \end{aligned}$$

which implies that

$$\|\overline{u} - u\| \leq k_2\|R(u)\|, \tag{3.9}$$

the right-hand inequality in (3.4) with $k_2 = (1/\alpha\rho)(1 + \beta)$.

Now from Assunption 1 and Lipschitz continuity of T, we have

$$\begin{aligned} \|R(u)\| &= \|u - J_{\varphi(u)}[u - \rho Tu]\| \\ &= \|u - \overline{u} + J_{\varphi(\overline{u})}[\overline{u} - \rho T\overline{u}] - J_{\varphi(u)}[u - \rho Tu]\| \\ &\leq \|u - \overline{u}\| + \|J_{\varphi(\overline{u})}[\overline{u} - \rho T\overline{u}] - J_{\varphi(\overline{u})}[u - \rho Tu]\| \\ &\quad + \|J_{\varphi(\overline{u})}[u - \rho Tu] - J_{\varphi(u)}[u - \rho Tu]\| \\ &\leq \|u - \overline{u}\| + \nu\|u - \overline{u}\| + \|u - \overline{u} + \rho(Tu - T\overline{u})\| \\ &\leq \{2 + \nu + \rho\beta\}\|u - \overline{u}\| = k_1\|u - \overline{u}\|, \end{aligned}$$

from which we have

$$(1/(k_1)\|R(u)\| \leq \|u - \overline{u}\|, \tag{3.10}$$

the left-most inequality in (3.4) with $k_1 = (2 + \nu + \rho\beta)$.

Combining (3.9) and (3.10), we obtain the required (3.4). ∎

Letting $u = 0$ in (3.4), we have

$$(1/k_1)\|R(0)\| \leq \|\overline{u}\| \leq k_2\|R(0)\|. \tag{3.11}$$

Combining (3.4) and (3.11), we obtain a relative error bound for any point $u \in H$.

Theorem 3.3. *Assume that all the assumptions of Theorem 3.2 hold. If $0 \neq \overline{u} \in H$ is the unique solution of (2.1), then*

$$c_1\|R(u)\|/\|R(0)\| \leq \|u - \overline{u}\|/\|\overline{u}\| \leq c_2\|R(u)\|/\|R(0)\|.$$

Note that the normal residue vector (merit function) $R(u)$ defined by (3.4) is nondifferentiable. To overcome the nondifferentiability, which is a serious drawback of the residue merit function, we consider another merit function associated with problem (2.1). This merit function can be viewed as a regularized merit function, see [18, 19]. We consider the function

$$\begin{aligned} M_\rho(u) &= \langle Tu, u - J_{\varphi(u)}[u-\rho Tu]\rangle + \varphi(u, J_{\varphi(u)}[u-\rho Tu]) \\ &\quad -\varphi(J_{\varphi(u)}[u-\rho Tu], J_{\varphi(u)}[u-\rho Tu]) \\ &\quad -(1/2\rho)\|u - J_{\varphi(u)}[u-\rho Tu]\|^2, \quad \forall u \in H. \end{aligned} \tag{3.12}$$

from which it follows that $M_\rho(u) \geq 0, \quad \forall u \in H.$

We now show that the function $M_\rho(u)$ defined by (3.12) is a merit function and this is the main motivation of our next result.

Theorem 3.4. $\forall u \in H$, *we have*

$$M_\rho(u) \geq (1/2\rho)\|R(u)\|^2. \tag{3.13}$$

In particular, we have $M_\rho(u) = 0$, *iff,* $u \in H$ *is a solution of (2.1).*

Proof. Setting $v = u, \quad u = J_{\varphi(u)}[u-\rho Tu]$ and $z = u - \rho Tu$ in (2.5), we have

$$\begin{aligned} \langle Tu &- (1/\rho)(u - J_{\varphi(u)}[u-\rho Tu]), u - J_{\varphi(u)}[u-\rho Tu]\rangle + \varphi(u, J_{\varphi(u)}[u-\rho Tu]) \\ &-\varphi(J_{\varphi(u)}[u-\rho Tu], J_{\varphi(u)}[u-\rho Tu]) \geq 0 \end{aligned}$$

which implies that

$$\begin{aligned} \langle Tu, R(u)\rangle &- \varphi(J_{\varphi(u)}[u-\rho Tu], J_{\varphi(u)}[u-\rho Tu]) \\ &+\varphi(u, J_{\varphi(u)}[u-\rho Tu]) \geq (1/\rho)\|R(u)\|^2. \end{aligned} \tag{3.14}$$

Combining (3.12) and (3.14), we have

$$\begin{aligned} M_\rho(u) &= \langle Tu, R(u)\rangle - \varphi(J_{\varphi(u)}[u-\rho Tu], J_{\varphi(u)}[u-\rho Tu]) \\ &\quad +\varphi(u, J_{\varphi(u)}[u-\rho Tu]) - (1/2\rho)\|R(u)\|^2 \\ &\geq (1/\rho)\|R(u)\|^2 - (1/2\rho)\|R(u)\|^2 \\ &= (1/2\rho)\|R(u)\|^2, \end{aligned}$$

the required result (3.13). Clearly we have $M_\rho(u) \geq 0, \quad \forall u \in H.$

Now if $M_\rho(u) = 0$, then clearly $R(u) = 0$. Hence by Lemma 3.1, we see that $u \in H$ is a solution of (2.1). Conversely, if $u \in H$ is a solution of (2.1), then $u = J_{\varphi(u)}[u-\rho Tu]$ by Lemma 3.1. Consequently, from (3.12), we see that $M_\rho(u) = 0$, the required result. ∎

From Theorem 3.4, we see that the function $M_\rho(u)$ defined by (3.12) is a merit function for the mixed quasivariational inequalities (2.1). We now derive the error bounds without using the Lipschitz continuity ofthe operator T.

Theorem 3.5. *Let* T *be a strongly monotone with a constant* $\alpha > 0$ *and the bifunction* $\varphi(.,.)$ *be a skew symmetric function. If* $\overline{u} \in H$ *is a solution of (2.1), then*

$$\|u - \overline{u}\|^2 \leq (2\rho)/(2\alpha\rho - 1)M_\rho(u), \quad \forall u \in H. \tag{3.15}$$

Proof. From (3.12), we have

$$\begin{aligned} M_\rho(u) &\geq \langle Tu, u-\overline{u}\rangle + \varphi(u,\overline{u}) - \varphi(\overline{u},\overline{u}) - (1/2\rho)\|u-\overline{u}\|^2 \\ &\geq \langle T\overline{u}, u-\overline{u}\rangle + \alpha\|u-\overline{u}\|^2 \\ &\quad +\varphi(u,\overline{u}) - \varphi(\overline{u},\overline{u}) - (1/2\rho)\|u-\overline{u}\|^2, \end{aligned} \tag{3.16}$$

where we have used the fact that the operator T is strongly monotone with a constant $\alpha > 0$. Taking $v = u$ in (3.5), we have

$$\langle T\overline{u}, u - \overline{u}\rangle + \varphi(u, \overline{u}) - \varphi(\overline{u}, \overline{u}) \geq 0. \tag{3.17}$$

From (3.16) and (3.17), we have

$$\begin{aligned} M_\rho(u) &\geq \alpha\|u - \overline{u}\|^2 - (1/2\rho)\|u - \overline{u}\|^2 \\ &= (\alpha - 1/2\rho)\|u - \overline{u}\|^2, \end{aligned}$$

from which the result (3.15) follows. ∎

We consider another function merit function associated with mixed quasivariational inequalities (2.1), which can be viewed as a difference of two regularized merit functions. Such type of the merit functions functions were introduced and studied by many authors for solving variational inequalities and complementarity problems; see [15, 16, 17]. Here we define the D-merit function by a formal difference of the regularized merit function defined by (3.12). To this end, we consider the following function

$$\begin{aligned} D_{\rho,\mu}(u) &= \langle Tu, J_{\varphi(u)}[u - \mu Tu] - J_{\varphi(u)}[u - \rho Tu]\rangle + \varphi(u, J_{\varphi(u)}[u - \mu Tu]) \\ &\quad -\varphi(u, J_{\varphi(u)}[u - \rho Tu]) + (1/2\mu)\|u - J_{\varphi(u)}[u - \mu Tu]\|^2 \\ &\quad -(1/2\rho)\|u - J_{\varphi(u)}[u - \rho Tu]\|^2 \\ &= \langle Tu, R_\rho(u) - R_\mu(u)\rangle + \varphi(u, J_{\varphi(u)}[u - \mu Tu]) \\ &\quad -\varphi(u, J_{\varphi(u)}[u - \rho Tu]) + (1/2\mu)\|R_\mu(u)\|^2 \\ &\quad -(1/2\rho)\|R_\rho(u)\|^2, \quad u \in H, \quad \rho > \mu > 0. \end{aligned} \tag{3.18}$$

It is clear that the $D_{\rho,\mu}(u)$ is everywhere finite. We now show that the function $D_{\rho,\mu}(u)$ defined by (3.18) is indeed a merit function for the mixed quasivariational inequalities (2.1) and this is the motivation of our next result.

Theorem 3.6. $\forall u \in H, \rho > \mu > 0,$ *we have*

$$(\rho - \mu)\|R_\rho(u)\|^2 \geq 2\rho\mu D_{\rho,\mu}(u) \geq (\rho - \mu)\|R_\mu(u)\|^2. \tag{3.19}$$

In particular, $D_{\rho,\mu}(u) = 0,$ *iff* $u \in H$ *solves problem (2.1).*

Proof. Taking $v = J_{\varphi(u)}[u - \mu Tu], u = J_{\varphi(u)}[u - \rho Tu]$ and $z = u - \rho Tu$ in (2.5), we have

$$\begin{aligned} &\langle J_{\varphi(u)}[u - \rho Tu] - u + \rho Tu, J_{\varphi(u)}[u - \mu Tu] - J_{\varphi(u)}[u - \rho Tu]\rangle \\ &+\rho\varphi(J_{\varphi(u)}[u - \mu Tu], J_{\varphi(u)}[u - \rho Tu]) - \rho\varphi(J_{\varphi(u)}[u - \rho Tu], J_{\varphi(u)}[u - \rho Tu]) \geq 0, \end{aligned}$$

which implies that

$$\begin{aligned} \langle Tu, R_\rho(u) - R_\mu(u)\rangle \quad + \quad &\varphi(J_{\varphi(u)}[u - \mu Tu], J_{\varphi(u)}[u - \rho Tu]) \\ &-\varphi(J_{\varphi(u)}[u - \rho Tu], J_{\varphi(u)}[u - \rho Tu]) \\ &\geq (1/\rho)\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle. \end{aligned} \tag{3.20}$$

From (3.18) and (3.20), we have

$$\begin{aligned} D_{\rho,\mu}(u) &\geq (1/\rho)\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle + (1/2\mu)\|R_\mu(u)\|^2 \\ &\quad -(1/2\rho)\|R_\rho(u)\|^2 \\ &= 1/2(1/\mu - 1/\rho)\|R_\mu(u)\|^2 + (1/\rho)\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle \\ &\quad -(1/2\rho)\|R_\rho(u) - R_\mu(u)\|^2 - (1/\rho)\langle R_\mu(u), R_\rho(u) - R_\mu(u)\rangle \\ &= 1/2(1/\mu - 1/\rho)\|R_\mu(u)\|^2 + (1/2\rho)\|R_\rho(u) - R_\mu(u)\|^2 \\ &\geq 1/2(1/\mu - 1/\rho)\|R_\mu(u)\|^2, \end{aligned} \tag{3.21}$$

which implies the right-most inequality in (3.19).

In a similar way, by taking $u = J_{\varphi(u)}[u - \mu Tu], z = u - \mu Tu$ and $v = J_{\varphi(u)}[u - \mu Tu]$ in (2.5), we have

$$\begin{aligned}&\langle J_{\varphi(u)}[u - \mu Tu] - u + \mu Tu, J_{\varphi(u)}[u - \mu Tu] - J_{\varphi(u)}[u - \mu Tu]\rangle \\ &+\mu\varphi(J_{\varphi(u)}[u - \mu Tu], J_{\varphi(u)}[u - \mu Tu]) - \mu\varphi(J_{\varphi(u)}[u - \mu Tu], J_{\varphi(u)}[u - \mu Tu]) \geq 0,\end{aligned}$$

which implies that

$$\begin{aligned}\langle Tu, R_\rho(u) - R_\mu(u)\rangle \quad - \quad & \varphi(J_{\varphi(u)}[u - \mu Tu], J_{\varphi(u)}[u - \mu Tu]) \\ & +\varphi(J_{\varphi(u)}[u - \mu Tu], J_{\varphi(u)}[u - \mu Tu]) \\ & \leq (1/\mu)\langle R_\mu(u), R_\rho(u) - R_\mu(u)\rangle. \end{aligned} \tag{3.22}$$

Consequently, from (3.18) and (3.22), we obtain

$$\begin{aligned} D_{\rho,\mu}(u) &\leq (1/\mu)\langle R_\mu(u), R_\rho(u) - R_\mu(u)\rangle + (1/2\mu)\|R_\mu(u)\|^2 \\ &\quad -(1/2\rho)\|R_\rho(u)\|^2 \\ &= 1/2(1/\mu - 1/\rho)\|R_\mu(u)\|^2 + (1/\rho)\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle \\ &\quad -(1/2\rho)\|R_\rho(u) - R_\mu(u)\|^2 - (1/\rho)\langle R_\mu(u), R_\rho(u) - R_\mu(u)\rangle \\ &= 1/2(1/\mu - 1/\rho)\|R_\rho(u)\|^2 - (1/2\mu)\|R_\rho(u) - R_\mu(u)\|^2 \\ &\leq 1/2(1/\mu - 1/\rho)\|R_\rho(u)\|^2, \end{aligned} \tag{3.23}$$

which implies the left-most inequality in (3.19).

Combining (3.21) and (3.23), we obtain (3.19), the required result. ∎

Using essentially the technique of Theorem 3.5, we can obtain the following result.

Theorem 3.7. *Let $\overline{u} \in H$ be a solution of (2.1). If the operator T is strongly monotone with constant $\alpha > 0$, then*

$$\|u - \overline{u}\|^2 \leq (2\rho\mu)/(\rho(2\mu\alpha + 1) - \mu)D_{\rho,\mu}, \quad \forall u \in H. \tag{3.24}$$

Proof. Let $\overline{u} \in H$ be a solution of (2.1). Then, taking $v = u$ in (3.5), we have

$$\langle T\overline{u}, u - \overline{u}\rangle + \varphi(u, \overline{u}) - \varphi(\overline{u}, \overline{u}) \geq 0. \tag{3.25}$$

Also from (3.18), (3.25) and strongly monotonicity of T, we have

$$\begin{aligned} D_{\rho,\mu}(u) &\geq \langle Tu, u - \overline{u}\rangle - \varphi(\overline{u}, \overline{u}) + \varphi(u, \overline{u}) \\ &\quad +(1/2\mu)\|u - \overline{u}\|^2 - (1/2\rho)\|u - \overline{u}\|^2 \\ &\geq \langle T\overline{u}, u - \overline{u}\rangle - \varphi(\overline{u}, \overline{u}) + \varphi(u, \overline{u}) \\ &\quad \alpha\|u - \overline{u}\|^2 + (1/2\mu)\|u - \overline{u}\|^2 - (1/2\rho)\|u - \overline{u}\|^2 \\ &\geq (\alpha + (1/2\mu) - (1/2\rho))\|u - \overline{u}\|^2, \end{aligned}$$

from which the required result (3.24) follows. ∎

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