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ITERATIVE APPROXIMATION OF COMMON FIXED POINTS OF A FINITE FAMILY OF ASYMPTOTICALLY HEMI-CONTRACTIVE TYPE MAPPINGS

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ABSTRACT. In this paper, we prove that the sequence of the modified Ishikawa-Xu, Ishikawa-Liu, Mann-Xu and Mann-Liu iterative types of a finite family of asymptotically hemi-contractive type mappings converges strongly to a common fixed point of the family in a real p-uniformly convex Banach space with $p > 1$. Our results improve and extend some recent results.

Key words and phrases: Asymptotically hemi-contractive type mappings, p-uniformly convex Banach spaces, Common fixed points, Ishikawa and Mann iterative sequences with errors.

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1. INTRODUCTION

Let E be a real Banach space, C be a nonempty closed subset of E, T be a self-mapping on C and $k \in [0,1]$, $p > 0$ be two constants. T is said to be *p-asymptotically contractive type mapping with respect to the constant k* if

$$\limsup_{n\to\infty} \sup_{x\in C}\{\|T^n x - T^n y\|^p - \|x-y\|^p - k\|(x-T^n x)-(y-T^n y)\|^p\} \leq 0 \tag{1.1}$$

for all $y \in C$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. T is called *p-asymptotically hemi-contractive type mapping with respect to the constant k* if $F(T) \neq \phi$ and for each $q \in F(T)$ we have

$$\limsup_{n\to\infty} \sup_{x\in C}\{\|T^n x - T^n q\|^p - \|x-q\|^p - k\|x-T^n x\|^p\} \leq 0. \tag{1.2}$$

These classes of mappings were introduced recently by Huang et al. [2]. If $p = 1$ and $k = 0$ in (1.1), then T is called *asymptotically nonexpansive type mapping* [4]. Moreover, they remarked that (1.1) and (1.2) contain the asymptotically pseudo-contractive mapping and asymptotically hemi-contractive mapping on a real Hilbert space as its special cases (see, for detail, [2, Remark 2.2]). In the sequel, we shall denote the set of all positive integers by $\mathbb{N}$ and $\overline{\mathbb{N}} := \mathbb{N} \cup \{0\}$.

T is called *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x-y\|, \tag{1.3}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

T is said to be *semi-compact* if C is closed and for any bounded sequence $\{x_n\}_{n=0}^{\infty}$ in C with $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_j}\}_{j=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$ such that $\lim_{j\to\infty} x_{n_j} = q \in C$.

In 2004, for a mapping T of C into itself, Huang et al. [2] considered the following modified Ishikawa iterative with errors in the sense of Xu [12] in C defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n-\gamma_n)x_n + \alpha_n T^n y_n + \gamma_n u_n, \\ y_n = (1-\beta_n-\delta_n)x_n + \beta_n T^n x_n + \delta_n v_n, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$, $\{\gamma_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are real sequences in [0,1] and $\{u_n\}$ and $\{v_n\}$ are arbitrary sequences in C. In particular, if $\beta_n = \delta_n = 0$ for all $n \in \overline{\mathbb{N}}$, then (1.4) reduces to the modified Mann iterative sequence with errors in the sense of Xu [12] as follows:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n-\gamma_n)x_n + \alpha_n T^n x_n + \gamma_n u_n. \end{cases} \tag{1.5}$$

Using (1.4) and (1.5), they proved strong convergence theorems for the iterative approximation of fixed points of p-asymptotically hemi-contractive type mappings with respect to the constant $k \in [0,1]$ as below:

Theorem 1.1. (see [2, Theorem 3.1]). *Let E be a real p-uniformly convex Banach space, $1 < p < \infty$, C be a nonempty bounded closed convex subset of E, and $T : C \to C$ be a semi-compact, uniformly L-Lipschitzian and p-asymptotically hemi-contractive type mapping with respect to a constant $k \in [0,1]$. Suppose that $\{x_n\}$ is the modified Mann iterative sequence with errors defined by (1.5). If*

(i) $k < ca(1-a_2)^{-1}, \quad a = a_1 a_2(a_1^{p-1} + a_2^{p-1})$;
(ii) $a_1 \leq \alpha_n + \gamma_n \leq 1 - a_2$;
(iii) $\sum_{n=0}^{\infty} \gamma_n < \infty$

for all $n \in \overline{\mathbb{N}}$ and some $a_1, a_2 \in (0,1)$, where c is the positive constant appearing in inequality (2.3), then $\{x_n\}$ converges strongly to some fixed points q of T in C.

Theorem 1.2. (see [2, Theorem 3.2]). *Let E be a real p-uniformly convex Banach space, $1 < p < \infty$, C be a nonempty bounded closed convex subset of E, and $T : C \to C$ be a semi-compact, uniformly L-Lipschitzian and p-asymptotically hemi-contractive type mapping with respect to a constant $k \in [0, 1]$. Suppose that $\{x_n\}$ is the modified Ishikawa iterative sequence with errors defined by (1.4). If*

(i) $[L(1-a_2)]^p < k < ca(1-a_2)^{-1}, \quad a = a_1 a_2 (a_1^{p-1} + a_2^{p-1})$;
(ii) $a_1 \le \alpha_n + \gamma_n, \quad \beta_n + \delta_n \le 1 - a_2$;
(iii) $\sum_{n=0}^{\infty} \gamma_n < \infty, \quad \sum_{n=0}^{\infty} \delta_n < \infty$

for all $n \in \overline{\mathbb{N}}$ and some $a_1, a_2 \in (0, 1)$, where c is the positive constant appearing in inequality (2.3), then $\{x_n\}$ converges strongly to some fixed points q of T in C.

In [10], Sun modified the implicit iteration process of Xu and Ori [13] and proved the modified implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings converges strongly to a common fixed point of the family in a uniformly convex Banach space. Let C be a nonempty closed convex subset of a Banach space E. Suppose that $T_i : C \to C$ is a given mapping for each $i = 1, 2, \dots, N$. Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\delta_n\}_{n=0}^{\infty}$ be real sequences in $[0, 1]$ and $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ be arbitrary sequences in C. Motivated and inspired by Sun's work, we now introduce the following two iteration processes generated by $\{T_i\}_{i=1}^{N}$. One is the sequence $\{x_n\}_{n=0}^{\infty}$ of the modified Ishikawa-Xu iterative type as follows: for $x_0 \in C$,

$$\begin{cases} x_1 = (1-\alpha_0-\gamma_0)x_0 + \alpha_0 T_1 y_0 + \gamma_0 u_0, \\ y_0 = (1-\beta_0-\delta_0)x_0 + \beta_0 T_1 x_0 + \delta_0 v_0, \end{cases}$$

$$\begin{cases} x_2 = (1-\alpha_1-\gamma_1)x_1 + \alpha_1 T_2 y_1 + \gamma_1 u_1, \\ y_1 = (1-\beta_1-\delta_1)x_1 + \beta_1 T_2 x_1 + \delta_1 v_1, \end{cases}$$

$$\vdots$$

$$\begin{cases} x_N = (1-\alpha_{N-1}-\gamma_{N-1})x_{N-1} + \alpha_{N-1} T_N y_{N-1} + \gamma_{N-1} u_{N-1}, \\ y_{N-1} = (1-\beta_{N-1}-\delta_{N-1})x_{N-1} + \beta_{N-1} T_N x_{N-1} + \delta_{N-1} v_{N-1}, \end{cases}$$

$$\begin{cases} x_{N+1} = (1-\alpha_N-\gamma_N)x_N + \alpha_N T_1^2 y_N + \gamma_N u_N, \\ y_N = (1-\beta_N-\delta_N)x_N + \beta_N T_1^2 x_N + \delta_N v_N, \end{cases}$$

$$\vdots$$

$$\begin{cases} x_{2N} = (1-\alpha_{2N-1}-\gamma_{2N-1})x_{2N-1} + \alpha_{2N-1} T_N^2 y_{2N-1} + \gamma_{2N-1} u_{2N-1}, \\ y_{2N-1} = (1-\beta_{2N-1}-\delta_{2N-1})x_{2N-1} + \beta_{2N-1} T_N^2 x_{2N-1} + \delta_{2N-1} v_{2N-1}, \end{cases}$$

$$\begin{cases} x_{2N+1} = (1-\alpha_{2N}-\gamma_{2N})x_{2N} + \alpha_{2N} T_1^3 y_{2N} + \gamma_{2N} u_{2N}, \\ y_{2N} = (1-\beta_{2N}-\delta_{2N})x_{2N} + \beta_{2N} T_1^3 x_{2N} + \delta_{2N} v_{2N}, \end{cases}$$

$$\vdots$$

which can be written in a compact form as

$$\begin{cases} x_n = (1-\alpha_{n-1}-\gamma_{n-1})x_{n-1} + \alpha_{n-1} T_n^m y_{n-1} + \gamma_{n-1} u_{n-1}, \\ y_{n-1} = (1-\beta_{n-1}-\delta_{n-1})x_{n-1} + \beta_{n-1} T_n^m x_{n-1} + \delta_{n-1} v_{n-1}, \end{cases} \tag{1.6}$$

where $n = (m-1)N + i$, $i = 1, 2, \ldots, N$ and the other is the following sequence $\{x_n\}_{n=0}^{\infty}$ of the modified Ishikawa-Liu iterative type (see, for example, [8] and [6]): for $x_0 \in C$,

$$\begin{cases} x_1 = (1-\alpha_0)x_0 + \alpha_0 T_1 y_0 + u_0, \\ y_0 = (1-\beta_0)x_0 + \beta_0 T_1 x_0 + v_0, \end{cases}$$

$$\begin{cases} x_2 = (1-\alpha_1)x_1 + \alpha_1 T_2 y_1 + u_1, \\ y_1 = (1-\beta_1)x_1 + \beta_1 T_2 x_1 + v_1, \end{cases}$$

$$\vdots$$

$$\begin{cases} x_N = (1-\alpha_{N-1})x_{N-1} + \alpha_{N-1} T_N y_{N-1} + u_{N-1}, \\ y_{N-1} = (1-\beta_{N-1})x_{N-1} + \beta_{N-1} T_N x_{N-1} + v_{N-1}, \end{cases}$$

$$\begin{cases} x_{N+1} = (1-\alpha_N)x_N + \alpha_N T_1^2 y_N + u_N, \\ y_N = (1-\beta_N)x_N + \beta_N T_1^2 x_N + v_N, \end{cases}$$

$$\vdots$$

$$\begin{cases} x_{2N} = (1-\alpha_{2N-1})x_{2N-1} + \alpha_{2N-1} T_N^2 y_{2N-1} + u_{2N-1}, \\ y_{2N-1} = (1-\beta_{2N-1})x_{2N-1} + \beta_{2N-1} T_N^2 x_{2N-1} + v_{2N-1}, \end{cases}$$

$$\begin{cases} x_{2N+1} = (1-\alpha_{2N})x_{2N} + \alpha_{2N} T_1^3 y_{2N} + u_{2N}, \\ y_{2N} = (1-\beta_{2N})x_{2N} + \beta_{2N} T_1^3 x_{2N} + v_{2N}, \end{cases}$$

$$\vdots$$

which can be written in a compact form as

$$\begin{cases} x_n = (1-\alpha_{n-1})x_{n-1} + \alpha_{n-1} T_n^m y_{n-1} + u_{n-1}, \\ y_{n-1} = (1-\beta_{n-1})x_{n-1} + \beta_{n-1} T_n^m x_{n-1} + v_{n-1}, \end{cases} \tag{1.7}$$

where $n = (m-1)N + i$, $i = 1, 2, \ldots, N$. Note that if $N = 1$, then (1.6) and (1.7) reduce to the modified Ishikawa iterative with errors in the sense of Xu and Liu, respectively. Moreover, if $\beta_n = \delta_n = 0$ in (1.6) for all $n \in \overline{\mathbb{N}}$, then the sequence $\{x_n\}_{n=0}^{\infty}$ of the modified Mann-Xu iterative type as follows:

$$x_n = (1-\alpha_{n-1}-\gamma_{n-1})x_{n-1} + \alpha_{n-1} T_n^m x_{n-1} + \gamma_{n-1} u_{n-1}, \tag{1.8}$$

with $n = (m-1)N + i$, $i = 1, 2, \ldots, N$. Additionally, if $\beta_n = v_n = 0$ in (1.7) for all $n \in \overline{\mathbb{N}}$, then the sequence $\{x_n\}_{n=0}^{\infty}$ of the modified Mann-Liu iterative type as follows:

$$x_n = (1-\alpha_{n-1})x_{n-1} + \alpha_{n-1} T_n^m x_{n-1} + u_{n-1}, \tag{1.9}$$

with $n = (m-1)N + i$, $i = 1, 2, \ldots, N$. Similarly, if $N = 1$, then (1.8) and (1.9) reduce to the modified Mann iterative with errors in the sense of Xu and Liu, respectively.

Using iteration schemes (1.6) and (1.8), under appropriate conditions on real sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ in $[0, 1]$, we improve Theorem 1.1 and 1.2 by removing the boundedness condition imposed on C and extending the map to a finite family of the p-asymptotically hemi-contractive type mappings. Moreover, we will also study the sequence of the modified Ishikawa-Liu and Mann-Liu iterative types for approximating common fixed points of a finite family of the p-asymptotically hemi-contractive type mappings in a real p-uniformly convex Banach space. In particular, our theorems hold in L_p, ℓ_p and $W^{1,p}$ spaces for $1 < p < \infty$.

2. Preliminaries and Lemmas

Let E be an arbitrary real Banach space. The modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) := \inf\{1 - \frac{1}{2}\|x+y\| : \|x\| = 1, \|y\| = 1, \|x-y\| \geq \varepsilon\}. \tag{2.1}$$

A Banach space E is called *uniformly convex* if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. Let J_p $(p > 1)$ denote the generalized duality mapping from E into 2^{E^*} given by

$$J_p(x) := \{f \in E^* : \langle x, f\rangle = \|x\|^p \text{ and } \|f\| = \|x\|^{p-1}\}, \tag{2.2}$$

where E^* denotes the dual space of E and $\langle .,. \rangle$ denotes the generalized duality pairing. In particular, J_2 is called *the normalized duality mapping* and it is usually denoted by J. It is well known that $J_p(x) = \|x\|^{p-2}J(x)$ if $x \neq 0$. A Banach space E is called p*-uniformly convex* if there exists a constant $d > 0$ such that $\delta_E(\varepsilon) \geq d\varepsilon^p$ for all $0 < \varepsilon \leq 2$. It is known (see e.g. [11]) that

$$L_p \text{ is } \begin{cases} \text{2-uniformly convex if } 1 < p \leq 2, \\ p\text{-uniformly convex if } p \geq 2. \end{cases}$$

In the sequel we shall need the following results.

Lemma 2.1. (see [11, Theorem 1]). *Let $p > 1$ be a given real number. Let E be a p-uniformly convex Banach space. Then, there exists a constant $c > 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - cW_p(\lambda)\|x-y\|^p, \tag{2.3}$$

for all $\lambda \in [0,1]$ and $x, y \in E$, where $W_p(\lambda) := \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$.

Lemma 2.2. (see [11, Corollary 1]). *Let $p > 1$ be a given real number. Then the following statements about a Banach space E are equivalent:*

(i) *E is p-uniformly convex;*
(ii) *There is a constant $s > 0$ such that for every $x, y \in E$, $f_x \in J_p(x)$, the following inequality holds:*

$$\|x+y\|^p \geq \|x\|^p + p\langle y, f_x\rangle + s\|y\|^p. \tag{2.4}$$

Remark 2.1. (see [1, Remark 1]). First, replacing x by $(x-y)$ in inequality (2.4), then y by $(-y)$ and using the Cauchy-Schwarz inequality, we have

$$\|x+y\|^p \leq \|x\|^p + p\|y\| \cdot \|x+y\|^{p-1}. \tag{2.5}$$

Lemma 2.3. (see [8, Lemma 1]). *Let $\{\varphi_n\}_{n=0}^\infty$, $\{\psi_n\}_{n=0}^\infty$ and $\{w_n\}_{n=0}^\infty$ be sequences of nonnegative real numbers satisfying the inequality*

$$\varphi_{n+1} \leq (1+w_n)\varphi_n + \psi_n, \quad \forall n \geq n_0 \tag{2.6}$$

where n_0 is some nonnegative integer. If $\sum_{n=0}^\infty w_n < \infty$ and $\sum_{n=0}^\infty \psi_n < \infty$, then $\lim_{n\to\infty} \varphi_n$ exists. In particular, if $\{\varphi_n\}_{n=1}^\infty$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty}\varphi_n = 0$.

3. Main Results

Now, we state and prove the following theorems:

Theorem 3.1. *Let C be a nonempty closed convex subset of a real p-uniformly convex Banach space E with $p > 1$. Let $T_i : C \to C$ be uniformly L-Lipschitzian and p-asymptotically hemicontractive type mapping with respect to a constant $k_i \in [0,1]$ for each $i = 1, 2, \ldots, N$ such that $\bigcap_{i=1}^N F(T_i) \neq \phi$. Let $\{u_n\}_{n=0}^\infty$ and $\{v_n\}_{n=0}^\infty$ be bounded sequences in C and $\{\alpha_n\}_{n=0}^\infty$, $\{\beta_n\}_{n=0}^\infty$, $\{\gamma_n\}_{n=0}^\infty$ and $\{\delta_n\}_{n=0}^\infty$ be sequences in $[0,1]$ satisfying the following conditions:*

(i) $[L(1-a_2)]^p < k < ca(1-a_2)^{-1}$, $a = a_1a_2(a_1^{p-1}+a_2^{p-1})$;
(ii) $a_1 \le \frac{\alpha_n}{1-\gamma_n}, \frac{\beta_n}{1-\delta_n} \le 1-a_2$, $0 \le \gamma_n, \delta_n < 1$;
(iii) $\sum_{n=0}^{\infty} \gamma_n < \infty$, $\sum_{n=0}^{\infty} \delta_n < \infty$

for all $n \in \overline{\mathbb{N}}$ *and some* $a_1, a_2 \in (0,1)$*, where* c *is the constant appearing in inequality (2.3) and* $k := \max_{1\le i\le N}\{k_i\}$*. Suppose that one member of the family* $\{T_i\}_{i=1}^N$ *is semi-compact. Then the sequence* $\{x_n\}_{n=0}^{\infty}$ *is defined by (1.6) converges strongly to a common fixed point of the family* $\{T_i\}_{i=1}^N$.

Proof. Take $q \in \bigcap_{i=1}^N F(T_i)$. Put $M := \max\{\sup_{n\in\overline{\mathbb{N}}} \|u_n - q\|, \sup_{n\in\overline{\mathbb{N}}} \|v_n - q\|\} < \infty$. By the definition of $\{x_n\}$ and Lemma 2.1, we have

$$\begin{aligned}
\|x_n - q\|^p &= \|(1-\alpha_{n-1}-\gamma_{n-1})(x_{n-1}-q) + \alpha_{n-1}(T_n^m y_{n-1}-q) + \gamma_{n-1}(u_{n-1}-q)\|^p \\
&= \Bigg\|(1-\gamma_{n-1})\left[\left(1-\frac{\alpha_{n-1}}{1-\gamma_{n-1}}\right)(x_{n-1}-q) + \frac{\alpha_{n-1}}{1-\gamma_{n-1}}(T_n^m y_{n-1}-q)\right] \\
&\quad +\gamma_{n-1}(u_{n-1}-q)\Bigg\|^p \\
&\le (1-\gamma_{n-1})\left\|\left(1-\frac{\alpha_{n-1}}{1-\gamma_{n-1}}\right)(x_{n-1}-q) + \frac{\alpha_{n-1}}{1-\gamma_{n-1}}(T_n^m y_{n-1}-q)\right\|^p \\
&\quad +\gamma_{n-1}M^p \\
&\le (1-\alpha_{n-1}-\gamma_{n-1})\|x_{n-1}-q\|^p + \alpha_{n-1}\|T_n^m y_{n-1}-q\|^p \\
&\quad -c(1-\gamma_{n-1})W_p\left(\frac{\alpha_{n-1}}{1-\gamma_{n-1}}\right)\|x_{n-1}-T_n^m y_{n-1}\|^p + \gamma_{n-1}M^p, \qquad (3.1)
\end{aligned}$$

where $T_n^m = T_i^m$ with $n = (m-1)N + i$ for each $i = 1, 2, \ldots, N$. Since each T_i is p-asymptotically hemi-contractive type mapping with respect to a constant k_i, it follows from (1.2) that there exists a positive integer n_i such that

$$\|T_i^m y_{n-1} - q\|^p \le \|y_{n-1}-q\|^p + k_i\|y_{n-1} - T_i^m y_{n-1}\|^p, \qquad (3.2)$$

$$\|T_i^m x_{n-1} - q\|^p \le \|x_{n-1}-q\|^p + k_i\|x_{n-1} - T_i^m x_{n-1}\|^p, \qquad (3.3)$$

for all $n \ge n_i$ and each $i = 1, 2, \ldots, N$. Let $\ell := \max\{n_i : i = 1, 2, \ldots, N\} < \infty$ and since $k = \max\{k_i : i = 1, 2, \ldots, N\}$. Then, by (3.2) and (3.3), we have

$$\|T_n^m y_{n-1} - q\|^p \le \|y_{n-1}-q\|^p + k\|y_{n-1} - T_n^m y_{n-1}\|^p, \qquad (3.4)$$

$$\|T_n^m x_{n-1} - q\|^p \le \|x_{n-1}-q\|^p + k\|x_{n-1} - T_n^m x_{n-1}\|^p, \qquad (3.5)$$

for all $n \ge \ell$.

Observe that for all $n \ge 1$ we have

$$\begin{aligned}
\|T_n^m y_{n-1} - q\| &= L\|y_{n-1}-q\| \\
&\le L[(1-\beta_{n-1}-\delta_{n-1})\|x_{n-1}-q\| + \beta_{n-1}\|T_n^m x_{n-1}-q\| \\
&\quad +\delta_{n-1}\|v_{n-1}-q\|] \\
&\le L(1+L)\|x_{n-1}-q\| + LM, \qquad (3.6)
\end{aligned}$$

$$\begin{aligned}
\|v_{n-1} - T_n^m y_{n-1}\|^p &\le [\|v_{n-1} - q\| + \|T_n^m y_{n-1} - q\|]^p \\
&\le [L(1+L)\|x_{n-1} - q\| + M(1+L)]^p \quad \text{(by (3.6))} \\
&\le [2\max\{L(1+L)\|x_{n-1} - q\|, M(1+L)\}]^p \\
&\le [2L(1+L)]^p\|x_{n-1} - q\|^p + [2M(1+L)]^p \\
&\le M_1\|x_{n-1} - q\|^p + M_1 \quad \text{for some constant } M_1 > 0, \qquad (3.7)\\
\|x_{n-1} - v_{n-1}\|^p &\le (\|x_{n-1} - q\| + \|v_{n-1} - q\|)^p \\
&\le (\|x_{n-1} - q\| + M)^p \\
&\le 2^p\|x_{n-1} - q\|^p + (2M)^p \\
&\le M_2\|x_{n-1} - q\|^p + M_2 \quad \text{for some constant } M_2 > 0. \qquad (3.8)
\end{aligned}$$

By the definition of $\{y_{n-1}\}$, Lemma 2.1, (3.5), (3.8) and (3.7), we obtain that for all $n \ge \ell$,

$$\begin{aligned}
\|y_{n-1} - q\|^p &= \|(1-\beta_{n-1}-\delta_{n-1})(x_{n-1}-q) + \beta_{n-1}(T_n^m x_{n-1}-q)+\delta_{n-1}(v_{n-1}-q)\|^p \\
&= \Bigg\|(1-\delta_{n-1})\left[\left(1-\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)(x_{n-1}-q) + \frac{\beta_{n-1}}{1-\delta_{n-1}}(T_n^m x_{n-1}-q)\right] \\
&\quad +\delta_{n-1}(v_{n-1}-q)\Bigg\|^p \\
&\le (1-\delta_{n-1})\left\|\left(1-\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)(x_{n-1}-q) + \frac{\beta_{n-1}}{1-\delta_{n-1}}(T_n^m x_{n-1}-q)\right\|^p \\
&\quad +\delta_{n-1}M^p \\
&\le (1-\beta_{n-1}-\delta_{n-1})\|x_{n-1}-q\|^p + \beta_{n-1}\|T_n^m x_{n-1}-q\|^p \\
&\quad -c(1-\delta_{n-1})W_p\left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)\|x_{n-1}-T_n^m x_{n-1}\|^p + \delta_{n-1}M^p \\
&\le (1-\delta_{n-1})\|x_{n-1}-q\|^p - \left[c(1-\delta_{n-1})W_p\left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right) - \beta_{n-1}k\right] \\
&\quad \times\|x_{n-1}-T_n^m x_{n-1}\|^p + \delta_{n-1}M^p, \qquad (3.9)
\end{aligned}$$

$$\begin{aligned}
\|x_{n-1}-y_{n-1}\|^p &= \|(1-\beta_{n-1}-\delta_{n-1})(x_{n-1}-x_{n-1}) + \beta_{n-1}(x_{n-1}-T_n^m x_{n-1}) \\
&\quad +\delta_{n-1}(x_{n-1}-v_{n-1})\|^p \\
&= \Bigg\|(1-\delta_{n-1})\Bigg[\left(1-\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)(x_{n-1}-x_{n-1}) \\
&\quad +\frac{\beta_{n-1}}{1-\delta_{n-1}}(x_{n-1}-T_n^m x_{n-1})\Bigg]+\delta_{n-1}(x_{n-1}-v_{n-1})\Bigg\|^p \\
&\le (1-\delta_{n-1})\Bigg\|\left(1-\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)(x_{n-1}-x_{n-1}) \\
&\quad +\frac{\beta_{n-1}}{1-\delta_{n-1}}(x_{n-1}-T_n^m x_{n-1})\Bigg\|^p + \delta_{n-1}\|x_{n-1}-v_{n-1}\|^p \\
&\le \left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)^p\|x_{n-1}-T_n^m x_{n-1}\|^p + \delta_{n-1}(M_2\|x_{n-1}-q\|^p + M_2), \qquad (3.10)
\end{aligned}$$

$$
\begin{aligned}
& \|y_{n-1}-T_n^m y_{n-1}\|^p \\
= \ & \|(1-\beta_{n-1}-\delta_{n-1})(x_{n-1}-T_n^m y_{n-1})+\beta_{n-1}(T_n^m x_{n-1}-T_n^m y_{n-1}) \\
& +\delta_{n-1}(v_{n-1}-T_n^m y_{n-1})\|^p \\
= \ & \left\|(1-\delta_{n-1})\left[\left(1-\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)(x_{n-1}-T_n^m y_{n-1})+\frac{\beta_{n-1}}{1-\delta_{n-1}}(T_n^m x_{n-1}-T_n^m y_{n-1})\right]\right. \\
& \left.+\delta_{n-1}(v_{n-1}-T_n^m y_{n-1})\right\|^p \\
\leq \ & (1-\delta_{n-1})\left\|\left(1-\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)(x_{n-1}-T_n^m y_{n-1})+\frac{\beta_{n-1}}{1-\delta_{n-1}}(T_n^m x_{n-1}-T_n^m y_{n-1})\right\|^p \\
& +\delta_{n-1}\|v_{n-1}-T_n^m y_{n-1}\|^p \\
\leq \ & (1-\beta_{n-1}-\delta_{n-1})\|x_{n-1}-T_n^m y_{n-1}\|^p+\beta_{n-1}\|T_n^m x_{n-1}-T_n^m y_{n-1}\|^p \\
& -c(1-\delta_{n-1})W_p\left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)\|x_{n-1}-T_n^m x_{n-1}\|^p+\delta_{n-1}(M_1\|x_{n-1}-q\|^p+M_1) \\
\leq \ & (1-\beta_{n-1}-\delta_{n-1})\|x_{n-1}-T_n^m y_{n-1}\|^p+\beta_{n-1}L^p\left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)^p\|x_{n-1}-T_n^m x_{n-1}\|^p \\
& +\beta_{n-1}L^p\delta_{n-1}(M_2\|x_{n-1}-q\|^p+M_2)-c(1-\delta_{n-1})W_p\left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right) \\
& \times\|x_{n-1}-T_n^m x_{n-1}\|^p+\delta_{n-1}(M_1\|x_{n-1}-q\|^p+M_1) \quad \text{(by (3.10))} \\
\leq \ & (1-\beta_{n-1}-\delta_{n-1})\|x_{n-1}-T_n^m y_{n-1}\|^p \\
& -\left[c(1-\delta_{n-1})W_p\left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)-\beta_{n-1}L^p\left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)^p\right]\|x_{n-1}-T_n^m x_{n-1}\|^p \\
& +\delta_{n-1}(M_3\|x_{n-1}-q\|^p+M_3) \quad \text{for some constant } M_3>0. \qquad (3.11)
\end{aligned}
$$

Substituting (3.9) and (3.11) into (3.4) and simplifying, we get

$$
\begin{aligned}
& \|T_n^m y_{n-1}-q\|^p \\
\leq \ & (1-\delta_{n-1}+\delta_{n-1}kM_3)\|x_{n-1}-q\|^p+k(1-\beta_{n-1}-\delta_{n-1})\|x_{n-1}-T_n^m y_{n-1}\|^p \\
& -\left\{c(1-\delta_{n-1})W_p\left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)-\beta_{n-1}k+k\left[c(1-\delta_{n-1})W_p\left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)\right.\right. \\
& \left.\left.-\beta_{n-1}L^p\left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)^p\right]\right\}\|x_{n-1}-T_n^m x_{n-1}\|^p \\
& +\delta_{n-1}(M^p+kM_3), \quad \text{for all } n\geq \ell. \qquad (3.12)
\end{aligned}
$$

Using (3.12) in (3.1), we then obtain that for all $n\geq \ell$,

$$
\begin{aligned}
& \|x_n-q\|^p \\
\leq \ & (1+\delta_{n-1}M_3)\|x_{n-1}-q\|^p \\
& -\left[c(1-\gamma_{n-1})W_p\left(\frac{\alpha_{n-1}}{1-\gamma_{n-1}}\right)-\alpha_{n-1}k(1-\beta_{n-1}-\delta_{n-1})\right]\|x_{n-1}-T_n^m y_{n-1}\|^p
\end{aligned}
$$

$$
\begin{aligned}
&-\alpha_{n-1}\Bigg\{c(1-\delta_{n-1})W_p\left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)-\beta_{n-1}k \\
&+k\left[c(1-\delta_{n-1})W_p\left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)-\beta_{n-1}L^p\left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)^p\right]\Bigg\}\|x_{n-1}-T_n^m x_{n-1}\|^p \\
&+(\delta_{n-1}+\gamma_{n-1})M_4 \quad \text{for some constant } M_4>0. \qquad (3.13)
\end{aligned}
$$

Observe that $W_p\left(\frac{\alpha_{n-1}}{1-\gamma_{n-1}}\right)\geq a>0$ and $W_p\left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)\geq a>0$. Then, by (i) and (ii), we have

$$
\begin{aligned}
& c(1-\gamma_{n-1})W_p\left(\frac{\alpha_{n-1}}{1-\gamma_{n-1}}\right)-\alpha_{n-1}k(1-\beta_{n-1}-\delta_{n-1}) \\
\geq\;& (1-\gamma_{n-1})\left[cW_p\left(\frac{\alpha_{n-1}}{1-\gamma_{n-1}}\right)-\left(\frac{\alpha_{n-1}}{1-\gamma_{n-1}}\right)k\right] \\
\geq\;& (1-\gamma_{n-1})[ca-(1-a_2)k]>0 \qquad (3.14)
\end{aligned}
$$

and

$$
\begin{aligned}
& c(1-\delta_{n-1})W_p\left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)-\beta_{n-1}k \\
& \quad +k\left[c(1-\delta_{n-1})W_p\left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)-\beta_{n-1}L^p\left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)^p\right] \\
=\;& c(1+k)(1-\delta_{n-1})W_p\left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)-k\beta_{n-1}\left[1+L^p\left(\frac{\beta_{n-1}}{1-\delta_{n-1}}\right)^p\right] \\
\geq\;& c(1+k)(1-\delta_{n-1})a-k(1-\delta_{n-1})(1-a_2)[1+L^p(1-a_2)^p] \\
\geq\;& c(1+k)(1-\delta_{n-1})a-ca(1-\delta_{n-1})[1+L^p(1-a_2)^p] \\
=\;& (1-\delta_{n-1})ca[k-L^p(1-a_2)^p]>0. \qquad (3.15)
\end{aligned}
$$

Put $\varepsilon:=\min\{ca-(1-a_2)k, ca[k-L^p(1-a_2)^p]\}>0$. Then, using (3.14) and (3.15) in (3.13), we obtain

$$
\begin{aligned}
\|x_n-q\|^p \leq\;& (1+\delta_{n-1}M_3)\|x_{n-1}-q\|^p-(1-\gamma_{n-1})\varepsilon\|x_{n-1}-T_n^m y_{n-1}\|^p \\
& -\alpha_{n-1}(1-\delta_{n-1})\varepsilon\|x_{n-1}-T_n^m x_{n-1}\|^p+(\delta_{n-1}+\gamma_{n-1})M_4 \\
\leq\;& (1+\delta_{n-1}M_3)\|x_{n-1}-q\|^p+(\delta_{n-1}+\gamma_{n-1})M_4, \quad \text{for all } n\geq \ell, \qquad (3.16)
\end{aligned}
$$

Since $\sum_{n=0}^{\infty}\delta_n<\infty$ and $\sum_{n=0}^{\infty}\gamma_n<\infty$, if follows from Lemma 2.3 that $\lim_{n\to\infty}\|x_n-q\|^p$ exists and hence $\{\|x_n-q\|^p\}$ is bounded.

Observe that for all $n\geq 1$ we have

$$
\begin{aligned}
\|x_{n-1}-T_n^m x_{n-1}\|^p \leq\;& (\|x_{n-1}-q\|+\|T_n^m x_{n-1}-q\|)^p \\
\leq\;& (1+L)^p\|x_{n-1}-q\|^p \qquad (3.17)
\end{aligned}
$$

and from (3.6) we have

$$
\begin{aligned}
\|x_{n-1}-T_n^m y_{n-1}\|^p \leq\;& (\|x_{n-1}-q\|+\|T_n^m y_{n-1}-q\|)^p \\
\leq\;& [(1+L+L^2)\|x_{n-1}-q\|+LM]^p \\
\leq\;& M_5\|x_{n-1}-q\|^p+M_5 \quad \text{for some constant } M_5>0. \qquad (3.18)
\end{aligned}
$$

Thus, $\{\|x_{n-1}-T_n^m x_{n-1}\|^p\}$ and $\{\|x_{n-1}-T_n^m y_{n-1}\|^p\}$ are bounded.

Let $M_6 := \max\{\sup_{n\in\bar{\mathbb{N}}} \|x_n - q\|^p, \sup_{n\in\bar{\mathbb{N}}} \|x_{n-1} - T_n^m x_{n-1}\|^p, \sup_{n\in\bar{\mathbb{N}}} \|x_{n-1} - T_n^m y_{n-1}\|^p\} < \infty$. Applying (ii), (iii) and (3.16), we get that

$$\begin{aligned} & \varepsilon\|x_{n-1} - T_n^m y_{n-1}\|^p + a_1\varepsilon\|x_{n-1} - T_n^m x_{n-1}\|^p \\ \leq\ & \varepsilon\|x_{n-1} - T_n^m y_{n-1}\|^p + (\alpha_{n-1} + \gamma_{n-1}a_1)\varepsilon\|x_{n-1} - T_n^m x_{n-1}\|^p \\ \leq\ & \gamma_{n-1}\varepsilon\|x_{n-1} - T_n^m y_{n-1}\|^p + (\alpha_{n-1}\delta_{n-1} + \gamma_{n-1}a_1)\varepsilon\|x_{n-1} - T_n^m x_{n-1}\|^p \\ & +(1 + \delta_{n-1}M_3)\|x_{n-1} - q\|^p - \|x_n - q\|^p + (\delta_{n-1} + \gamma_{n-1})M_4 \\ \leq\ & \|x_{n-1} - q\|^p - \|x_n - q\|^p + \delta_{n-1}[(\varepsilon+M_3)M_6 + M_4] + \gamma_{n-1}[(\varepsilon+\varepsilon a_1)M_6 + M_4] \\ \to\ & 0 \text{ as } n \to \infty. \end{aligned}$$

This implies that $\lim_{n\to\infty} \|x_{n-1} - T_n^m y_{n-1}\| = 0$ and $\lim_{n\to\infty} \|x_{n-1} - T_n^m x_{n-1}\| = 0$. Since

$$x_n - x_{n-1} = \alpha_{n-1}(T_n^m y_{n-1} - x_{n-1}) + \gamma_{n-1}(u_{n-1} - x_{n-1})$$

and $\{u_{n-1} - x_{n-1}\}$ is bounded. Thus $\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0$. Hence, $\lim_{n\to\infty} \|x_n - x_{n+i}\| = 0$ for each $i = 1, 2, \ldots, N$. For all $n > N$, we have $T_n = T_{n-N}$ so that

$$\begin{aligned} & \|T_n^{m-1}x_n - x_n\| \\ \leq\ & \|T_n^{m-1}x_n - T_{n-N}^{m-1}x_{(n-N)-1}\| + \|T_{n-N}^{m-1}x_{(n-N)-1} - x_{(n-N)-1}\| \\ & +\|x_{(n-N)-1} - x_n\| \\ \leq\ & (1 + L)\|x_n - x_{(n-N)-1}\| + \|T_{n-N}^{m-1}x_{(n-N)-1} - x_{(n-N)-1}\|. \end{aligned} \tag{3.19}$$

Hence,

$$\begin{aligned} & \|x_n - T_n x_n\| \\ \leq\ & \|x_n - x_{n-1}\| + \|x_{n-1} - T_n^m x_{n-1}\| + \|T_n^m x_{n-1} - T_n^m x_n\| + \|T_n^m x_n - T_n x_n\| \\ \leq\ & (1 + L)\|x_n - x_{n-1}\| + \|x_{n-1} - T_n^m x_{n-1}\| + L\|T_n^{m-1}x_n - x_n\| \\ \leq\ & (1 + L)\|x_n - x_{n-1}\| + L(1 + L)\|x_n - x_{(n-N)-1}\| + \|x_{n-1} - T_n^m x_{n-1}\| \\ & +L\|T_{n-N}^{m-1}x_{(n-N)-1} - x_{(n-N)-1}\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.20}$$

Thus, for each $i = 1, 2, \ldots, N$, we have

$$\begin{aligned} & \|x_n - T_{n+i}x_n\| \\ \leq\ & \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \|T_{n+i}x_{n+i} - T_{n+i}x_n\| \\ \leq\ & (1 + L)\|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.21}$$

It follows that $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for each $i = 1, 2, \ldots, N$.

Since one member of $\{T_i\}_{i=1}^N$ is semi-compact, then there exists a subsequence $\{x_{n_j}\}$ of the sequence $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $z \in C$ and hence

$$\|z - T_i z\| = \lim_{j\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0$$

for each $i = 1, 2, \ldots, N$. That is, $z \in \bigcap_{i=1}^N F(T_i)$. Replacing the q in inequality (3.16) by z we obtain that

$$\|x_n - z\|^p \leq (1 + \delta_{n-1}M_3)\|x_{n-1} - z\|^p + (\delta_{n-1} + \gamma_{n-1})M_4, \tag{3.22}$$

for all $n \geq \ell$. Since $\{x_{n_j}\}$ converges strongly to z and (iii) holds, it follows from (3.22) and Lemma 2.3 that $\lim_{n\to\infty} \|x_n - z\| = 0$, i.e., $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$. This completes the proof. ∎

Theorem 3.2. *Let E be a real p-uniformly convex Banach space with $p > 1$. Let $T_i : E \to E$ be uniformly L-Lipschitzian and p-asymptotically hemi-contractive type mapping with respect to a constant $k_i \in [0,1]$ for each $i = 1, 2, \ldots, N$ such that $\bigcap_{i=1}^N F(T_i) \neq \phi$.*

Let $\{u_n\}_{n=0}^\infty$ and $\{v_n\}_{n=0}^\infty$ be sequences in E such that $\sum_{n=0}^\infty \|u_n\| < \infty$ and $\sum_{n=0}^\infty \|v_n\| < \infty$ and $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ be sequences in $[0,1]$ satisfying the following conditions:

(i) $[L(1-a_2)]^p < k < ca(1-a_2)^{-1}$, $a = a_1 a_2 (a_1^{p-1} + a_2^{p-1})$;

(ii) $a_1 \leq \alpha_n, \beta_n \leq 1 - a_2$,

for all $n \in \overline{\mathbb{N}}$ and some $a_1, a_2 \in (0,1)$, where c is the constant appearing in inequality (2.3) and $k := \max_{1 \leq i \leq N}\{k_i\}$. Suppose that one member of the family $\{T_i\}_{i=1}^N$ is semi-compact. Then the sequence $\{x_n\}_{n=0}^\infty$ is defined by (1.7) converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^N$.

Proof. Take $q \in \bigcap_{i=1}^N F(T_i)$. Since $\|u_n\| \to 0$ and $\|v_n\| \to 0$ as $n \to \infty$.

Thus, there exist $n_0 \in \overline{\mathbb{N}}$ and $e \in (0,1)$ such that

$$1 - p\|u_{n-1}\| > e \quad \text{and} \quad 1 - p\|v_{n-1}\| > e \tag{3.23}$$

for all $n \geq n_0$. By the definition of $\{x_n\}$, Lemma 2.1 and (2.5), we have

$$\begin{aligned} \|x_n - q\|^p &= \|(1-\alpha_{n-1})(x_{n-1} - q + u_{n-1}) + \alpha_{n-1}(T_n^m y_{n-1} - q + u_{n-1})\|^p \\ &\leq (1-\alpha_{n-1})\|x_{n-1} - q + u_{n-1}\|^p + \alpha_{n-1}\|T_n^m y_{n-1} - q + u_{n-1}\|^p \\ &\quad - cW_p(\alpha_{n-1})\|x_{n-1} - T_n^m y_{n-1}\|^p \\ &\leq (1-\alpha_{n-1})\|x_{n-1} - q\|^p + p(1-\alpha_{n-1})\|u_{n-1}\|\,\|x_{n-1} - q + u_{n-1}\|^{p-1} \\ &\quad + \alpha_{n-1}\|T_n^m y_{n-1} - q\|^p + p\alpha_{n-1}\|u_{n-1}\|\,\|T_n^m y_{n-1} - q + u_{n-1}\|^{p-1} \\ &\quad - cW_p(\alpha_{n-1})\|x_{n-1} - T_n^m y_{n-1}\|^p, \quad \text{for all } n \geq 1. \end{aligned} \tag{3.24}$$

Observe that, by (3.23), (2.5) and Lemma 2.1, for all $n \geq n_0$ we have

$$\begin{aligned} \|x_{n-1} - q + u_{n-1}\|^{p-1} &\leq 1 + \|x_{n-1} - q + u_{n-1}\|^p \\ &\leq 1 + \|x_{n-1} - q\|^p + p\|u_{n-1}\|\,\|x_{n-1} - q + u_{n-1}\|^{p-1} \\ &\leq e^{-1}(1 + \|x_{n-1} - q\|^p), \end{aligned} \tag{3.25}$$

$$\begin{aligned} \|y_{n-1} - q\|^p &= \|(1-\beta_{n-1})(x_{n-1} - q) + \beta_{n-1}(T_n^m x_{n-1} - q) + v_{n-1}\|^p \\ &\leq \|(1-\beta_{n-1})(x_{n-1} - q) + \beta_{n-1}(T_n^m x_{n-1} - q)\|^p \\ &\quad + p\|v_{n-1}\|\,\|y_{n-1} - q\|^{p-1} \\ &\leq (1-\beta_{n-1})\|x_{n-1} - q\|^p + \beta_{n-1}\|T_n^m x_{n-1} - q\|^p \\ &\quad + p\|v_{n-1}\|\,(1 + \|y_{n-1} - q\|^p) \\ &\leq (1 + L^p)\|x_{n-1} - q\|^p + p\|v_{n-1}\|\,(1 + \|y_{n-1} - q\|^p) \\ &\leq e^{-1}(1 + L^p)\|x_{n-1} - q\|^p + e^{-1}p\|v_{n-1}\|, \end{aligned} \tag{3.26}$$

$$\begin{aligned} &\|T_n^m y_{n-1} - q + u_{n-1}\|^{p-1} \\ &\leq 1 + \|T_n^m y_{n-1} - q + u_{n-1}\|^p \\ &\leq 1 + \|T_n^m y_{n-1} - q\|^p + p\|u_{n-1}\|\,\|T_n^m y_{n-1} - q + u_{n-1}\|^{p-1} \\ &\leq e^{-1}(1 + L^p\|y_{n-1} - q\|^p) \\ &\leq e^{-1}[1 + L^p e^{-1}(1 + L^p)\|x_{n-1} - q\|^p + L^p e^{-1} p\|v_{n-1}\|] \quad \text{(by (3.26))} \\ &\leq M_7(1 + \|x_{n-1} - q\|^p + \|v_{n-1}\|), \quad \text{for some constant } M_7 > 0. \end{aligned} \tag{3.27}$$

By the definition of $\{y_{n-1}\}$, Lemma 2.1 and (2.5), we get

$$\begin{aligned} \|y_{n-1}-q\|^p &= \|(1-\beta_{n-1})(x_{n-1}-q+v_{n-1})+\beta_{n-1}(T_n^m x_{n-1}-q+v_{n-1})\|^p \\ &\le (1-\beta_{n-1})\|x_{n-1}-q+v_{n-1}\|^p+\beta_{n-1}\|T_n^m x_{n-1}-q+v_{n-1}\|^p \\ &\quad -cW_p(\beta_{n-1})\|x_{n-1}-T_n^m x_{n-1}\|^p \\ &\le (1-\beta_{n-1})\|x_{n-1}-q\|^p+p(1-\beta_{n-1})\|v_{n-1}\|\,\|x_{n-1}-q+v_{n-1}\|^{p-1} \\ &\quad +\beta_{n-1}\|T_n^m x_{n-1}-q\|^p+p\beta_{n-1}\|v_{n-1}\|\,\|T_n^m x_{n-1}-q+v_{n-1}\|^{p-1} \\ &\quad -cW_p(\beta_{n-1})\|x_{n-1}-T_n^m x_{n-1}\|^p, \quad \text{for all } n\ge 1, \end{aligned} \tag{3.28}$$

$$\begin{aligned} &\|y_{n-1}-T_n^m y_{n-1}\|^p \\ =\ & \|(1-\beta_{n-1})(x_{n-1}-T_n^m y_{n-1}+v_{n-1})+\beta_{n-1}(T_n^m x_{n-1}-T_n^m y_{n-1}+v_{n-1})\|^p \\ \le\ & \|(1-\beta_{n-1})\|x_{n-1}-T_n^m y_{n-1}+v_{n-1}\|^p+\beta_{n-1}\|T_n^m x_{n-1}-T_n^m y_{n-1}+v_{n-1}\|^p \\ & -cW_p(\beta_{n-1})\|x_{n-1}-T_n^m x_{n-1}\|^p \\ \le\ & (1-\beta_{n-1})\|x_{n-1}-T_n^m y_{n-1}\|^p+p(1-\beta_{n-1})\|v_{n-1}\|\,\|x_{n-1}-T_n^m y_{n-1}+v_{n-1}\|^{p-1} \\ & +\beta_{n-1}\|T_n^m x_{n-1}-T_n^m y_{n-1}\|^p+p\beta_{n-1}\|v_{n-1}\|\,\|T_n^m x_{n-1}-T_n^m y_{n-1}+v_{n-1}\|^{p-1} \\ & -cW_p(\beta_{n-1})\|x_{n-1}-T_n^m x_{n-1}\|^p, \quad \text{for all } n\ge 1. \end{aligned} \tag{3.29}$$

Observe that for all $n\ge n_0$, we have

$$\begin{aligned} \|x_{n-1}-q+v_{n-1}\|^{p-1} &\le 1+\|x_{n-1}-q+v_{n-1}\|^p \\ &\le 1+\|x_{n-1}-q\|^p+p\|v_{n-1}\|\,\|x_{n-1}-q+v_{n-1}\|^{p-1} \\ &\le e^{-1}(1+\|x_{n-1}-q\|^p), \end{aligned} \tag{3.30}$$

$$\begin{aligned} \|T_n^m x_{n-1}-q+v_{n-1}\|^{p-1} &\le 1+\|T_n^m x_{n-1}-q+v_{n-1}\|^p \\ &\le 1+\|T_n^m x_{n-1}-q\|^p+p\|v_{n-1}\|\,\|T_n^m x_{n-1}-q+v_{n-1}\|^{p-1} \\ &\le e^{-1}(1+L^p\|x_{n-1}-q\|^p), \end{aligned} \tag{3.31}$$

$$\begin{aligned} \|x_{n-1}-T_n^m y_{n-1}\|^p &\le (\|x_{n-1}-q\|+\|T_n^m y_{n-1}-q\|)^p \\ &\le [(1+L+L^2)\|x_{n-1}-q\|+LM]^p \quad \text{(by (3.6))} \\ &\le M_8(\|x_{n-1}-q\|^p+1), \quad \text{for some constant } M_8>0, \end{aligned} \tag{3.32}$$

$$\begin{aligned} &\|x_{n-1}-T_n^m y_{n-1}+v_{n-1}\|^{p-1} \\ \le\ & 1+\|x_{n-1}-T_n^m y_{n-1}+v_{n-1}\|^p \\ \le\ & 1+\|x_{n-1}-T_n^m y_{n-1}\|^p+p\|v_{n-1}\|\,\|x_{n-1}-T_n^m y_{n-1}+v_{n-1}\|^{p-1} \\ \le\ & e^{-1}[1+M_8(\|x_{n-1}-q\|^p+1)] \quad \text{(by (3.32))}, \end{aligned} \tag{3.33}$$

$$\begin{aligned} \|x_{n-1}-y_{n-1}\|^p &\le (\|x_{n-1}-q\|+\|y_{n-1}-q\|)^p \\ &\le 2^p(\|x_{n-1}-q\|^p+\|y_{n-1}-q\|^p) \\ &\le 2^p[1+e^{-1}(1+L^p)]\,\|x_{n-1}-q\|^p+2^pe^{-1}p\|v_{n-1}\| \quad \text{(by (3.26))} \\ &\le M_9(\|x_{n-1}-q\|^p+\|v_{n-1}\|), \quad \text{for some constant } M_9>0, \end{aligned} \tag{3.34}$$

$$\begin{aligned} \|x_{n-1}-y_{n-1}\|^{p-1} &\le 1+\|x_{n-1}-y_{n-1}\|^p \\ &\le 1+M_9(\|x_{n-1}-q\|^p+\|v_{n-1}\|) \quad \text{(by (3.34))}, \end{aligned} \tag{3.35}$$

$$\begin{aligned}
& \|T_n^m x_{n-1} - T_n^m y_{n-1} + v_{n-1}\|^{p-1} \\
\leq\ & 1 + \|T_n^m x_{n-1} - T_n^m y_{n-1} + v_{n-1}\|^p \\
\leq\ & 1 + \|T_n^m x_{n-1} - T_n^m y_{n-1}\|^p + p\|v_{n-1}\| \, \|T_n^m x_{n-1} - T_n^m y_{n-1} + v_{n-1}\|^{p-1} \\
\leq\ & e^{-1}(1 + L^p\|x_{n-1} - y_{n-1}\|^p) \\
\leq\ & e^{-1}[1 + L^p M_9(\|x_{n-1} - q\|^p + \|v_{n-1}\|)] \quad \text{(by (3.34))} \\
\leq\ & M_{10}(1 + \|x_{n-1} - q\|^p + \|v_{n-1}\|), \quad \text{for some constant } M_{10} > 0, \qquad (3.36)
\end{aligned}$$

$$\begin{aligned}
& \|T_n^m x_{n-1} - T_n^m y_{n-1}\|^p \\
\leq\ & L^p\|x_{n-1} - y_{n-1}\|^p \\
=\ & L^p\|\beta_{n-1}(x_{n-1} - T_n^m x_{n-1}) - v_{n-1}\|^p \\
\leq\ & L^p\beta_{n-1}^p\|x_{n-1} - T_n^m x_{n-1}\|^p + pL^p\|v_{n-1}\| \, \|x_{n-1} - y_{n-1}\|^{p-1} \qquad (3.37) \\
\leq\ & L^p\beta_{n-1}^p\|x_{n-1} - T_n^m x_{n-1}\|^p + pL^p\|v_{n-1}\|[1 + M_9(\|x_{n-1} - q\|^p + \|v_{n-1}\|)] \\
& \text{(by (3.35))}
\end{aligned}$$

In virtue of (3.2) and (3.3), we infer that for all $n \geq \ell$ (3.4) and (3.5) hold for some $\ell \geq n_0$. Using (3.5), (3.30) and (3.31) in (3.28), we conclude that for all $n \geq \ell$,

$$\begin{aligned}
& \|y_{n-1} - q\|^p \\
\leq\ & (1 - \beta_{n-1})\|x_{n-1} - q\|^p + p\|v_{n-1}\|e^{-1}(1 + \|x_{n-1} - q\|^p) \\
& + \beta_{n-1}\|x_{n-1} - q\|^p + k\beta_{n-1}\|x_{n-1} - T_n^m x_{n-1}\|^p \\
& + p\|v_{n-1}\|e^{-1}(1 + L^p\|x_{n-1} - q\|^p) - cW_p(\beta_{n-1})\|x_{n-1} - T_n^m x_{n-1}\|^p \\
=\ & [1 + pe^{-1}(1 + L^p)\|v_{n-1}\|] \, \|x_{n-1} - q\|^p \\
& - [cW_p(\beta_{n-1}) - k\beta_{n-1}] \, \|x_{n-1} - T_n^m x_{n-1}\|^p + 2pe^{-1}\|v_{n-1}\| \\
\leq\ & (1 + M_{11}\|v_{n-1}\|)\|x_{n-1} - q\|^p - [cW_p(\beta_{n-1}) - k\beta_{n-1}]\|x_{n-1} - T_n^m x_{n-1}\|^p \\
& + M_{11}\|v_{n-1}\|, \quad \text{for some constant } M_{11} > 0. \qquad (3.38)
\end{aligned}$$

Moreover, using (3.33), (3.37) and (3.36) in (3.29), we get that

$$\begin{aligned}
& \|y_{n-1} - T_n^m y_{n-1}\|^p \\
\leq\ & (1 - \beta_{n-1})\|x_{n-1} - T_n^m y_{n-1}\|^p + p\|v_{n-1}\|e^{-1}[1 + M_8(\|x_{n-1} - q\|^p + 1)] \\
& + \beta_{n-1}^{p+1}L^p\|x_{n-1} - T_n^m x_{n-1}\|^p + pL^p\|v_{n-1}\|[1 + M_9(\|x_{n-1} - q\|^p + \|v_{n-1}\|)] \\
& + p\|v_{n-1}\|M_{10}(1 + \|x_{n-1} - q\|^p + \|v_{n-1}\|) - cW_p(\beta_{n-1})\|x_{n-1} - T_n^m x_{n-1}\|^p \\
=\ & p(e^{-1}M_8 + L^p M_9 + M_{10})\|v_{n-1}\| \, \|x_{n-1} - q\|^p + (1 - \beta_{n-1})\|x_{n-1} - T_n^m y_{n-1}\|^p \\
& - [cW_p(\beta_{n-1}) - \beta_{n-1}^{p+1}L^p] \, \|x_{n-1} - T_n^m x_{n-1}\|^p \\
& + p[e^{-1}(1 + M_8) + L^p(1 + M_9\|v_{n-1}\|) + M_{10}(1 + \|v_{n-1}\|)] \, \|v_{n-1}\| \\
\leq\ & M_{12}\|v_{n-1}\| \, \|x_{n-1} - q\|^p + (1 - \beta_{n-1})\|x_{n-1} - T_n^m y_{n-1}\|^p \\
& - [cW_p(\beta_{n-1}) - \beta_{n-1}^{p+1}L^p] \, \|x_{n-1} - T_n^m x_{n-1}\|^p + M_{12}\|v_{n-1}\|, \qquad (3.39)
\end{aligned}$$

for some constant $M_{12} > 0$ because $\{\|v_n\|\}_{n=0}^{\infty}$ is bounded.

Thus, substituting (3.38) and (3.39) into (3.4), we have

$$\begin{aligned}
& \|T_n^m y_{n-1} - q\|^p \\
\leq\ & [1 + (M_{11} + kM_{12})\|v_{n-1}\|] \, \|x_{n-1} - q\|^p + (M_{11} + kM_{12}) \, \|v_{n-1}\| \\
& - \{cW_p(\beta_{n-1}) - k\beta_{n-1} + k[cW_p(\beta_{n-1}) - \beta_{n-1}^{p+1}L^p]\} \, \|x_{n-1} - T_n^m x_{n-1}\|^p \\
& + k(1 - \beta_{n-1})\|x_{n-1} - T_n^m y_{n-1}\|^p, \qquad (3.40)
\end{aligned}$$

for all $n \geq \ell$. Now using (3.25), (3.40) and (3.27) in (3.24), we obtain that

$$\begin{aligned}
\|x_n - q\|^p \leq\ & (1-\alpha_{n-1})\|x_{n-1}-q\|^p + p\|u_{n-1}\|e^{-1}(1+\|x_{n-1}-q\|^p) \\
& + \alpha_{n-1}[1+(M_{11}+kM_{12})\|v_{n-1}\|]\,\|x_{n-1}-q\|^p + (M_{11}+kM_{12})\,\|v_{n-1}\| \\
& - \alpha_{n-1}\{cW_p(\beta_{n-1}) - k\beta_{n-1} + k[cW_p(\beta_{n-1}) - \beta_{n-1}^{p+1}L^p]\}\,\|x_{n-1}-T_n^m x_{n-1}\|^p \\
& + k\alpha_{n-1}(1-\beta_{n-1})\,\|x_{n-1}-T_n^m y_{n-1}\|^p \\
& + p\|u_{n-1}\|M_7(1+\|x_{n-1}-q\|^p + \|v_{n-1}\|) - cW_p(\alpha_{n-1})\|x_{n-1}-T_n^m y_{n-1}\|^p \\
\leq\ & [1+p(e^{-1}+M_7)\|u_{n-1}\| + (M_{11}+kM_{12})\|v_{n-1}\|]\,\|x_{n-1}-q\|^p \\
& - [cW_p(\alpha_{n-1}) - k\alpha_{n-1}(1-\beta_{n-1})]\,\|x_{n-1}-T_n^m y_{n-1}\|^p \\
& - \alpha_{n-1}\{cW_p(\beta_{n-1}) - k\beta_{n-1} + k[cW_p(\beta_{n-1}) - \beta_{n-1}^{p+1}L^p]\}\,\|x_{n-1}-T_n^m x_{n-1}\|^p \\
& p[e^{-1}+M_7(1+\|v_{n-1}\|)]\,\|u_{n-1}\| + (M_{11}+kM_{12})\,\|v_{n-1}\|, \qquad (3.41)
\end{aligned}$$

for all $n \geq \ell$.

Observe that $W_p(\alpha_{n-1}) \geq a > 0$ and $W_p(\beta_{n-1}) \geq a > 0$. Thus, by (i) and (ii), we get

$$cW_p(\alpha_{n-1}) - k\alpha_{n-1}(1-\beta_{n-1}) \geq ca - k(1-a_2) > 0 \qquad (3.42)$$

and

$$\begin{aligned}
& cW_p(\beta_{n-1}) - k\beta_{n-1} + k[cW_p(\beta_{n-1}) - \beta_{n-1}^{p+1}L^p] \\
=\ & c(1+k)W_p(\beta_{n-1}) - k\beta_{n-1}(1+\beta_{n-1}^p L^p) \\
\geq\ & c(1+k)a - k(1-a_2)[1+(1-a_2)^p L^p] \\
\geq\ & ca[k-(1-a_2)^p L^p] > 0. \qquad (3.43)
\end{aligned}$$

Put ε as in Theorem 3.1. Using (3.42) and (3.43) in (3.41) yields that for all $n \geq \ell$ we have

$$\begin{aligned}
\|x_n - q\|^p \leq\ & [1+M_{13}(\|u_{n-1}\|+\|v_{n-1}\|)]\,\|x_{n-1}-q\|^p - \varepsilon\|x_{n-1}-T_n^m y_{n-1}\|^p \\
& - a_1\varepsilon\|x_{n-1}-T_n^m x_{n-1}\|^p + M_{13}(\|u_{n-1}\|+\|v_{n-1}\|) \qquad (3.44)
\end{aligned}$$

for some constant $M_{13} > 0$. This implies that

$$\|x_n - q\|^p \leq [1+M_{13}(\|u_{n-1}\|+\|v_{n-1}\|)]\,\|x_{n-1}-q\|^p + M_{13}(\|u_{n-1}\|+\|v_{n-1}\|) \qquad (3.45)$$

for all $n \geq \ell$. Since $\sum_{n=0}^{\infty}\|u_n\| < \infty$ and $\sum_{n=0}^{\infty}\|v_n\| < \infty$, it follows from Lemma 2.3 that $\lim_{n\to\infty}\|x_n - q\|^p$ exists. So, by (3.44), we have

$$\begin{aligned}
& \varepsilon\|x_{n-1}-T_n^m y_{n-1}\|^p + a_1\varepsilon\|x_{n-1}-T_n^m x_{n-1}\|^p \\
\leq\ & [1+M_{13}(\|u_{n-1}\|+\|v_{n-1}\|)]\,\|x_{n-1}-q\|^p - \|x_n-q\|^p + M_{13}(\|u_{n-1}\|+\|v_{n-1}\|) \\
\to\ & 0 \text{ as } n\to\infty.
\end{aligned}$$

This implise that $\lim_{n\to\infty}\|x_{n-1}-T_n^m y_{n-1}\| = 0$ and $\lim_{n\to\infty}\|x_{n-1}-T_n^m x_{n-1}\| = 0$. Since

$$x_n - x_{n-1} = \alpha_{n-1}(T_n^m y_{n-1} - x_{n-1}) + u_{n-1}$$

and $\lim_{n\to\infty}\|u_n\| = 0$. Thus $\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0$. The rest of the proof follows the lines similar to Theorem 3.1 and is, therefore, omitted. This completes the proof. ∎

Theorem 3.3. *Let C be a nonempty closed convex subset of a real p-uniformly convex Banach space E with $p > 1$. Let $T_i : C \to C$ be uniformly L-Lipschitzian and p-asymptotically hemicontractive type mapping with respect to a constant $k_i \in [0,1]$ for each $i = 1, 2, \dots, N$ such that $\bigcap_{i=1}^N F(T_i) \neq \phi$. Let $\{u_n\}_{n=0}^{\infty}$ be a bounded sequence in C and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ be sequence in $[0,1]$ satisfying following the conditions:*

(i) $k < ca(1-a_2)^{-1}$, $a = a_1 a_2(a_1^{p-1} + a_2^{p-1})$;
(ii) $a_1 \leq \frac{\alpha_n}{1-\gamma_n} \leq 1 - a_2$, $0 \leq \gamma_n < 1$;

(iii) $\sum_{n=0}^{\infty} \gamma_n < \infty$,

for all $n \in \overline{\mathbb{N}}$ *and some* $a_1, a_2 \in (0,1)$*, where* c *is the constant appearing in inequality (2.3) and* $k := \max_{1\le i\le N}\{k_i\}$*. Suppose that one member of the family* $\{T_i\}_{i=1}^N$ *is semi-compact. Then the sequence* $\{x_n\}_{n=0}^{\infty}$ *is defined by (1.8) converges strongly to a common fixed point of the family* $\{T_i\}_{i=1}^N$*.*

Proof. Take $q \in \bigcap_{i=1}^N F(T_i)$. Put $M := \sup_{n\in\overline{\mathbb{N}}} \|u_n - q\| < \infty$. By the definition of $\{x_n\}$ and Lemma 2.1, we have

$$\begin{aligned}
& \|x_n - q\|^p \\
= \;& \|(1-\alpha_{n-1}-\gamma_{n-1})(x_{n-1}-q) + \alpha_{n-1}(T_n^m x_{n-1}-q) + \gamma_{n-1}(u_{n-1}-q)\|^p \\
= \;& \left\|(1-\gamma_{n-1})\left[\left(1-\frac{\alpha_{n-1}}{1-\gamma_{n-1}}\right)(x_{n-1}-q) + \frac{\alpha_{n-1}}{1-\gamma_{n-1}}(T_n^m x_{n-1}-q)\right]\right. \\
& \left. +\gamma_{n-1}(u_{n-1}-q)\right\|^p \\
\le \;& (1-\gamma_{n-1})\left\|\left(1-\frac{\alpha_{n-1}}{1-\gamma_{n-1}}\right)(x_{n-1}-q) + \frac{\alpha_{n-1}}{1-\gamma_{n-1}}(T_n^m x_{n-1}-q)\right\|^p + \gamma_{n-1}M^p \\
\le \;& (1-\alpha_{n-1}-\gamma_{n-1})\|x_{n-1}-q\|^p + \alpha_{n-1}\|T_n^m x_{n-1}-q\|^p \\
& -c(1-\gamma_{n-1})W_p\left(\frac{\alpha_{n-1}}{1-\gamma_{n-1}}\right)\|x_{n-1}-T_n^m x_{n-1}\|^p + \gamma_{n-1}M^p. \qquad (3.46)
\end{aligned}$$

Substituting (3.5) into (3.46) gives

$$\begin{aligned}
\|x_n - q\|^p \le\;& \|x_{n-1}-q\|^p - \left[c(1-\gamma_{n-1})W_p\left(\frac{\alpha_{n-1}}{1-\gamma_{n-1}}\right) - k\alpha_{n-1}\right] \\
& \times\|x_{n-1}-T_n^m x_{n-1}\|^p + \gamma_{n-1}M^p \qquad (3.47)
\end{aligned}$$

for all $n \ge n_1$ and some $n_1 \in \overline{\mathbb{N}}$. Since $W_p\left(\frac{\alpha_{n-1}}{1-\gamma_{n-1}}\right) \ge a > 0$ and let $\varepsilon = ca - (1-a_2)k > 0$, it follows from (3.47) that

$$\|x_n - q\|^p \le \|x_{n-1}-q\|^p - (1-\gamma_{n-1})\varepsilon\|x_{n-1}-T_n^m x_{n-1}\|^p + \gamma_{n-1}M^p, \qquad (3.48)$$

for all $n \ge n_1$. Now, the rest of the proof follows the lines similar to Theorem 3.1 and is, therefore, omitted. This completes the proof. ∎

Theorem 3.4. *Let* E *be a real* p*-uniformly convex Banach space with* $p > 1$*. Let* $T_i : E \to E$ *be uniformly* L*-Lipschitzian and* p*-asymptotically hemi-contractive type mapping with respect to a constant* $k_i \in [0,1]$ *for each* $i = 1, 2, \ldots, N$ *such that* $\bigcap_{i=1}^N F(T_i) \neq \phi$*. Let* $\{u_n\}_{n=0}^{\infty}$ *be a sequence in* E *such that* $\sum_{n=0}^{\infty}\|u_n\| < \infty$ *and* $\{\alpha_n\}_{n=0}^{\infty}$ *be a sequence in* $[0,1]$ *satisfying the following conditions:*

(i) $k < ca(1-a_2)^{-1}$, $a = a_1a_2(a_1^{p-1} + a_2^{p-1})$*;*
(ii) $a_1 \le \alpha_n \le 1 - a_2$,

for all $n \in \overline{\mathbb{N}}$ *and some* $a_1, a_2 \in (0,1)$*, where* c *is the constant appearing in inequality* (2.3) *and* $k := \max_{1\le i\le N}\{k_i\}$*. Suppose that one member of the family* $\{T_i\}_{i=1}^N$ *is semi-compact. Then the sequence* $\{x_n\}_{n=0}^{\infty}$ *is defined by* (1.9) *converges strongly to a common fixed point of the family* $\{T_i\}_{i=1}^N$*.*

Proof. Take $q \in \bigcap_{i=1}^N F(T_i)$. Since $\|u_n\| \to 0$ as $n \to \infty$. Thus, there exist $n_0 \in \overline{\mathbb{N}}$ and $e \in (0,1)$ such that $1 - p\|u_{n-1}\| > e$ for all $n \ge n_0$. By the definition of $\{x_n\}$, Lemma 2.1 and

(2.5), we get

$$\begin{aligned}\|x_n - q\|^p &= \|(1-\alpha_{n-1})(x_{n-1}-q+u_{n-1}) + \alpha_{n-1}(T_n^m x_{n-1} - q + u_{n-1})\|^p \\ &\leq (1-\alpha_{n-1})\|x_{n-1}-q+u_{n-1}\|^p + \alpha_{n-1}\|T_n^m x_{n-1}-q+u_{n-1}\|^p \\ &\quad -cW_p(\alpha_{n-1})\|x_{n-1}-T_n^m x_{n-1}\|^p \\ &\leq (1-\alpha_{n-1})\|x_{n-1}-q\|^p + p(1-\alpha_{n-1})\|u_{n-1}\|\,\|x_{n-1}-q+u_{n-1}\|^{p-1} \\ &\quad +\alpha_{n-1}\|T_n^m x_{n-1}-q\|^p + p\alpha_{n-1}\|u_{n-1}\|\,\|T_n^m x_{n-1}-q+u_{n-1}\|^{p-1} \\ &\quad -cW_p(\alpha_{n-1})\|x_{n-1}-T_n^m x_{n-1}\|^p, \quad \text{for all } n \geq 1. \end{aligned} \tag{3.49}$$

Replacing the v_{n-1} in inequality (3.31) by u_{n-1} we obtain that for all $n \geq n_0$, we have

$$\|T_n^m x_{n-1} - q + u_{n-1}\|^{p-1} \leq e^{-1}(1 + L^p\|x_{n-1}-q\|^p). \tag{3.50}$$

Using (3.25), (3.5) and (3.50) in (3.49) we have

$$\begin{aligned}\|x_n - q\|^p &\leq (1-\alpha_{n-1})\|x_{n-1}-q\|^p + p\|u_{n-1}\|e^{-1}(1+\|x_{n-1}-q\|^p) \\ &\quad +\alpha_{n-1}\|x_{n-1}-q\|^p + \alpha_{n-1}k\|x_{n-1}-T_n^m x_{n-1}\|^p \\ &\quad +p\|u_{n-1}\|e^{-1}(1+L^p\|x_{n-1}-q\|^p) - cW_p(\alpha_{n-1})\|x_{n-1}-T_n^m x_{n-1}\|^p \\ &= [1+pe^{-1}(1+L^p)\|u_{n-1}\|]\,\|x_{n-1}-q\|^p \\ &\quad -[cW_p(\alpha_{n-1}) - \alpha_{n-1}k]\,\|x_{n-1}-T_n^m x_{n-1}\|^p + 2pe^{-1}\|u_{n-1}\|, \end{aligned} \tag{3.51}$$

for all $n \geq n_1$ and some $n_1 \geq n_0$. Since $W_p(\alpha_{n-1}) \geq a > 0$ and let $\varepsilon = ca - (1-a_2)k > 0$. Hence, from (3.51), we conclude that

$$\begin{aligned}\|x_n - q\|^p &\leq [1+pe^{-1}(1+L^p)\|u_{n-1}\|]\,\|x_{n-1}-q\|^p - \varepsilon\|x_{n-1}-T_n^m x_{n-1}\|^p \\ &\quad +2pe^{-1}\|u_{n-1}\|, \end{aligned} \tag{3.52}$$

for all $n \geq n_1$. Then, the rest of the proof follows the lines similar to Theorem 3.2 and is, therefore, omitted. This completes the proof. ∎

Remark 3.1. Our Theorem 3.1 and 3.3 improve and extend the corresponding results in Schu [9, Theorem 1.5] and Liu [5, Theorem 1, 2 and 3] to the more general iteration scheme with errors; and from Hilbert spaces to the much more general Banach spaces that include the L_p, ℓ_p and $W^{1,p}$ spaces for $1 < p < \infty$.

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