The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 3, Issue 1, Article 10, pp. 1-8, 2006

PARAMETER DEPENDENCE OF THE SOLUTION OF SECOND ORDER NONLINEAR ODE'S VIA PEROV'S FIXED POINT THEOREM

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Received 1 February 2005; accepted 18 May 2005; published 24 April 2006.

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ABSTRACT. Using the Perov's fixed point theorem, the smooth dependence by parameter of the solution of a two point boundary value problem corresponding to nonlinear second order ODE's is obtained.

Key words and phrases: Two point boundary value problem, Nonlinear second order differential equations, Perov's fixed point theorem, Smooth dependence by parameter.

2000 *Mathematics Subject Classification.* Primary 45J05, Secondary 47H10, 34A12.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Consider the following two point boundary value problem corresponding to the nonlinear second order ODE :

$$\left\{ \begin{array}{ll} y''(x) = f(x, y(x), y'(x)), & x \in [a,b] \\ y(a) = \alpha, \quad y(b) = \beta, & \alpha, \beta \in \mathbb{R}. \end{array} \right. \tag{1.1}$$

The existence and uniqueness in $C^2[a,b]$ of the solution of (1.1) is studied in [2] using a fixed point theorem on vector valued generalized metric spaces, which have an equivalent enunciation with Theorem 2.1.

It is known that the problem (1.1) is equivalent with the following integro-differential equation (see [1], [2] and [5]) :

$$y(x) = \frac{x-a}{b-a} \cdot \beta + \frac{b-x}{b-a} \cdot \alpha - \int_a^b G(x,s) f(s, y(s), y'(s)) \, ds, \tag{1.2}$$

$x \in [a,b]$.

If we are interested by the parameter dependence of the solution of equation (1.2), then this equation becomes,

$$y(x,\lambda) = \frac{x-a}{b-a} \cdot \beta + \frac{b-x}{b-a} \cdot \alpha - \int_a^b G(x,s) f(s, y(s,\lambda), y'_s(s,\lambda), \lambda) \, ds, \tag{1.3}$$

$x \in [a,b]$, $\lambda \in [c,d]$, where $G(x,s)$ is the well known Green's function.

Using the Perov's fixed point theorem (see [3], [4] and [8]) and a result of I. A. Rus (see [8]), we obtain here the smooth dependence, of the solution of (1.3) and of his derivative, by the parameter λ. A similar result of smooth dependence by the end points a and b of the solution of fredholm integral equations which use an ideea of Sotomayor (see [9]) was obtained by I. A. Rus in [8].

2. PRELIMINARIES

Let X be a nonempty set and $A : X \to X$ an operator. The fixed points set of A will be

$$F_A = \{x \in X : A(x) = x\}.$$

Definition 2.1. (Rus, [6] or [7]) Let (X,d) be a metric space. An operator $A : X \to X$ is Picard operator if there exists $x^* \in X$ such that :
(a) $F_A = \{x^*\}$,
(b) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^*, for all $x_0 \in X$, where $A^0 = 1_X$, $A^1 = A$, $A^n = A \circ A^{n-1}$, $\forall n \in \mathbb{N}^*$.

Definition 2.2. (Rus, [6] or [7]) Let (X,d) be a metric space. An operator $A : X \to X$ is weakly Picard operator if the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of A.

For the following notion, let $X \neq \emptyset$, $n \in \mathbb{N}$ and $d : X \times X \to \mathbb{R}^n_+$ where,

$$\mathbb{R}^n_+ = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_i \geq 0, \quad \forall i = \overline{1,n}\}.$$

Definition 2.3. (in [4] or [2]) The pair (X,d) is generalized metric space iff the function d have the following properties :
(gm1) $d(x,y) \geq 0$, $\forall x, y \in X$ and $d(x,y) = 0 \Longleftrightarrow x = y$

(gm2) $d(y,x) = d(x,y), \forall x,y \in X$
(gm3) $d(x,y) \leq d(x,z) + d(z,y), \forall x,y,z \in X.$
The function d is called generalized metric.

The euclidean space $\mathbb{R}^n$ is ordered by the relation :

$$x \leq y \Longleftrightarrow x_i \leq y_i, \quad \forall i = \overline{1,n},$$

for $x = (x_1,...,x_n), y = (y_1,...,y_n) \in \mathbb{R}^n$.

A generalized metric space is complet if any fundamental sequence in X is convergent. Let $M_n(\mathbb{R}_+)$ the set of matrices with all elements positive.

Definition 2.4. (in [4]) Let (X,d) be a generalized metric space. A map $T : X \to X$ satisfy a generalized Lipschitz inequality if there exists a matrix $A \in M_n(\mathbb{R}_+)$ such that :

$$d(T(x),T(y)) \leq Ad(x,y), \quad \forall x,y \in X.$$

Theorem 2.1. (*Perov,* [3], [4]) *Let (X,d) be a generalized metric space and $A : X \to X$ a mapping which have the generalized Lipschitz inequality property with a matrix $Q \in M_n(\mathbb{R}_+)$. If all eigenvalues of Q lies in the open unit ball from the complex plane, then :*
(i) the operator A has a unique fixed point $x^* \in X$
(ii) for any $x_0 \in X$, the sequence $(x_m)_{m\in\mathbb{N}} \subset X$ defined by $x_m = A(x_{m-1}), \forall m \in \mathbb{N}^*$, is convergent to x^*
(iii) the following inequality holds :

$$d(x_m,x^*) \leq Q^m(I_n - Q)^{-1} \cdot d(x_0,x_1), \forall m \in \mathbb{N}^*. \tag{2.1}$$

Using this Perov's fixed point theorem, I. A. Rus obtains the result :

Theorem 2.2. (*of fiber generalized contractions, Rus* [8]) *Let (X,d) be a metric space (generalized or not) and (Y,ρ) be a complete generalized metric space ($\rho(x,y) \in \mathbb{R}^n_+$). Let $A : X \times Y \to X \times Y$ be a continuous operator and $C : X \times Y \to Y$ an operator. Suppose that :*
(i) $B : X \to X$ is a weakly Picard operator
(ii) $A(x,y) = (B(x), C(x,y))$, for all $x \in X, y \in Y$
(iii) there exists a matrix $Q \in M_n(\mathbb{R}_+)$, with $Q^m \to 0$ as $m \to \infty$, such that

$$\rho(C(x,y_1), C(x,y_2)) \leq Q \cdot \rho(y_1,y_2),$$

for all $x \in X$, y_1 and $y_2 \in Y$.
Then, the operator A is weakly Picard operator. Moreover, if B is Picard operator, then A is Picard operator.

Remark 2.1. (see [4]) For a matrix $Q \in M_n(\mathbb{R}_+)$, the following properties are equivalent :
(i) $Q^m \to 0$ as $m \to \infty$,
(ii) all eigenvalues of Q lies in the open unit ball from the complex plane.

3. THE MAIN RESULT

Deriving the equation (1.3) in respect with x we obtain,

$$y'_x(x,\lambda) = \frac{\beta - \alpha}{b - a} - \int_a^b \frac{\partial G}{\partial x}(x,s)\, f\left(s, y(s,\lambda), y'_s(s,\lambda), \lambda\right) ds, \tag{3.1}$$

$x \in [a,b]$, $\lambda \in [c,d]$.

Denoting $z = y'_x$, from (1.3) and (3.1), we can take in consideration the system of integral equations,

$$(3.2)\quad \begin{cases} y(x,\lambda) = \frac{x-a}{b-a}\cdot\beta + \frac{b-x}{b-a}\cdot\alpha - \int_a^b G(x,s)\, f(s, y(s,\lambda), z(s,\lambda), \lambda)\, ds \\ z(x,\lambda) = \frac{\beta-\alpha}{b-a} - \int_a^b \frac{\partial G}{\partial x}(x,s)\, f(s, y(s,\lambda), z(s,\lambda), \lambda)\, ds, \quad x\in[a,b], \lambda\in[c,d], \end{cases}$$

where

$$G(x,s) = \begin{cases} \frac{(s-a)(b-x)}{b-a}, & \text{if } s \le x \\ \frac{(x-a)(b-s)}{b-a}, & \text{if } s \ge x \end{cases}$$

and

$$\frac{\partial G}{\partial x}(x,s) = \begin{cases} -\frac{(s-a)}{b-a}, & \text{if } s < x \\ \frac{(b-s)}{b-a}, & \text{if } s > x. \end{cases}$$

Let

$$C([a,b]\times[c,d]) = \{y : [a,b]\times[c,d] \to \mathbb{R} \mid y \text{ continuous } \}$$

and on this set we define the Chebyshev's norm,

$$\|u\|_C = \max\{|u(x,y)| : x\in[a,b], \quad y\in[c,d]\}.$$

Let

$$X = Y = C([a,b]\times[c,d]) \times C([a,b]\times[c,d])$$

and on this product space we consider the generalized metric $d_C : X\times X \to \mathbb{R}^2$, defined by

$$d_C((y_1,z_1),(y_2,z_2)) = (\|y_1-y_2\|_C \ , \ \|z_1-z_2\|_C), \quad \forall (y_1,z_1),(y_2,z_2)\in X.$$

We define the operators,

$$B : X \to X, \qquad C : X\times X \to X$$

and $A : X\times X \to X\times X$, by

$$B(y,z) = (B_1(y,z), B_2(y,z))$$

$$(3.3)\quad B_1(y,z)(x,\lambda) = \frac{x-a}{b-a}\cdot\beta + \frac{b-x}{b-a}\cdot\alpha - \int_a^b G(x,s)\, f(s, y(s,\lambda), z(s,\lambda), \lambda)\, ds$$

$$(3.4)\quad B_2(y,z)(x,\lambda) = \frac{\beta-\alpha}{b-a} - \int_a^b \frac{\partial G}{\partial x}(x,s)\, f(s, y(s,\lambda), z(s,\lambda), \lambda)\, ds$$

$$C((y,z),(u,v)) = (C_1((y,z),(u,v)), C_2((y,z),(u,v)))$$

$$C_1((y,z),(u,v))(x,\lambda) = -\int_a^b G(x,s)\cdot[\frac{\partial f}{\partial \lambda}(s, y(s,\lambda), z(s,\lambda), \lambda) +$$

$$(3.5)\quad + \frac{\partial f}{\partial x}(s, y(s,\lambda), z(s,\lambda), \lambda)\cdot u(s,\lambda) + \frac{\partial f}{\partial y}(s, y(s,\lambda), z(s,\lambda), \lambda)\cdot v(s,\lambda)]ds$$

$$C_2((y,z),(u,v))(x,\lambda) = -\int_a^b \frac{\partial G}{\partial x}(x,s)\cdot[\frac{\partial f}{\partial \lambda}(s, y(s,\lambda), z(s,\lambda), \lambda) +$$

$$+\frac{\partial f}{\partial x}\left(s,y\left(s,\lambda\right),z\left(s,\lambda\right),\lambda\right)\cdot u(s,\lambda)+\frac{\partial f}{\partial y}\left(s,y\left(s,\lambda\right),z\left(s,\lambda\right),\lambda\right)\cdot v(s,\lambda)]ds \tag{3.6}$$

and

$$A((y,z),(u,v))=(B(y,z),C((y,z),(u,v))).$$

For $(y,z)\in X$ fixed, consider the system,

$$\begin{cases} u(x,\lambda)=-\int\limits_a^b G\left(x,s\right)\cdot[\frac{\partial f}{\partial\lambda}\left(s,y\left(s,\lambda\right),z\left(s,\lambda\right),\lambda\right)+ \\ +\frac{\partial f}{\partial x}\left(s,y\left(s,\lambda\right),z\left(s,\lambda\right),\lambda\right)\cdot u(s,\lambda)+\frac{\partial f}{\partial y}\left(s,y\left(s,\lambda\right),z\left(s,\lambda\right),\lambda\right)\cdot v(s,\lambda)]ds \\ \\ v(x,\lambda)=-\int\limits_a^b \frac{\partial G}{\partial x}\left(x,s\right)\cdot[\frac{\partial f}{\partial\lambda}\left(s,y\left(s,\lambda\right),z\left(s,\lambda\right),\lambda\right)+ \\ +\frac{\partial f}{\partial x}\left(s,y\left(s,\lambda\right),z\left(s,\lambda\right),\lambda\right)\cdot u(s,\lambda)+\frac{\partial f}{\partial y}\left(s,y\left(s,\lambda\right),z\left(s,\lambda\right),\lambda\right)\cdot v(s,\lambda)]ds. \end{cases} \tag{3.7}$$

We will suppose that the following conditions hold:
(C_1) (continuity) : $f\in C([a,b]\times\mathbb{R}^2\times[c,d])$;
(C_2) (boundedness) : there exist $M>0$ such that

$$|f(x,u,v,\lambda)|\le M,\quad\forall(x,u,v,\lambda)\in[a,b]\times\mathbb{R}^2\times[c,d]$$

(C_3) (Lipschitz) : there exist $L_1>0,\ L_2>0$ such that

$$\left|\frac{\partial f(x,u,v,\lambda)}{\partial u}\right|\le L_1,\quad\left|\frac{\partial f(x,u,v,\lambda)}{\partial v}\right|\le L_2,\quad\forall(x,u,v,\lambda)\in[a,b]\times\mathbb{R}^2\times[c,d].$$

(C_4) (smoothness) : $f(x,\cdot,\cdot,\cdot)\in C^1(\mathbb{R}^2\times[c,d]),\quad\forall x\in[a,b]$.

Theorem 3.1.

(a) *With the conditions (C_1), (C_2), (C_3), if $L_1(b-a)^2<1$ and $L_2(b-a)<\frac{3}{4}$, then the system (3.2) of integral equations has in X an unique solution (y^*,z^*) such that $y^*,z^*\in C^1([a,b]\times[c,d]), y^*(\cdot,\lambda)\in C^2[a,b],\quad\forall\lambda\in[c,d]$ and $\frac{\partial}{\partial x}y^*=z^*$.*
(b) *With the conditions $(C_1)-(C_4)$, if $L_1(b-a)^2<1$ and $L_2(b-a)<\frac{3}{4}$, then the pair $(\frac{\partial}{\partial\lambda}y^*,\frac{\partial}{\partial\lambda}z^*)$ is the unique solution in $C([a,b]\times[c,d])\subset Y$ of the system (3.7) for the fixed pair $(y,z)=(y^*,z^*)$.*

Proof. By the conditions (C_1) and (C_2) follows that $B(X)\subset C([a,b]\times[c,d])$. Elementary calculus lead to :

$$\begin{gathered}|B_1(y_1,z_1)\left(x,\lambda\right)-B_1(y_2,z_2)\left(x,\lambda\right)|\le\\ \le(b-a)\left\|G\right\|_C\cdot(L_1\left\|y_1-y_2\right\|_C\ +L_2\ \left\|z_1-z_2\right\|_C)\le\\ \le\frac{(b-a)^2}{4}\cdot(L_1\left\|y_1-y_2\right\|_C\ +L_2\ \left\|z_1-z_2\right\|_C),\quad\forall(x,\lambda)\in[a,b]\times[c,d]\end{gathered}$$

and

$$\begin{gathered}|B_2(y_1,z_1)\left(x,\lambda\right)-B_2(y_2,z_2)\left(x,\lambda\right)|\le\\ \le(b-a)\left\|\frac{\partial G}{\partial x}\right\|_C\cdot(L_1\left\|y_1-y_2\right\|_C\ +L_2\ \left\|z_1-z_2\right\|_C)\le\\ \le(b-a)(L_1\left\|y_1-y_2\right\|_C\ +L_2\ \left\|z_1-z_2\right\|_C),\quad\forall(x,\lambda)\in[a,b]\times[c,d].\end{gathered}$$

Then,

$$d_C(B(y_1,z_1),B(y_2,z_2))\le\begin{pmatrix}\frac{1}{4}L_1(b-a)^2 & \frac{1}{4}L_2(b-a)^2\\ L_1(b-a) & L_2(b-a)\end{pmatrix}\cdot d_C((y_1,z_1),(y_2,z_2)),$$

$\forall (y_1, z_1), (y_2, z_2) \in X$.
Since the eigenvalues of the matrix,

$$Q = \begin{pmatrix} \frac{1}{4}L_1(b-a)^2 & \frac{1}{4}L_2(b-a)^2 \\ L_1(b-a) & L_2(b-a) \end{pmatrix}$$

are $\lambda_1 = 0$ and $\lambda_2 = \frac{1}{4}L_1(b-a)^2 + L_2(b-a)$, from the conditions $L_1(b-a)^2 < 1$ and $L_2(b-a) < \frac{3}{4}$ we infer that $|\lambda_2| < 1$ and so, $Q^m \to 0$ when $m \to \infty$.
Applying the Perov's fixed point Theorem 2.1, we conclude that the operator B has in $C([a,b] \times [c,d])$ a unique fixed point (y^*, z^*) and the sequence (y_m, z_m) converges uniformly to (y^*, z^*) in $C([a,b] \times [c,d])$ for any $(y_0, z_0) \in X$, where

$$y_m(x,\lambda) = \frac{x-a}{b-a} \cdot \beta + \frac{b-x}{b-a} \cdot \alpha - \int_a^b G(x,s)\, f(s, y_{m-1}(s,\lambda), z_{m-1}(s,\lambda), \lambda)\, ds,$$

$$z_m(x,\lambda) = \frac{\beta-\alpha}{b-a} - \int_a^b \frac{\partial G}{\partial x}(x,s)\, f(s, y_{m-1}(s,\lambda), z_{m-1}(s,\lambda), \lambda)\, ds, \tag{3.8}$$

$\forall m \in \mathbb{N}^*, \quad \forall (x,\lambda) \in [a,b] \times [c,d]$.
Consequently,

$$y^*(x,\lambda) = \frac{x-a}{b-a} \cdot \beta + \frac{b-x}{b-a} \cdot \alpha - \int_a^b G(x,s)\, f(s, y^*(s,\lambda), z^*(s,\lambda), \lambda)\, ds, \tag{3.9}$$

and

$$z^*(x,\lambda) = \frac{\beta-\alpha}{b-a} + \int_a^x \frac{s-a}{b-a} \cdot f(s, y^*(s,\lambda), z^*(s,\lambda), \lambda)\, ds -$$

$$- \int_x^b \frac{b-s}{b-a} \cdot f(s, y^*(s,\lambda), z^*(s,\lambda), \lambda)\, ds, \tag{3.10}$$

$\forall (x,\lambda) \in [a,b] \times [c,d]$.
With the condition (C_1) we see that

$$\frac{\partial}{\partial x} y^*(x,\lambda) = \frac{\beta-\alpha}{b-a} + \int_a^x \frac{s-a}{b-a} \cdot f(s, y^*(s,\lambda), z^*(s,\lambda), \lambda)\, ds -$$

$$- \int_x^b \frac{b-s}{b-a} \cdot f(s, y^*(s,\lambda), z^*(s,\lambda), \lambda)\, ds$$

and therefore $\frac{\partial}{\partial x} y^* = z^*$. With the same condition we infer that $y^*, z^* \in C^1([a,b] \times [c,d])$.
Because $\frac{\partial}{\partial x} y^* = z^*$ we conclude that $y^*(\cdot,\lambda) \in C^2[a,b], \quad \forall \lambda \in [c,d]$.
It is easy to see that $y^*(a,\lambda) = \alpha,\ y^*(b,\lambda) = \beta, \quad \forall \lambda \in [c,d]$ and

$$\frac{\partial}{\partial x} z^*(x,\lambda) = f(x, y^*(x,\lambda), z^*(x,\lambda), \lambda), \quad \forall (x,\lambda) \in [a,b] \times [c,d],$$

that is,

$$\frac{\partial^2 y^*}{\partial x^2}(x,\lambda) = f(x, y^*(x,\lambda), \frac{\partial y^*}{\partial x}(x,\lambda), \lambda), \quad \forall (x,\lambda) \in [a,b] \times [c,d].$$

(b) Consider the operator

$$C((y^*, z^*), \cdot) : Y \to Y,$$

which in the conditions (C_1), (C_2) and (C_4), is well defined. The condition (C_4) permits to consider the Lipschitz constants from the condition (C_3) as

$$L_1 = \left\| \frac{\partial f}{\partial y} \right\| \text{ and } L_2 = \left\| \frac{\partial f}{\partial z} \right\|.$$

From elementary calculus we obtain,

$$|C_1((y^*, z^*), (u_1, v_1))(x, \lambda) - C_1((y^*, z^*), (u_2, v_2))(x, \lambda)| \leq$$
$$\leq \int_a^b |G(x, s)| (L_1 |u_1(s, \lambda) - u_2(s, \lambda)| + L_2 |v_1(s, \lambda) - v_2(s, \lambda)|) ds \leq$$
$$\leq \frac{(b-a)^2}{4} (L_1 \|u_1 - u_2\|_C + L_2 \|v_1 - v_2\|_C), \quad \forall (x, \lambda) \in [a, b] \times [c, d]$$

and

$$|C_2((y^*, z^*), (u_1, v_1))(x, \lambda) - C_2((y^*, z^*), (u_2, v_2))(x, \lambda)| \leq$$
$$\leq (b-a)(L_1 \|u_1 - u_2\|_C + L_2 \|v_1 - v_2\|_C), \quad \forall (x, \lambda) \in [a, b] \times [c, d].$$

Then,

$$d_C(C((y^*, z^*), (u_1, v_1)), C((y^*, z^*), (u_2, v_2))) \leq Q d_C((u_1, v_1), (u_2, v_2)),$$

$\forall (u_1, v_1), (u_2, v_2) \in Y$ and $Q^m \to 0$ for $m \to \infty$.

From Theorem 2.2 we infer that the operator A has in $X \times Y$ a unique fixed point $((y^*, z^*), (u^*, v^*)) \in X \times Y$, that is

$$C((y^*, z^*), (u^*, v^*)) = (u^*, v^*).$$

Moreover, for $y_0 \in C^2([a, b] \times [c, d])$, $z_0 = \frac{\partial}{\partial x} y_0$, $u_0 = \frac{\partial}{\partial \lambda} y_0$, $v_0 = \frac{\partial}{\partial \lambda} z_0$ the sequence defined by

$$((x_m, y_m), (u_m, v_m))_m = (A^m((y_0, z_0), (u_0, v_0)))_m$$

converges uniformly to $((y^*, z^*), (u^*, v^*))$ in $X \times Y$.

From the conditions (C_1), (C_2) and (C_3) we ifer that $y_m \in C^2([a, b] \times [c, d]),\ z_m \in C^1([a, b] \times [c, d]),\ u_m, v_m \in C([a, b] \times [c, d]), \quad \forall m \in \mathbb{N}$ and

$$y_m \rightrightarrows y^*, \quad z_m = \frac{\partial y_m}{\partial x} \rightrightarrows z^*$$

$$u_m = \frac{\partial y_m}{\partial \lambda} \rightrightarrows u^* \text{ and } v_m = \frac{\partial z_m}{\partial \lambda} \rightrightarrows v^*,$$

that is,

$$z^* = \frac{\partial y^*}{\partial x}, \quad u^* = \frac{\partial y^*}{\partial \lambda}, \quad v^* = \frac{\partial z^*}{\partial \lambda}.$$

∎

Corollary 3.2. *Under the conditions of Theorem 3.1 the two point boundary value problem*

$$\begin{cases} \frac{\partial^2}{\partial x^2} y(x, \lambda) = f(x, y(x, \lambda), \frac{\partial}{\partial x} y(x, \lambda), \lambda), & (x, \lambda) \in [a, b] \times [c, d]. \\ y(a, \lambda) = \alpha, \quad y(b, \lambda) = \beta, & \forall \lambda \in [c, d] \end{cases}$$

has in

$$C^{2,1}([a, b] \times [c, d]) = \{y : [a, b] \times [c, d] \to \mathbb{R} \mid y \in C^1([a, b] \times [c, d]) \text{ and } y(\cdot, \lambda) \in C^2[a, b], \quad \forall \lambda \in [c, d]\}$$

a unique solution y^* such that y^* and his partial derivative in respect by x are smooth dependent by the parameter λ.

Proof. Follows directly by the proof of Theorem 3.1. ∎

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