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ON THE HOHOV CONVOLUTION OF THE CLASS $S_p(\alpha, \beta)$

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ABSTRACT. Let $F(a, b; c; z)$ be the Gaussian hypergeometric function and $I_{a,b;c}(f) = zF(a, b; c; z) * f(z)$ be the Hohlov operator defined on the class $\mathcal{A}$ of all normalized analytic functions. We determine conditions on the parameters a, b, c such that $I_{a,b;c}(f)$ will be in the class of parabolic starlike functions $S_p(\alpha, \beta)$. Our results extend several earlier results.

Key words and phrases: Gaussian hypergeometric function, Parabolic starlike function, Uniformly starlike function.

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1. INTRODUCTION

Let $\mathcal{A}$ be the class of functions f *normalized* by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are *analytic* in the *open* unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$, the class of functions which are also *univalent* in U.

A function $f \in \mathcal{A}$ is said to be starlike of order $\alpha,\ 0 \leq \alpha < 1,$ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, z \in U. \tag{1.2}$$

This class is denoted by $S^*(\alpha)$ where $S^*(0) = S^*$, the class of functions that are starlike.

A function $f \in \mathcal{A}$ is said to be in the class $K(\alpha)$, the class of functions which are convex univalent of order α, if $zf'(z) \in$ S*(α). Clearly $K(0) = K$, the class of convex univalent functions.

A function $f \in \mathcal{A}$ is said to be uniformly convex in U if $f(z)$ is a normalized convex function and has the property that for every circular arc γ contained in the unit disc U, with center ζ also in U, the image arc $f(\gamma)$ is a convex arc . This class was introduced by Goodman [4]. Kanas and Wiśniowska [7] defined the class $k-UCV$ as

$$k-UCV := \left\{f \in \mathcal{A} : Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right|, (0 \leq k < \infty)\right\}. \tag{1.3}$$

Note that the class $k-UCV$ is an extension of the class UCV studied by Goodman [4, 5]. The Class UCV describes geometrically the domain of values of the expression $1 + \frac{zf''(z)}{f'(z)}, z \in U$ as a parabolic region $\Omega = \left\{\omega \in \mathbb{C} : (Im(\omega))^2 < 2Re(\omega) - 1\right\}$. A one variable characterization for the class UCV was independently given by F. Rønning [14], and Ma and Minda [11] as $f \in UCV$ if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, z \in U. \tag{1.4}$$

Using the analytic condition (1.4), Rønning [14] defined a new class called S_p consisting of functions $f \in \mathcal{A}$ satisfying

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|, z \in U. \tag{1.5}$$

Kanas and Wiśniowska [8] extended the class S_p as

$$k-ST := \left\{f \in \mathcal{A} : Re\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right|, (0 \leq k < \infty)\right\}. \tag{1.6}$$

The various properties of the class $k-UCV$ and $k-ST$ was extensively studied by Kanas and Srivastava [9]. Bharathi et al. [1] defined a new class $UCV(\alpha, \beta)$ as follows.
For $\alpha \geq 0, 0 \leq \beta < 1,$

$$UCV(\alpha,\beta) := \left\{f \in \mathcal{A} : Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \alpha\left|\frac{zf''(z)}{f'(z)}\right| + \beta, z \in U\right\}. \tag{1.7}$$

They gave a sufficient condition for a function to be in $UCV(\alpha, \beta)$ in terms of the coefficient of the function. The class $S_p(\alpha, \beta)$ was also defined in [1], as

$$S_p(\alpha,\beta) := \left\{ f \in \mathcal{A} : Re\left\{ \frac{zf'(z)}{f(z)} \right\} \geq \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta, z \in U \right\}. \tag{1.8}$$

Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, $g(z) = \sum\limits_{n=0}^{\infty} b_n z^n$. Then the Hadamard product or convolution of $f(z)$ and $g(z)$, written as $(f * g)(z)$ is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n. \tag{1.9}$$

The Gaussian hypergeometric function $F(a, b; c; z) = {}_2F_1(a, b; c; z)$ is defined by

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \tag{1.10}$$

where a, b, c are complex numbers and $c \neq 0, -1, -2, ...$ and $(a)_n$, the Pochhammer symbol (or ascending factorial) defined by

$$(a)_n = \begin{cases} 1 & \text{for } n = 0 \\ a\ (a+1) \ldots (a+n-1) & \text{for } n = 1, 2, 3\ldots . \end{cases} \tag{1.11}$$

It is known that

$$F(a,b;c;1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, Re(c-a-b) > 0. \tag{1.12}$$

Also the function $F(a, b; c; z)$ is bounded if $Re(c - a - b) > 0$. For $f \in \mathcal{A}$, we recall the operator $I_{a,b;c}(f)$ of Hohlov [6] which maps $\mathcal{A}$ into itself defined by

$$[I_{a,b;c}(f)](z) = zF(a,b;c;z) * f(z) \tag{1.13}$$

where * denotes the Hadamard's convolution.

Let $\tau \in \mathbb{C} \setminus \{0\}$, $-1 \leq B < A \leq 1$. A function $f \in \mathcal{A}$ is said to be in the class $\Re^\tau(A, B)$ if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A-B)\tau - B[f'(z) - 1]} \right| < 1, z \in U. \tag{1.14}$$

The class $\Re^\tau(A, B)$ was introduced by Dixit and Pal [3] . For the choices of $\tau = e^{-i\eta} \cos\eta \left(-\frac{\pi}{2} < \eta < \frac{\pi}{2}\right)$, $A = 1 - 2\beta$ $(0 \leq \beta < 1)$ and $B = -1$, the class $\Re^\tau(A, B)$ reduces to the class $\Re_\eta(\gamma)$ by Ponnusamy and Rønning [13], where

$$\Re_\eta(\gamma) := \left\{ f \in \mathcal{A} : Re\left(e^{i\eta}(f'(z) - \gamma)\right) > 0; \left(z \in U; -\frac{\pi}{2} < \eta < \frac{\pi}{2}; 0 \leq \gamma < 1\right) \right\}. \tag{1.15}$$

Parvatham and Prabhakaran [12] obtained sufficient condition for the Gaussian hypergeometric function $zF(a, b; c; z)$ to be in the class S_p. The class $S_p(\alpha, \beta)$ was discussed in detail by Swaminathan [17]. Motivated essentially by the aforementioned works, we obtain sufficient condition for the operator $I_{a,b;c}(f)$ to be in the class $S_p(\alpha, \beta)$.

2. THE CLASS $UCV(\alpha, \beta)$

We state the following results which will be used to prove our main results.

Theorem 2.1. [9, Theorem 2.3, p. 21] *A function $f(z)$ of the form (1.1) is in $UCV(\alpha, \beta)$ if it satisfies the condition*

$$\sum_{n=2}^{\infty} n\left(n(1+\alpha)-(\alpha+\beta)\right)|a_n| \leq 1-\beta. \tag{2.1}$$

Theorem 2.2. [1, Theorem 2.6, p. 23], *A function $f(z)$ of the form (1.1) is in $S_p(\alpha, \beta)$ if it satisfies the condition*

$$\sum_{n=2}^{\infty} \left(n(1+\alpha)-(\alpha+\beta)\right)|a_n| \leq 1-\beta. \tag{2.2}$$

Lemma 2.3. [1, Lemma 2, p. 767] *(i) For $a \neq 1, b \neq 1$ and $c \neq 1$ with $c > \max[0, a+b-1]$,*

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_{n+1}} = \frac{1}{(a-1)(b-1)}\left[\frac{\Gamma(c-a-b+1)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} - (c-1)\right].$$

(ii) For $a, b > 0$ and $c > a+b+3$

$$\begin{aligned}&\sum_{n=0}^{\infty}(n+1)^3\frac{(a)_n(b)_n}{(c)_n(1)_n}\\ =\ &\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{(a)_3(b)_3}{(c-a-b-3)_3}+6\frac{(a)_2(b)_2}{(c-a-b-2)_2}+7\frac{ab}{(c-a-b)}+1\right].\end{aligned}$$

(iii) For $a, b > 0$ and $c > a+b+2$

$$\sum_{n=0}^{\infty}(n+1)^2\frac{(a)_n(b)_n}{(c)_n(1)_n} = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{(a)_2(b)_2}{(c-a-b-2)_2}+3\frac{ab}{(c-a-b)}\right].$$

(iv) For $a, b > 0$ and $c > a+b+1$

$$\sum_{n=0}^{\infty}(n+1)\frac{(a)_n(b)_n}{(c)_n(1)_n} = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{ab}{(c-a-b-1)}\right].$$

Lemma 2.4. [3] *If $f \in \Re^\tau(A, B)$ is of form (1.1), then*

$$|a_n| \leq (A-B)\frac{|\tau|}{n}, n \neq 1. \tag{2.3}$$

Theorem 2.5. *Let $f \in \Re^\tau(A, B)$, $a, b \in \mathbb{C} \setminus \{0\}$ and $c > |a|+|b|+1$. If*

$$(A-B)|\tau|\left\{\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[\left(\frac{1+\alpha}{1-\beta}\right)\left(\frac{|ab|}{c-|a|-|b|-1}\right)+1\right]-1\right\} \leq 1 \tag{2.4}$$

then $I_{a,b;c}(f) \in UCV(\alpha, \beta)$.

Proof.

$$zF(a,b;c;z) = z + \sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n$$

and

$$I_{a,b;c}(f) = z + \sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n z^n. \tag{2.5}$$

By the Theorem 2.1, we only need to show that

$$\sum_{n=2}^{\infty} n\left(n(1+\alpha)-(\alpha+\beta)\right)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \le 1-\beta. \tag{2.6}$$

In view of Lemma 2.4, we have

$$\sum_{n=2}^{\infty} n\left[n(1+\alpha)-(\alpha+\beta)\right]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right|$$

$$\le (A-B)\left|\tau\right|\sum_{n=2}^{\infty}\left[n(1+\alpha)-(\alpha+\beta)\right]\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

$$=(A-B)\left|\tau\right|\left[(1+\alpha)\sum_{n=1}^{\infty}\left[\frac{(|a|)_n(|b|)_n}{(c)_n(1)_{n-1}}\right]+(1-\beta)\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}\right].$$

By using the Pochhammer symbol $(a)_n = a(a+1)_{n-1}$, we get the right hand side of the above expression as

$$(A-B)\left|\tau\right|\left[(1+\alpha)\frac{|ab|}{c}\sum_{n=1}^{\infty}\frac{(|a|+1)_{n-1}(|b|+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}}+(1-\beta)\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}\right].$$

From (1.12), the above expression becomes

$$(A-B)\left|\tau\right|\left((1+\alpha)\frac{|ab|}{c}\frac{\Gamma(c-|a|-|b|-1)\Gamma(c+1)}{\Gamma(c-|a|)\Gamma(c-|b|)}+(1-\beta)\left[\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right]\right).$$

The above quantity is bounded above by $1-\beta$ if and only if (2.4) holds . ∎

Taking $\alpha=1,\beta=0$ and $f\in\Re_\eta(\gamma)$ in Theorem 2.5, we get the following result of Kim and Ponnusamy [10].

Corollary 2.6. *Let $f\in\Re_\eta(\gamma)$, $a,b\in\mathbb{C}\setminus\{0\}$ and $c>|a|+|b|+1$,*

$$2(1-\gamma)\cos\eta\left\{\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[2\left(\frac{|ab|}{c-|a|-|b|-1}\right)+1\right]-1\right\}\le 1, \tag{2.7}$$

then $I_{a,b;c}(f)\in UCV$.

Theorem 2.7. *Let $f\in\mathcal{S}$, $a,b\in\mathbb{C}\setminus\{0\}$ and $c>|a|+|b|+3$. If*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}$$

$$\left[(1-\beta)+(1+\alpha)\frac{(|a|)_3(|b|)_3}{(c-|a|-|b|-3)_3}+\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2}(6+5\alpha-\beta)\right]+$$

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|a|)}\left[\frac{|ab|}{(c-|a|-|b|-3)}(7+4\alpha-3\beta)\right]\le 2(1-\beta) \tag{2.8}$$

then, $I_{a,b;c}(f)\in UCV(\alpha,\beta)$.

Proof. Since $f\in\mathcal{S}$, we have

$$|a_n|\le n. \tag{2.9}$$

An application of Lemma 2.1, together with (2.9) gives the Theorem 2.7. ∎

Taking $f(z)=\frac{z}{1-z}$ in Theorem 2.7, we have the following result obtained by Swaminathan [17].

Corollary 2.8. *If $a, b > 0$ and $c > a + b + 2$ then a sufficient condition for $zF(a, b; c; z)$ to be in $UCV(\alpha, \beta)$ is that*

$$\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\left(\frac{3+2\alpha\ -\beta}{1-\beta}\right)\left(\frac{ab}{c-a-b-1}\right)+\left(\frac{1+\alpha}{1-\beta}\right)\left(\frac{(a)_2(b)_2}{(c-a-b)_2}\right)\right]$$
$$\leq \ 2.$$

Taking $f(z) = \frac{z}{1-z}, \alpha = 1$ in Theorem 2.7, we get the following result of Cho et al. [2].

Corollary 2.9. *If $a, b > 0$ and $c > a + b + 2$ then a sufficient condition for $zF(a, b; c; z)$ to be in $UCV(\beta)$ is that*

$$\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\left(\frac{5-\beta}{1-\beta}\right)\left(\frac{ab}{c-a-b-1}\right)+\left(\frac{2}{1-\beta}\right)\left(\frac{(a)_2(b)_2}{(c-a-b)_2}\right)\right]$$
$$\leq \ 2.$$

Taking $f(z) = \frac{z}{1-z}, \alpha = 1, \beta = 0$ in Theorem 2.7, we get the following result of Kim and Ponnusamy [10].

Corollary 2.10. *If $a, b \in \mathbb{C}\backslash\{0\}$ and $c > |a|+|b|+1$, then a sufficient condition for $zF(a, b; c; z)$ to be in UCV is that*

$$\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+5\left(\frac{|ab|}{c-|a|-|b|-1}\right)+2\left(\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|)_2}\right)\right] \leq 2.$$

3. THE CLASS $S_p(\alpha, \beta)$

Theorem 3.1. *Let $f \in \Re^\tau(A, B)$, $a, b \in \mathbb{C}\setminus\{0\}$ with $|a| \neq 1, |b| \neq 1$. If $c > \max\{0, |a| + |b|\}$, then the sufficient condition for $I_{a,b;c}(f)$ to be in the class $S_p(\alpha, \beta)$ is*

$$(1+\alpha)\left(\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right)-\frac{\alpha+\beta}{(|a|-1)(|b|-1)}\left(\frac{\Gamma(c-|a|-|b|+1)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\right)$$

$$+(\alpha+\beta)\left[\frac{(c-1)}{(|a|-1)(|b|-1)}+1\right] \leq \frac{1-\beta}{(A-B)|\tau|}. \tag{3.1}$$

Proof.

$$zF(a, b; c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n$$

and

$$I_{a,b;c}(f) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n. \tag{3.2}$$

By the Theorem 2.2, we only need to show that

$$\sum_{n=2}^{\infty} (n(1+\alpha) - (\alpha+\beta)) \left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n\right| \leq 1-\beta. \tag{3.3}$$

In view of Lemma 2.4, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}(n(1+\alpha)-(\alpha+\beta))\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \\
\leq\ & (A-B)\,|\tau|\left[\sum_{n=2}^{\infty}\{n(1+\alpha)-(\alpha+\beta)\}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\frac{1}{n}\right] \\
=\ & (A-B)\,|\tau|\left[(1+\alpha)\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}-(\alpha+\beta)\left(\sum_{n=0}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_{n+1}}-1\right)\right].
\end{aligned}$$

An application of Lemma 2.3 gives the Theorem 3.1. ∎

Theorem 3.2. *Let $f(z)\in R_\eta(\gamma)$, $a,b\in\mathbb{C}\setminus\{0\}$ with $|a|\neq 1,|b|\neq 1$. If $c>max\,\{0,|a|+|b|-1\}$, then the sufficient condition for $I_{a,b;c}(f)$ to be in the class $k-ST$ is*

$$\begin{aligned}
&\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+\frac{k(|ab|-c+1)}{(|a|-1)(|b|-1)}\right] \\
\leq\ & \frac{1}{2(1-\beta)\cos\eta}+k\left[\frac{(1-c)}{(|a|-1)(|b|-1)}\right]+1.
\end{aligned}$$

Proof. Taking $\beta=0$, $A=1-2\gamma$, $B=-1$, and $\alpha=k$ $(0\leq k<\infty)$, in Theorem 3.1, we get the desired result. This is the result obtained by Kanas and Srivastava [9]. ∎

Corollary 3.3. *If $f(z)\in R_\eta(\gamma),\alpha=1$ and $\beta=0$ in Theorem 3.1, we get the sufficient condition for $I_{a,b;c}(f)$ to be in the class of S_p obtained by Parvatham and Prabhakaran* [12].

Theorem 3.4. *Let $f\in S$, $a,b\in\mathbb{C}\setminus\{0\}$ and $c>|a|+|b|+2$. If*

$$\begin{aligned}
&\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+\frac{(1+\alpha)}{(1-\beta)}\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2}+\frac{(3+2\alpha-\beta)}{(1-\beta)}\frac{|ab|}{(c-|a|-|b|-1)}\right] \\
\leq\ & 2,
\end{aligned}$$

then $I_{a,b;c}(f)\in S_p(\alpha,\beta)$.

Proof. Since $f\in\mathcal{S}$,

$$|a_n|\leq n. \tag{3.4}$$

An application of Theorem 2.2 together with (3.4) , gives the Theorem 3.4. ∎

If $\alpha=k$ $(0\leq k<\infty)$, $\beta=0$ in Theorem 3.4, we get the following result of Kanas and Srivastava [9].

Corollary 3.5. *Let $f\in S$, $a,b\in\mathbb{C}\setminus\{0\}$ and $c>|a|+|b|+2$. If*

$$\begin{aligned}
&\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+(1+\alpha)\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2}+(3+2\alpha)\frac{|ab|}{(c-|a|-|b|-1)}\right] \\
\leq\ & 2,
\end{aligned}$$

then $I_{a,b;c}(f)\in k-ST$.

If $\alpha=1,\beta=0$ in Theorem 3.4, we get the following result obtained by Parvatham and Prabhakaran [12].

Corollary 3.6. *Let* $f \in \mathcal{S}$ *,* $a, b \in \mathbb{C} \setminus \{0\}$ *and* $c > |a| + |b| + 2$. If

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1 + 2\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2} + 5\frac{|ab|}{(c-|a|-|b|-1)}\right] \leq 2, \tag{3.5}$$

then $I_{a,b;c}(f) \in S_p$.

Taking $f(z) = \frac{z}{1-z}$ in Theorem 3.4, we have the following result obtained by Swaminathan [17].

Theorem 3.7. *If* $a, b > 0$ *and* $c > a + b + 1$*, then a sufficient condition for* $zF(a, b; c; z)$ *to be in* $S_p(\alpha, \beta)$ *is that*

$$\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left[1 + \left(\frac{1+\alpha}{1-\beta}\right)\left(\frac{ab}{c-a-b-1}\right)\right] \leq 2.$$

Corollary 3.8. *Taking* $f(z) = \frac{z}{1-z}, \alpha = 0$ *in Theorem 3.4, we get the sufficient condition for* $zF(a, b; c; z)$ *to be in the class of starlike functions of order* β *obtained by Silverman* [16].

Corollary 3.9. *Taking* $f(z) = \frac{z}{1-z}, \alpha = 0$ *and* $\beta = 1$ *in Theorem 3.4, we get the sufficient condition for* $zF(a, b; c; z)$ *to be in the class of parabolic starlike functions* S_p *obtained by Parvatham and Prabhakaran* [12].

REFERENCES

[1] R. BHARATI, R. PARVATHAM and A. SWAMINATHAN, On subclasses of uniformly convex functions and corresponding class of starlike functions, *Tamkang J. Math.*, **28** (1997), no. 1, pp. 17-32.

[2] N. E. CHO, S. Y. WOO and S. OWA, Uniform convexity properties for hypergeometric functions, *Fract. Calc. Appl. Anal.*, **5** (2002), no. 3, pp. 303-313.

[3] K. K. DIXIT and S. K. PAL, On a class of univalent functions related to complex order, *Indian J. Pure Appl. Math.*, **26** (1995), no. 9, pp. 889-896.

[4] A. W. GOODMAN, On uniformly convex functions, *Ann. Polon. Math.*, **56** (1991), no. 1, pp. 87-92.

[5] A. W. GOODMAN, On uniformly starlike functions, *J. Math. Anal. Appl.*, **155** (1991), no. 2, pp. 364-370.

[6] JU. E. HOHLOV and, Operators and operations on the class of univalent functions, *Izv. Vyssh. Uchebn. Zaved. Mat.*, no. 10 (1978), pp. 83-89.

[7] S. KANAS and A. WIŚNIOWSKA, Conic regions and k-uniform convexity, *J. Comput. Appl. Math.*, **105** (1999), no. 1-2, pp. 327-336.

[8] S. KANAS and A. WIŚNIOWSKA, Conic domains and starlike functions, *Rev. Roumaine Math. Pures Appl.*, **45** (2000), no. 4, pp. 647-657.

[9] S. KANAS and H. M. SRIVASTAVA, Linear operators associated with k-uniformly convex functions, *Integral Transform. Spec. Funct.*, **9** (2000), no. 2, pp. 121-132.

[10] Y. C. KIM and S. PONNUSAMY, Sufficiency for Gaussian hypergeometric functions to be uniformly convex, *Int. J. Math. Math. Sci.*, **22** (1999), no. 4, pp. 765–773.

[11] W. C. MA and D. MINDA, Uniformly convex functions, *Ann. Polon. Math.*, **57** (1992), no. 2, pp. 165-175.

[12] R. PARVATHAM and D. J. PRABHAKARAN, On the Hohlov convolution operator of the class S_p, *Far East J. Math. Sci. (FJMS)*, Special Volume, Part II, 2001, pp. 217-228.

[13] S. PONNUSAMY and F. RØNNING, Starlikeness properties for convolutions involving hypergeometric series, *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, **52** (1998), no. 1, pp. 141-155.

[14] F. RØNNING, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.*, **118** (1993), no. 1, pp. 189-196.

[15] S. RUSCHEWEYH, Convolutions in geometric function theory, *Presses Univ. Montréal, Montreal*, Que., 1982.

[16] H. SILVERMAN, Starlike and convexity properties for hypergeometric functions, *J. Math. Anal. Appl.*, **172** (1993), no. 2, pp. 574-581.

[17] A. SWAMINATHAN, Hypergeometric functions in the parabolic domain, *Tamsui Oxf. J. Math. Sci.*, **20** (2004), no. 1, pp. 1-16.