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THE SUCCESSIVE APPROXIMATIONS METHOD AND ERROR ESTIMATION IN TERMS OF AT MOST THE FIRST DERIVATIVE FOR DELAY ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. We present here a numerical method for first order delay ordinary differential equations, which use the Banach's fixed point theorem, the sequence of successive approximations and the trapezoidal quadrature rule. The error estimation of the method uses a recent result of P. Cerone and S.S. Dragomir about the remainder of the trapezoidal quadrature rule for Lipchitzian functions and for functions with continuous first derivative.

Key words and phrases: Delay ordinary differential equations, Method of successive approximations, Trapezoidal quadrature rule.

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1. INTRODUCTION

Consider the initial value problem :

$$(1.1)\qquad \begin{cases} x'(t) &= f(t,x(t),x(t-\tau)),\ t\in[0,T] \\ x(t) &= \varphi(t),\ t\in[-\tau,0], \end{cases}$$

where $\tau>0$ is the constant delay and $T>0$ is such that $T=l\cdot\tau$, with $l\in\mathbb{N}^*$.

Such problems appear as models in economy and biology. As applications of the delay differential equations in biology we can mention the models from [9], [15], [26] and [10]. For instance, as in [26], we can study the Nicholson's model (from 1957) for the time evolution of some species of blowflies,

$$N'(t)=r\cdot N(t)\cdot[1-\frac{N(t-\tau)}{K}].$$

On the other hand, the arterial concentration of CO_2 in lungs which is governed by the equation,

$$x'(t)=a-\frac{b\cdot x(t)-x^m(t-\tau)}{1+x^m(t-\tau)}$$

and the production of white cells in blood which is governed by the equation,

$$x'(t)=\frac{a\cdot x(t-\tau)}{b+x^m(t-\tau)}-c\cdot x\left(t\right).$$

Another model from haematopoiesis can be found in [15]. The models in economy governed by delay differential equations can be found in [4], [14] and [29]. For instance, in [29] has been generalized the model

$$\begin{cases} x'(t) &= x(t)\cdot[\frac{a}{b+x^n(t)}-\frac{c\cdot x^m(t-\tau)}{d+x^m(t-\tau)}],\ t\geq 0 \\ x(t) &= \varphi(t),\ t\in[-\tau,0],\ a,b,c,m\in\mathbb{R},\ \tau>0,\ n\geq 1 \end{cases}$$

of the naïve consumer, from [4], obtaining the model

$$\begin{cases} x'(t) &= x(t)\cdot[f(x(t))-g(x(t-\tau))],\ t\geq 0 \\ x(t) &= \varphi(t),\ t\in[-\tau,0], \end{cases}$$

for which the authors obtained results about the existence and uniqueness of the positive bounded solution. A general model for market price fluctuations has the form,

$$\begin{cases} x'(t) &= x(t)\cdot F(x(t),x(t-\tau)),\ t\geq 0 \\ x(t) &= \varphi(t),\ t\in[-\tau,0]. \end{cases}$$

Such models has been also studied in [29] from the existence and uniqueness of the positive bounded solution point of view.

To approximate the solution of the initial value problem (1.1) numerical methods has been elaborated in the last 30 years. The classical methods can be found in [1], [5], [19], [20], [21], [22], [30] and [31]. Some papers (see [8], [2], [18], [24] and [25]) use spline functions to approximate the solution of (1.1). For instance, in [25] the authors use the method of steps for the problem,

$$\begin{cases} x'(t) &= f(t,x(t),x(t-\omega)),\ t\in[a,b],\ \omega>0 \\ x(t) &= \varphi(t),\ t\in[a-\omega,a]. \end{cases}$$

At the first step, the solution of the Cauchy's problem,

$$\begin{cases} x'(t) &= f(t,x(t),\varphi(t-\omega)),\ t\in[a,a+\omega] \\ x(a) &= \varphi(a), \end{cases}$$

has been approximated by a spline function of even degree and at the general step were solved the initial value problems,

$$\begin{cases} x'(t) & = & f(t,x(t),x(t-\omega)),\ t\in[a+k\omega,a+(k+1)\omega] \\ x(t) & = & s_x^k(t),\ t\in[a+(k-1)\omega,a+k\omega], \end{cases}$$

where $s_x^k(t)$ is the spline function of even degree constructed on the previous interval $[a+(k-1)\omega,\ a+k\omega]$.

Applications of the approximation theory for delay differential equations can be found in [18] and [23]. Using Hermite polynomial interpolation in [27] a numerical method for delay differential equations has been obtained. The procedures based on the one step method and on the collocation method have been obtained in [3] and [16], respectively. The numerical methods for delay differential equations based on the Runge-Kutta procedure have been investigated in [6], [12] and [13].

2. EXISTENCE, UNIQUENESS AND APPROXIMATION OF THE POSITIVE SOLUTION

Consider the functional space

$$X=C[-\tau,T]=\{x:[-\tau,T]\longrightarrow\mathbb{R}\mid x \text{ is continuous on } [-\tau,T]\ \}$$

with the metric generated by the Bielecki's norm,

$$\|x\|_B=\max\{|x(t)|\cdot e^{-\theta\cdot(t+\tau)}:t\in[-\tau,T]\},$$

where $\theta>0$ is convenable chosen. With this metric the space X became a complete metric space.

Theorem 2.1. *Suppose that* $f\in C([-\tau,T]\times\mathbb{R}\times\mathbb{R}), \varphi\in C[-\tau,0]$ *and the following conditions are fulfilled :*
(i) $f(t,u,v)\geq 0,\quad \forall t\in[-\tau,T],\quad \forall u,v\geq 0,$ *and* $\varphi(t)\geq 0,\quad \forall t\in[-\tau,0]$;
(ii) there exists $M>0$ *such that ,* $|f(t,u,v)|\leq M,\quad \forall (t,u,v)\in[-\tau,T]\times\mathbb{R}\times\mathbb{R}$;
(iii) there exists $L>0$ *such that,* $\forall u_1,u_2,v_1,v_2\in\mathbb{R}$, *we have :*

$$|f(t,u_1,v_1)-f(t,u_2,v_2)|\leq\frac{L}{2LT+1}\cdot(|u_1-u_2|+|v_1-v_2|),\quad \forall t\in[-\tau,T].$$

Then the initial value problem (1.1) has in the space X *a unique positive and bounded solution* x^**, which can be approximated by the sequence of successive approximations :*

$$x_0(t)=\begin{cases}\varphi(t), & t\in[-\tau,0]\\ \varphi(0), & t\in[0,T],\end{cases} \tag{2.1}$$

$$x_m(t)=\varphi(t),\qquad \forall t\in[-\tau,0],\qquad \forall m\in\mathbb{N}^*, \tag{2.2}$$

$$x_m(t)=\varphi(0)+\int_0^t f(s,x_{m-1}(s),x_{m-1}(s-\tau))ds,\quad \forall t\in[0,T],\qquad \forall m\in\mathbb{N}^*, \tag{2.3}$$

with the error estimation,

$$|x^*(t)-x_m(t)|\leq\frac{4LT+1}{2LT+1}\cdot\left(\frac{1}{2}\right)^m\cdot\|x_1-x_0\|_B\quad ,\forall t\in[0,T],\forall m\in\mathbb{N}^*. \tag{2.4}$$

Moreover, this solution is differentiable with continuous first derivative on $[0,T]$.

Proof. The problem (1.1) is equivalent on $C[-\tau,T]$ with the initial value problem,

$$x(t)=\begin{cases}\varphi(0)+\int\limits_0^t f(s,x(s),x(s-\tau))ds, & t\in[0,T]\\ \varphi(t), & t\in[0,T],\end{cases} \tag{2.5}$$

which suggest to define the operator, $A:X\longrightarrow X$,

$$A(x)(t)=\begin{cases}\varphi(0)+\int\limits_0^t f(s,x(s),x(s-\tau))ds, & t\in[0,T]\\ \varphi(t), & t\in[0,T],\end{cases}\quad,\quad \forall x\in X.$$

Elementary calculus leads to,

$$|A(x)(t)-A(y)(t)|\le\int\limits_0^t|f(s,x(s),x(s-\tau))-f(s,y(s),y(s-\tau))|\,ds$$

$$\le L\left\|x-y\right\|_B\int\limits_0^t[e^{\theta(s+\tau)}+e^{\theta s}]ds\le\frac{2L}{\theta}\left\|x-y\right\|_B\int\limits_0^t\theta\cdot e^{\theta(s+\tau)}ds$$

$$=\frac{2L}{\theta}\left\|x-y\right\|_B[e^{\theta(t+\tau)}-e^{\theta\tau}]<\frac{2L}{\theta}\left\|x-y\right\|_B\cdot e^{\theta(t+\tau)},\qquad\forall t\in[0,T].$$

Then,

$$\left\|A(x)-A(y)\right\|_B\le\frac{2L}{\theta}\left\|x-y\right\|_B,\qquad\forall x,y\in X,$$

and choosing $\theta=4LT+1$, it follows that the operator A is a contraction. Then, on applying the Banach's fixed point theorem (see [28], [32]) we conclude that the initial value problem (2.5) has a unique solution $x^*\in X\cap C^1[0,T]$ (since f is continuous). If $\varphi\in C^1[-\tau,0]$ and

$$\varphi'(0)=f(0,\varphi(0),\varphi(-\tau))$$

then $x^*\in C^1[-\tau,T]$. From the conditions (i) and (ii) we deduce that

$$0\le x^*(t)\le\max(MT,\max\{\varphi(t):t\in[-\tau,0]\}),\qquad\forall t\in[-\tau,T].$$

Using the properties of the sequence of successive approximations from the Banach's fixed point theorem (see [17], [28]), since $\frac{2L}{\theta}=\frac{2LT}{4LT+1}<\frac{1}{2}$, we obtain the estimation (2.4). ∎

Remark 2.1. The estimation (2.4) permits to approximate the solution x^* on $[0,T]$, by the sequence of successive approximations (2.1), (2.2), (2.3). To compute the integrals from (2.3) we can use a quadrature rule.

3. The Numerical Method

To compute the integrals from (2.3) we will apply the trapezoidal quadrature rule with a new remainder estimation recently obtained by P. Cerone and S.S. Dragomir in [11], presented below.

Proposition 3.1. *The following inequalities holds :*

$$\left|\int\limits_a^b f(x)dx-\frac{b-a}{2}[f(a)+f(b)]\right|\le\begin{cases}\frac{(b-a)^2}{4}\cdot L, & f\in Lip[a,b]\\ \frac{(b-a)^2}{4}\cdot\left\|f'\right\|_C, & f\in C^1[a,b],\end{cases} \tag{3.1}$$

where L is the Lipschitz constant of f, if f is Lipschitzian.

We have used the notations :

$$Lip[a,b] = \{f : [a,b] \longrightarrow \mathbb{R} \mid f \text{ Lipschitzian } \}$$

$$C^1[a,b] = \{f : [a,b] \longrightarrow \mathbb{R} \mid f \text{ differentiable with } f' \in C[a,b]\}$$

and

$$\|f'\|_C = \sup_{t\in[a,b]} |f'(t)| .$$

Considering an uniform partition of the interval $[a, b]$,

$$\Delta : a = x_0 < x_1 < ... < x_{n-1} < x_n = b \tag{3.2}$$

with

$$x_i = a + \frac{(b-a)i}{n}, \qquad \forall i = \overline{0,n}, \quad n \in \mathbb{N}^*.$$

On can state the result from [11]:

Corollary 3.2. *For the trapezoidal quadrature rule we have,*

$$\left| \int_a^b f(x)dx - \frac{b-a}{2n}[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)] \right|$$

$$\leq \begin{cases} \frac{(b-a)^2}{4n} \cdot L, & f \in Lip[a,b] \\ \frac{(b-a)^2}{4n} \cdot \|f'\|, & f \in C^1[a,b]. \end{cases} \tag{3.3}$$

The followings results are known (see for instance [7]) :

Proposition 3.3. *If $f \in C^1[a,b]$ with Lipschitzian first derivative, having the Lipschitz constant $L' > 0$, then the following trapezoidal quadrature formula holds :*

$$\left| \int_a^b f(x)dx - \frac{b-a}{2}[f(a) + f(b)] \right| \leq \frac{(b-a)^3}{12} \cdot L'.$$

Corollary 3.4. *If $f \in C^1[a,b]$ with Lipschitzian first derivative, having the Lipschitz constant $L' > 0$, then, for the uniform partition (3.2), the following trapezoidal quadrature rule holds :*

$$\left| \int_a^b f(x)dx - \frac{b-a}{2n}[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)] \right| \leq \frac{(b-a)^3}{12n^2} \cdot L'.$$

To apply the above quadrature rules for the integrals from (2.3) we define the functions $F_m : [0,T] \longrightarrow \mathbb{R}, \quad m \in \mathbb{N}$, by

$$F_m(t) = f(t, x_m(t), x_m(t-\tau)), \quad \forall t \in [0,T]. \tag{3.4}$$

Now, consider the uniform partition of $[-\tau, T]$,

$$\Delta_n : -\tau = t_0 < t_1 < ... < t_{n-1} < t_n = 0 < t_{n+1} < ... < t_q = T,$$

with $t_{i+1} = t_i + \frac{\tau}{n}, \quad \forall i = \overline{0, q-1}, \quad n \in \mathbb{N}^*, \quad q = (l+1)n.$

In the conditions of the Theorem 2.1 we can see that $x_m \in C^1[-\tau, T], \quad \forall m \in \mathbb{N}^*$. On the knots of the partition Δ_n let,

$$x_0(t_k) = \begin{cases} \varphi(t_k), & k = \overline{0,n} \\ \varphi(0), & k = \overline{n+1,q}, \end{cases} \tag{3.5}$$

and

$$x_m(t_k) = \varphi(t_k), \qquad k = \overline{0,n}, \quad \forall m \in \mathbb{N}^*. \tag{3.6}$$

Applying the trapezoidal quadrature rule to the integrals from (2.3), on the knots t_k, $k = \overline{n+1,q}$, since

$$\frac{t_k}{2(k-n)} = \frac{\frac{\tau}{n}(k-n)}{2(k-n)} = \frac{\tau}{2n}, \qquad \forall k = \overline{n+1,q},$$

we obtain the following numerical method :

$$x_m(t_k) = \varphi(0) + \int_0^{t_k} f(s, x_{m-1}(s), x_{m-1}(s-\tau))ds = \varphi(0) + \int_0^{t_k} F_{m-1}(s)ds$$

$$= \varphi(0) + \frac{\tau}{2n}[f(0,\varphi(0),\varphi(-\tau)) + 2\sum_{j=1}^{k-n-1} f(t_{n+j}, x_{m-1}(t_{n+j}), x_{m-1}(t_j))$$

$$+ f(t_k, x_{m-1}(t_k), x_{m-1}(t_{k-n}))] + R_{m,k}, \quad \forall k = \overline{n+1,q}, \quad \forall m \in \mathbb{N}^*. \tag{3.7}$$

For the remainder $R_{m,k}$, according to Corollary 3.2 and 3.4, we have,

$$|R_{m,k}| \leq \begin{cases} \frac{\tau^2}{4n} \cdot \overline{L}, & \text{if } F_{m-1} \in Lip[0,T], \quad \forall m \in \mathbb{N}^* \\ \frac{\tau^2}{4n} \cdot \left\| F'_{m-1} \right\|_C & \text{if } F_{m-1} \in C^1[0,T], \quad \forall m \in \mathbb{N}^* \\ \frac{\tau^3}{12n^2} \cdot L', & \text{if } F'_{m-1} \in Lip[0,T], \quad \forall m \in \mathbb{N}^*, \end{cases} , \forall k = \overline{n+1,q}, \tag{3.8}$$

where $\overline{L} > 0$ (respectively $L' > 0$), is an upper bound of the Lipschitz constants of the functions F_{m-1} and F'_{m-1}, $m \in \mathbb{N}^*$, respectively.

The relations (3.7) lead to the following algorithm :

$$x_1(t_k) = \varphi(0) + \frac{\tau}{2n}[f(0,\varphi(0),\varphi(-\tau)) + 2\sum_{j=1}^{k-n-1} f(t_{n+j}, x_0(t_{n+j}), x_0(t_j))$$

$$+ f(t_k, x_0(t_k), x_0(t_{k-n}))] + R_{1,k} \text{ (by notation)}$$

$$= \overline{x_1(t_k)} + R_{1,k}, \qquad \forall k = \overline{n+1,q}, \tag{3.9}$$

$$x_2(t_k) = \varphi(0) + \frac{\tau}{2n}[f(0,\varphi(0),\varphi(-\tau)) + 2\sum_{j=1}^{k-n-1} f(t_{n+j}, \overline{x_1(t_{n+j})} + R_{1,n+j}, \overline{x_1(t_j)}$$

$$+ R_{1,j}) + f(t_k, \overline{x_1(t_k)} + R_{1,k}, \overline{x_1(t_{k-n})} + R_{1,k-n})] + R_{2,k} = \varphi(0)$$

$$+ \frac{\tau}{2n}[f(0,\varphi(0),\varphi(-\tau)) + 2\sum_{j=1}^{k-n-1} f(t_{n+j}, \overline{x_1(t_{n+j})}, \overline{x_1(t_j)})$$

$$+ f(t_k, \overline{x_1(t_k)}, \overline{x_1(t_{k-n})})] + \overline{R_{2,k}} \text{ (by notation)}$$

$$= \overline{x_2(t_k)} + \overline{R_{2,k}}, \qquad \forall k = \overline{n+1,q}. \tag{3.10}$$

By induction, for $m \in \mathbb{N}^*$, $m \geq 3$, we obtain :

$$x_m(t_k) = \varphi(0) + \frac{\tau}{2n}[f(0,\varphi(0),\varphi(-\tau)) + 2\sum_{j=1}^{k-n-1} f(t_{n+j}, \overline{x_{m-1}(t_{n+j})}$$

$$+ \overline{R_{m-1,n+j}}, \overline{x_{m-1}(t_j)} + \overline{R_{m-1,j}}) + f(t_k, \overline{x_{m-1}(t_k)} + \overline{R_{m-1,k}}, \overline{x_{m-1}(t_{k-n})}$$

$$+ \overline{R_{m-1,k-n}})] + R_{m,k} = \varphi(0) + \frac{\tau}{2n}[f(0,\varphi(0),\varphi(-\tau))$$

$$+2\sum_{j=1}^{k-n-1} f(t_{n+j},\overline{x_{m-1}(t_{n+j})},\overline{x_{m-1}(t_j)})+f(t_k,\overline{x_{m-1}(t_k)},\overline{x_{m-1}(t_{k-n})})]$$

$$+\overline{R_{m,k}}\ \text{(by notation)}$$

$$=\overline{x_m(t_k)}+\overline{R_{m,k}},\qquad \forall k=\overline{n+1,q},\quad \forall m\geq 3. \tag{3.11}$$

To estimate the remainders, applying the Lipschitz property (iii) from Theorem 2.1, we obtain the recurrent inequalities :

$$\left|\overline{R_{2,k}}\right|\leq |R_{2,k}|+\frac{\tau}{2n}[\frac{L}{2LT+1}(|R_{1,k}|+|R_{1,k-n}|)+2\sum_{j=1}^{k-n-1}\frac{L}{2LT+1}(|R_{1,n+j}|$$

$$+|R_{1,j}|)]<|R_2|+\frac{\tau}{2n}[2(k-n-1)+1]\cdot\frac{2L}{2LT+1}\left|R_1\right|,\quad \forall k=\overline{n+1,q},$$

where,

$$|R_1|=\max\{|R_{1,i}|:i=\overline{1,k}\}\leq\begin{cases}\frac{\tau^2}{4n}\cdot\overline{L}\\ \frac{\tau^2}{4n}\cdot\left\|F'_{m-1}\right\|_C\\ \frac{\tau^3}{12n^2}\cdot L'\end{cases}$$

and

$$|R_2|=\max\{|R_{2,i}|:i=\overline{1,k}\}\leq\begin{cases}\frac{\tau^2}{4n}\cdot\overline{L}\\ \frac{\tau^2}{4n}\cdot\left\|F'_{m-1}\right\|_C\\ \frac{\tau^3}{12n^2}\cdot L'.\end{cases} \tag{3.12}$$

Then,

$$\left|\overline{R_{2,k}}\right|\leq |R_2|+\frac{2\tau}{2n}(k-n)\cdot\frac{2L}{2LT+1}\left|R_1\right|\leq |R_2|$$

$$+\frac{\tau l\cdot n}{n}\cdot\frac{2L}{2LT+1}\left|R_1\right|=(1+\frac{2LT}{2LT+1})\cdot\left|R_2\right|,\quad \forall k=\overline{n+1,q}. \tag{3.13}$$

Moreover, for any $m\in\mathbb{N}^*, m\geq 3$, and $k=\overline{n+1,q}$, we obtain :

$$\left|\overline{R_{m,k}}\right|\leq |R_{m,k}|+\frac{\tau}{2n}[\frac{L}{2LT+1}(\left|\overline{R_{m-1,k}}\right|+\left|\overline{R_{m-1,k-n}}\right|)$$

$$+2\sum_{j=1}^{k-n-1}\frac{L}{2LT+1}(\left|\overline{R_{m-1,n+j}}\right|+\left|\overline{R_{m-1,j}}\right|)]\leq |R_{m,k}|$$

$$+\frac{\tau}{2n}[2(k-n-1)+1]\frac{L}{2LT+1}\left|R_{m-1}\right|<|R_{m,k}|+\frac{2LT}{2LT+1}\left|R_{m-1}\right|, \tag{3.14}$$

where,

$$|R_{m-1}|=\max\{\left|\overline{R_{m,i}}\right|:i=\overline{1,k}\}.$$

4. MAIN RESULTS

Consider the following Lipschitz conditions :
$\exists\gamma > 0,\ \delta > 0$ such that,

$$|f(t_1,u,v) - f(t_2,u,v)| \le \gamma\,|t_1 - t_2|\,,\quad \forall t_1,t_2 \in [0,T],\quad \forall u,v \in \mathbb{R}, \tag{4.1}$$

and

$$|\varphi(t_1) - \varphi(t_2)| \le \delta\,|t_1 - t_2|\,,\quad \forall t_1,t_2 \in [-\tau,0]. \tag{4.2}$$

We can see that the Lipschitz condition (iii) from Theorem 2.1 lead to the inequality,

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \le L(|u_1-u_2| + |v_1-v_2|), \forall t\in[0,T], \forall u_1,u_2,v_1,v_2 \in \mathbb{R}.$$

Proposition 4.1. *In the conditions of the Theorem 2.1, if the Lipschitz properties (4.1) and (4.2) are fulfilled then the functions $F_m,\ m\in\mathbb{N}$, are Lipschitzian with the same Lipschitz constant $\overline{L} = \gamma + 2L\cdot\max(\delta,M)$.*

Proof. Firstly,

$$\begin{aligned}|F_0(t_1) - F_0(t_2)| &= |f(t_1,x_0(t_1),x_0(t_1-\tau)) - f(t_2,x_0(t_2),x_0(t_2-\tau))|\\ &\le \gamma\,|t_1-t_2| + L(|x_0(t_1)-x_0(t_2)| + |x_0(t_1-\tau) - x_0(t_2-\tau)|)\\ &\le (\gamma+\delta L)\,|t_1-t_2|\,,\quad \forall t_1,t_2\in[0,T].\end{aligned}$$

By induction for $m\in\mathbb{N}^*$, we obtain,

$$\begin{aligned}&|F_m(t_1) - F_m(t_2)| \le |f(t_1,x_m(t_1),x_m(t_1-\tau)) - f(t_2,x_m(t_1),x_m(t_1-\tau))|\\ &+ |f(t_2,x_m(t_1),x_m(t_1-\tau)) - f(t_2,x_m(t_2),x_m(t_2-\tau))| \le \gamma\,|t_1-t_2|\\ &+L[|x_m(t_1)-x_m(t_2)| + |x_m(t_1-\tau)-x_m(t_2-\tau)|] \le \gamma\,|t_1-t_2|\\ &+L[\int_{t_1}^{t_2} |f(s,x_{m-1}(s),x_{m-1}(s-\tau))|\,ds + \max\{|\varphi(t_1-\tau)-\varphi(t_2-\tau)|\,,\\ &\left|\varphi(t_1-\tau) - \varphi(0) - \int_0^{t_2-\tau} f(s,x_{m-1}(s),x_{m-1}(s-\tau))ds\right|,\\ &\int_{t_1-\tau}^{t_2-\tau} |f(s,x_{m-1}(s),x_{m-1}(s-\tau))|\,ds\}] \le \gamma\,|t_1-t_2| + LM\cdot|t_1-t_2|\\ &+L\cdot\max(\delta,M)\cdot|t_1-t_2| \le [\gamma + 2L\cdot\max(\delta,M)]\cdot|t_1-t_2|\,,\ \forall m\in\mathbb{N}^*, \forall t_1,t_2\in[0,T].\end{aligned}$$

Consequently, the functions F_m, are Lipschitzian with the Lipschitz constant $\overline{L} = \gamma + 2L\cdot\max(\delta,M),\quad \forall\, m\in\mathbb{N}$. ∎

In the conditions of the above theorem the remainder estimations (3.8) becomes :

$$|R_{m,k}| \le \frac{\tau^2}{4n}\cdot\overline{L},\qquad \forall\, m\in\mathbb{N},\ \forall k = \overline{n+1,q}. \tag{4.3}$$

Theorem 4.2. *In the conditions of the above Theorem, the sequence $(\overline{x_m(t_k)})_{m\in\mathbb{N}^*}$ approximates the solution x^* of the initial value problem (1.1) on the knots $t_k,\ k = \overline{n+1,q}$ of the uniform partition Δ_n, with the error estimation :*

$$\left|\overline{x_m(t_k)} - x^*(t_k)\right| \le \frac{4LT+1}{2LT+1}\cdot\left(\frac{1}{2}\right)^m\cdot\|x_1 - x_0\|_B + (2LT+1)\frac{\tau^2}{4n}\cdot\overline{L}, \tag{4.4}$$

$\forall m\in\mathbb{N}^*, m\ge 2,\quad \forall k = \overline{n+1,q}.$

Proof. From the inequalities (3.12), (3.13) and (4.3) we obtain,

$$\left|\overline{R_{2,k}}\right| \leq (1+\frac{2LT}{2LT+1})\frac{\tau^2}{4n}\cdot\overline{L}, \qquad \forall k=\overline{n+1,q}.$$

From the recurrent inequalities (3.14) we obtain successively, by induction, that

$$\left|\overline{R_{3,k}}\right| \leq |R_{3,k}| + \frac{2LT}{2LT+1}|R_2| \leq \frac{\tau^2}{4n}\cdot\overline{L} + \frac{2LT}{2LT+1}(1+\frac{2LT}{2LT+1})\frac{\tau^2}{4n}\cdot\overline{L}$$

$$= [1+\frac{2LT}{2LT+1}+\left(\frac{2LT}{2LT+1}\right)^2]\frac{\tau^2}{4n}\cdot\overline{L}, \qquad \forall k=\overline{n+1,q},$$

$$\left|\overline{R_{m,k}}\right| \leq |R_{m,k}| + \frac{2LT}{2LT+1}|R_{m-1}|$$

$$\leq [1+\frac{2LT}{2LT+1}+...+\left(\frac{2LT}{2LT+1}\right)^{m-1}]\frac{\tau^2}{4n}\cdot\overline{L}$$

$$< (2LT+1)\frac{\tau^2}{4n}\cdot\overline{L}, \qquad ,\forall m\in\mathbb{N}^*, m\geq 3, \quad \forall k=\overline{n+1,q}. \tag{4.5}$$

The estimation (4.4) follows now, from the inequalities (2.4) and (4.5). ∎

Theorem 4.3. *In the conditions of the Theorem 2.1, if* $f\in C^1([-\tau,T]\times\mathbb{R}\times\mathbb{R})$, $\varphi\in C^1[-\tau,0]$ *and* $\varphi'(0)=f(0,\varphi(0),\varphi(-\tau))$*, then the terms of the sequence of successive approximations lies in* $C^1[-\tau,T]$ *and the sequence* $(\overline{x_m(t_k)})_{m\in\mathbb{N}^*}$ *approximates the solution* x^* *of the initial value problem (1.1) on the knots* t_k, $k=\overline{n+1,q}$ *of the uniform partition* Δ_n*, with the error estimation :*

$$\left|\overline{x_m(t_k)}-x^*(t_k)\right| \leq \frac{4LT+1}{2LT+1}\cdot\left(\frac{1}{2}\right)^m\cdot\|x_1-x_0\|_B$$

$$+(2LT+1)\frac{\tau^2}{4n}\cdot M_1(1+M+K), \tag{4.6}$$

$\forall m\in\mathbb{N}^*, m\geq 2, \quad \forall k=\overline{n+1,q}$, *where* $K=\max(\|\varphi'\|_C, M)$.

Proof. Let $K=\max(\|\varphi'\|_C, M)$ and

$$M_1=\max(\left\|\frac{\partial f}{\partial t}\right\|_C, \left\|\frac{\partial f}{\partial x}\right\|_C, \left\|\frac{\partial f}{\partial y}\right\|_C).$$

Since $f\in C^1([-\tau,T]\times\mathbb{R}\times\mathbb{R})$, $\varphi\in C^1[-\tau,0]$, we get that

$$F_m'(t)=\frac{\partial f}{\partial t}(t,x_m(t),x_m(t-\tau))+\frac{\partial f}{\partial x}(t,x_m(t),x_m(t-\tau))\cdot x_m'(t)$$

$$+\frac{\partial f}{\partial y}(t,x_m(t),x_m(t-\tau))\cdot x_m'(t-\tau)=\frac{\partial f}{\partial t}(t,x_m(t),x_m(t-\tau))$$

$$+\frac{\partial f}{\partial x}(t,x_m(t),x_m(t-\tau))\cdot f(t,x_{m-1}(t),x_{m-1}(t-\tau))$$

$$+\frac{\partial f}{\partial y}(t,x_m(t),x_m(t-\tau))\cdot\begin{cases}\varphi'(t-\tau), & t\in[0,\tau]\\ f(t-\tau,x_{m-1}(t-\tau),x_{m-1}(t-2\tau)) & ,t\in[\tau,T].\end{cases}$$

Then $F_m\in C[0,T]$ and

$$\|F_m'\|_C \leq M_1(1+M+K), \qquad \forall m\in\mathbb{N}^*.$$

On the other hand, for any $t \in [0,T]$ we have,

$$|F_0'(t)| \leq \left| \frac{\partial f}{\partial t}(t, x_0(t), x_0(t-\tau)) \right| + \left| \frac{\partial f}{\partial y}(t, x_0(t), x_0(t-\tau)) \right| \cdot |\varphi'(t-\tau)| \, .$$

Therefore,

$$\|F_m'\|_C \leq M_1(1+M+K), \qquad \forall m \in \mathbb{N}.$$

In this case, the estimation (3.8) is,

$$|R_{m,k}| \leq \frac{\tau^2}{4n} \cdot M_1(1+M+K), \qquad \forall\, m \in \mathbb{N},\ \forall k = \overline{n+1,q}.$$

From this moment, the proof is analogous with the proof of the Theorem 4.2. ∎

In the conditions of the Theorem 4.3, we consider the Lipschitz properties of the partial derivatives of first order :

$$\left| \frac{\partial f}{\partial t}(t_1,u,v) - \frac{\partial f}{\partial t}(t_2,u,v) \right| \leq \gamma_1 |t_1 - t_2| \, , \quad \forall t_1,t_2 \in [0,T], \quad \forall u,v \in \mathbb{R}, \tag{4.7}$$

$$\left| \frac{\partial f}{\partial t}(t,u_1,v) - \frac{\partial f}{\partial t}(t,u_2,v) \right| \leq \gamma_2 |u_1 - u_2| \, , \quad \forall t \in [0,T], \quad \forall u_1,u_2,v \in \mathbb{R}, \tag{4.8}$$

$$\left| \frac{\partial f}{\partial t}(t,u,v_1) - \frac{\partial f}{\partial t}(t,u,v_2) \right| \leq \gamma_3 |v_1 - v_2| \, , \quad \forall t \in [0,T], \quad \forall v_1,v_2,u \in \mathbb{R}, \tag{4.9}$$

$$\left| \frac{\partial f}{\partial x}(t_1,u,v) - \frac{\partial f}{\partial x}(t_2,u,v) \right| \leq \alpha_1 |t_1 - t_2| \, , \quad \forall t_1,t_2 \in [0,T], \quad \forall u,v \in \mathbb{R}, \tag{4.10}$$

$$\left| \frac{\partial f}{\partial x}(t,u_1,v) - \frac{\partial f}{\partial x}(t,u_2,v) \right| \leq \alpha_2 |u_1 - u_2| \, , \quad \forall t \in [0,T], \quad \forall u_1,u_2,v \in \mathbb{R}, \tag{4.11}$$

$$\left| \frac{\partial f}{\partial x}(t,u,v_1) - \frac{\partial f}{\partial x}(t,u,v_2) \right| \leq \alpha_3 |v_1 - v_2| \, , \quad \forall t \in [0,T], \quad \forall v_1,v_2,u \in \mathbb{R}, \tag{4.12}$$

$$\left| \frac{\partial f}{\partial y}(t_1,u,v) - \frac{\partial f}{\partial y}(t_2,u,v) \right| \leq \beta_1 |t_1 - t_2| \, , \quad \forall t_1,t_2 \in [0,T], \quad \forall u,v \in \mathbb{R}, \tag{4.13}$$

$$\left| \frac{\partial f}{\partial y}(t,u_1,v) - \frac{\partial f}{\partial y}(t,u_2,v) \right| \leq \beta_2 |u_1 - u_2| \, , \quad \forall t \in [0,T], \quad \forall u_1,u_2,v \in \mathbb{R}, \tag{4.14}$$

$$\left| \frac{\partial f}{\partial y}(t,u,v_1) - \frac{\partial f}{\partial y}(t,u,v_2) \right| \leq \beta_3 |v_1 - v_2| \, , \quad \forall t \in [0,T], \quad \forall v_1,v_2,u \in \mathbb{R}, \tag{4.15}$$

with $\alpha_i > 0, \beta_i > 0, \gamma_i > 0,\ \forall i = \overline{1,3}$ and a Lipschitz condition on the initial interval :

$$|\varphi'(t_1) - \varphi'(t_2)| \leq \gamma' \cdot |t_1 - t_2| \, , \qquad \forall t_1,t_2 \in [-\tau, 0], \tag{4.16}$$

with $\gamma' > 0$.

Theorem 4.4. *In the conditions of the Theorem 4.3 and under the assumptions (4.7)-(4.16), the first derivatives of the functions $F_m,\ m \in \mathbb{N}$, are Lipschitzian with the same Lipschitz constant $L' > 0$.*

Proof. Firstly,

$$|F_0'(t_1) - F_0'(t_2)| \le \left|\frac{\partial f}{\partial t}(t_1, x_0(t_1), x_0(t_1-\tau)) - \frac{\partial f}{\partial t}(t_2, x_0(t_2), x_0(t_2-\tau))\right|$$

$$+\left|\frac{\partial f}{\partial y}(t_1, x_0(t_1), x_0(t_1-\tau)) \cdot \varphi'(t_1-\tau) - \frac{\partial f}{\partial y}(t_2, x_0(t_2), x_0(t_2-\tau)) \cdot \varphi'(t_2-\tau)\right|$$

$$\le \gamma_1 \cdot |t_1 - t_2| + \beta_1 \cdot |\varphi(t_1-\tau) - \varphi(t_2-\tau)| + M_1\gamma' \cdot |t_1-t_2|$$

$$+\|\varphi'\|_C \left[\gamma_3 \cdot |t_1-t_2| + \beta_3 \cdot |\varphi(t_1-\tau) - \varphi(t_2-\tau)|\right]$$

$$\le [\gamma_1 + \beta_1\delta + K(\gamma_3 + \beta_3\delta) + M_1\gamma'] \cdot |t_1 - t_2|, \qquad \forall t_1, t_2 \in [0,T].$$

Let

$$L_0' = \gamma_1 + \beta_1\delta + K(\gamma_3 + \beta_3\delta) + M_1\gamma'.$$

We obtain,

$$|F_m'(t_1) - F_m'(t_2)| \le \left|\frac{\partial f}{\partial t}(t_1, x_m(t_1), x_m(t_1-\tau)) - \frac{\partial f}{\partial t}(t_2, x_m(t_2), x_m(t_2-\tau))\right|$$

$$+|f(t_1, x_{m-1}(t_1), x_{m-1}(t_1-\tau))| \cdot | \frac{\partial f}{\partial x}(t_1, x_m(t_1), x_m(t_1-\tau))$$

$$-\frac{\partial f}{\partial x}(t_2, x_m(t_2), x_m(t_2-\tau)) \mid + \left|\frac{\partial f}{\partial x}(t_2, x_m(t_2), x_m(t_2-\tau))\right|$$

$$\cdot |f(t_1, x_{m-1}(t_1), x_{m-1}(t_1-\tau)) - f(t_2, x_{m-1}(t_2), x_{m-1}(t_2-\tau))|$$

$$| \frac{\partial f}{\partial y}(t_1, x_m(t_1), x_m(t_1-\tau)) - \frac{\partial f}{\partial y}(t_2, x_m(t_2), x_m(t_2-\tau)) |$$

$$\cdot \max\left\{|\varphi'(t_1-\tau)|, \quad |f(t_1-\tau, x_{m-1}(t_1-\tau), x_{m-1}(t_1-2\tau))|\right\}$$

$$+\left|\frac{\partial f}{\partial y}(t_2, x_m(t_2), x_m(t_2-\tau))\right| \cdot \max\{|\varphi'(t_1-\tau) - \varphi'(t_2-\tau)|,$$

$$| f(t_1-\tau, x_{m-1}(t_1-\tau), x_{m-1}(t_1-2\tau))$$

$$-f(t_2-\tau, x_{m-1}(t_2-\tau), x_{m-1}(t_2-2\tau)) |\}$$

$$\le \gamma_1 \cdot |t_1-t_2| + \alpha_1 \cdot |x_m(t_1) - x_m(t_2)| + \beta_1 \cdot |x_m(t_1-\tau) - x_m(t_2-\tau)|$$

$$+M \cdot (\gamma_2 \cdot |t_1-t_2| + \alpha_2 \cdot |x_m(t_1) - x_m(t_2)| + \beta_2 \cdot |x_m(t_1-\tau) - x_m(t_2-\tau)|)$$

$$+K \cdot [\gamma_3 \cdot |t_1-t_2| + \alpha_3 \cdot |x_m(t_1) - x_m(t_2)| + \beta_3 \cdot |x_m(t_1-\tau) - x_m(t_2-\tau)|]$$

$$+M_1\overline{L} \cdot |t_1-t_2| + M_1 \cdot \max\{\gamma' \cdot |t_1-t_2|, \quad \overline{L} \cdot |t_1-t_2|\}$$

$$\le L' \cdot |t_1-t_2|, \qquad \forall t_1, t_2 \in [0,T],$$

where,

$$L' = \gamma_1 + \alpha_1 \cdot \max(\delta, M) + \beta_1 \cdot \max(\delta, M) + M\gamma_2 + M\alpha_2 \cdot \max(\delta, M)$$

$$+M\beta_2 \cdot \max(\delta, M) + M_1\overline{L} + K\gamma_3 + K\alpha_3 \cdot \max(\delta, M)$$

$$+K\beta_3 \cdot \max(\delta, M) + M_1 \cdot \max(\gamma', \overline{L}).$$

Since $L_0' \le L'$, hence we obtain,

$$|F_m'(t_1) - F_m'(t_2)| \le L' \cdot |t_1-t_2|, \quad \forall t_1, t_2 \in [0,T], \quad \forall m \in \mathbb{N}.$$

∎

Corollary 4.5. *In the conditions of the Theorem 4.4 the sequence* $(\overline{x_m(t_k)})_{m\in\mathbb{N}^*}$ *approximates the solution* x^* *of the initial value problem (1.1) on the knots* $t_k,\ k=\overline{n+1,q}$ *of the uniform partition* Δ_n*, with the error estimate :*

$$\left|\overline{x_m(t_k)} - x^*(t_k)\right| \le \frac{4LT+1}{2LT+1}\cdot\left(\frac{1}{2}\right)^m \cdot \|x_1 - x_0\|_B + (2LT+1)\frac{\tau^3}{12n^2}\cdot L', \tag{4.17}$$

$\forall m\in\mathbb{N}^*, m\ge 2, \quad \forall k=\overline{n+1,q}.$

Proof. In this case, the estimation (3.8) is,

$$|R_{m,k}| \le \frac{\tau^3}{12n^2}\cdot L', \qquad \forall\, m\in\mathbb{N},\ \forall k=\overline{n+1,q}.$$

Using the result from the Theorem 4.3 and the method of the proof from Theorem 4.2 we obtain the estimation (4.17). ∎

Remark 4.1. If $f\in C^2([-\tau,T]\times\mathbb{R}\times\mathbb{R})$ and $\varphi\in C^2[-\tau,0]$, then using the numerical method (3.7)-(3.11) and the classic trapezoidal quadrature rule we obtain that $F_m\in C^2[0,T],\ \forall\, m\in\mathbb{N}$, with $\|F_m''\|\le M''$ and the error estimations is :

$$\left|\overline{x_m(t_k)} - x^*(t_k)\right| \le \frac{4LT+1}{2LT+1}\cdot\left(\frac{1}{2}\right)^m \cdot \|x_1 - x_0\|_B + (2LT+1)\frac{\tau^3}{12n^2}\cdot M'',$$

$\forall m\in\mathbb{N}^*, m\ge 2, \quad \forall k=\overline{n+1,q}.$

Remark 4.2. Note that the numerical method presented in this paper works in the case when the function f have at most a continuous first order partial derivative. The known methods from literature (see [2], [3], [5], [6], [8]-[10], [12], [13], [16], [18]-[27], [30], [31]) on states in the cases when the function f have partial derivatives of high order.

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