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SOME APPROXIMATION FOR THE LINEAR COMBINATIONS OF MODIFIED BETA OPERATORS

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ABSTRACT. In this paper, we propose a sequence of new positive linear operators β_n to study the ordinary approximation of unbounded functions by using some properties of the Steklov means.

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Dedicated to **Dr Vijay Gupta**, Netaji Subhas Institute of Technology, Dwarka, New Delhi, India.

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1. INTRODUCTION

A family of linear positive operators, from a mapping $C[0,\infty)$ into $C[0,\infty)$, the class of all bounded and continuous functions on $[0,\infty)$, is called Beta operators which is denoted by V_n and defined as

$$(V_n f)(x) = \frac{1}{n}\sum_{v=0}^{\infty} b_{n,v}(x) f\left(\frac{v}{n+1}\right),$$

where

$$b_{n,v}(x) = \frac{1}{B(v+1,n)}\frac{x^v}{(1+x)^{n+v+1}}, \qquad x\in[0,\infty)$$

and $B(v+1,n)$ denotes the Beta function given by $\Gamma(v+1)\cdot\Gamma(n)/\Gamma(v+n+1)$.

Let f be a function defined on $[0,\infty)$ then we define a sequence of linear positive operators β_n as

$$(1.1)\qquad (\beta_n f)(x) = \frac{1}{n}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty} b_{n,v-1}(t) f(t)dt + (1+x)^{-n-1} f(0).$$

The operators (1.1) may also be written as

$$\beta_n(f(t);x) = \int_0^{\infty} K_n(t,x) f(t)dt,$$

where

$$K_n(t,x) = \frac{1}{n}\sum_{v=1}^{\infty} b_{n,v}(x) b_{n,v-1}(t) + (1+x)^{-n-1}\delta(t),$$

$\delta(t)$ being the Dirac delta function.

A lot of work has been done on such type operators by Gupta et al. (see e.g. [6], [7], [8], [9]). To approximate Lebesgue integrable functions on $[0,\infty)$, the space $C_\alpha[0,\infty)$ is normed by

$$\|f\|_{C_\alpha} = \sup_{0\le t<\alpha} |f(t)|(1+t)^{-\alpha}.$$

Note that the order of approximation by these operators (1.1) is at best $O(n^{-1})$, howsoever smooth the function may be. Thus to improve the order of approximation, we use the technique of linear combination of the operators (1.1). Actually May [3] and Rathore [5] first considered the linear combinations to improve the rate of convergence for exponential type operators, which include the well known Bernstein, Szasz and Baskakov operators as special cases. We consider the linear combination of operators (1.1) as described below.

For $d_0, d_1, d_2, ..., d_k$ arbitrary but fixed distinct positive integers, the linear combination $\beta_n(f,k,x)$ of $\beta_{d_j n}(f;x), j=0,1,2,...,n$ is defined by

$$(1.2)\qquad \beta_n(f,k,x) = \sum_{j=0}^{k} C(j,x)\beta_{d_j n}(f;x),$$

where

$$C(j,k) = \begin{cases} \prod\limits_{\substack{i=0\\ i\ne j}}^{k} \frac{d_j}{d_j-d_i}, & for\ k\ne 0 \\ 1, & for\ k=0 \end{cases}$$

Let $m \in \boldsymbol{N}$ and $0 < a < b < \infty$. For $f \in L_p[a,b],\ 1 \leq p < \infty$, the m^{th} order integral modulus of smoothness of f is defined as:

$$\omega_m(f,\tau,p,[a,b]) = \sup_{0<\delta\leq\tau} \left\| \triangle_\delta^m f(t) \right\|_{L_p[a,b-m\delta]},$$

where $\triangle_\delta^m f(t)$ is the m^{th} order forward difference with length δ and $0 < \tau \leq \frac{(b-a)}{m}$.

The spaces $AC[a,b]$ and $BV[a,b]$ are defined as the classes of absolutely continuous functions and functions of bounded variation over $[a,b]$, respectively. The seminorm $\|f\|_{BV[a,b]}$ is defined by the total variation of f on $[a,b]$.

Throughout this paper, we assume $0 < a_1 < a_3 < a_2 < b_2 < b_3 < b_1 < \infty$, $I_i = [a_i, b_i],\ i = 1,2,3$ and C denotes a positive constant, not necessarily the same at all occurrence.

The main purpose of this paper to give some results in terms of higher order modulus of continuity in ordinary approximation by the operators (1.1). First, we study the basic pointwise convergence theorem and then proceed to give the degree of approximation.

2. BASIC RESULTS

In the section, we shall mention some definitions and certain lemmas to prove our main theorems.

Lemma 2.1. *For* $m \in \boldsymbol{N} \cup \{0\}$ *if*

$$\mu_{n,m}(t) = \frac{1}{n} \sum_{v=1}^{\infty} b_{n,v-1}(t) \int_0^\infty b_{n,v}(x)(x-t)^m dx$$

then

$$\mu_{n,0}(t) = 1, \quad \mu_{n,1}(t) = \frac{2(1+x)}{n-1} \tag{2.1}$$

and

$$\begin{aligned}(n-m-1)\mu_{n,m+1}(t) &= t(1+t)\mu'_{n,m}(t) \\ &\quad + [(m+2) + 2x(m+1)]\mu_{n,m}(t) + 2mt(1+t)\mu_{n,m-1}(t).\end{aligned} \tag{2.2}$$

Consequently,

(i) $\mu_{n,m}(t)$ *is a polynomial in x of degree* $\leq m$.

(ii) $\mu_{n,m}(t) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right)$, *where* $[\alpha]$ *denotes the integral part of* α.

Proof. We can easily obtain (2.1) by using the definition of $\mu_{n,m}(t)$. For the proof of (2.2), we proceed as follows. First

$$t(1+t)\mu'_{n,m}(t) = \frac{1}{n} \sum_{v=1}^{\infty} t(1+t) b'_{n,v-1}(t) \int_0^\infty b_{n,v}(x)(x-t)^m dx - mt(1+t)\mu_{n,m-1}(t).$$

Now, using the relation twice and integrating by parts

$$x(1+x)b'_{n,v}(x) = [v-(n+1)x]\, b_{n,v}(x)$$

$$
\begin{aligned}
t(1+t)\mu'_{n,m}(t) &= \frac{1}{n}\sum_{v=1}^{\infty}[(v-1)-(n+1)x+(n+1)(x-t)]b_{n,v-1}(t) \\
&\quad \cdot \int_0^\infty b_{n,v}(x)(x-t)^m dx - mt(1+t)\mu_{n,m-1}(t) \\
&= \frac{1}{n}\sum_{v=1}^{\infty}[(v-1)-(n+1)x]b_{n,v-1}(t)\int_0^\infty b_{n,v}(x)(x-t)^m dx \\
&\quad + (n+1)\mu_{n,m+1}(t) - mt(1+t)\mu_{n,m-1}(t) \\
&= \frac{1}{n}\sum_{v=1}^{\infty} b_{n,v-1}(x)\int_0^\infty [v-(n+1)x]b_{n,v}(x)(x-t)^m dx \\
&\quad + (n+1)\mu_{n,m+1}(t) - mt(1+t)\mu_{n,m-1}(t) - \mu_{n,m}(t) \\
&= \frac{1}{n}\sum_{v=1}^{\infty} b_{n,v-1}(x)\int_0^\infty x(1+x)b'_{n,v}(x)(x-t)^m dx \\
&\quad + (n+1)\mu_{n,m+1}(t) - mt(1+t)\mu_{n,m-1}(t) - \mu_{n,m}(t) \\
&= \frac{1}{n}\sum_{v=1}^{\infty} b_{n,v-1}(x)\int_0^\infty [(1+2x)(x-t)-(x-t)^2 \\
&\quad + t(1+t)]b'_{n,v}(x)(x-t)^m dx + (n+1)\mu_{n,m+1}(t) \\
&\quad - mt(1+t)\mu_{n,m-1}(t) - \mu_{n,m}(t) \\
&= \frac{(1+2x)}{n}\sum_{v=1}^{\infty} b_{n,v-1}(t)\int_0^\infty b'_{n,v}(x)(x-t)^{m+1} dx \\
&\quad - \frac{1}{n}\sum_{v=1}^{\infty} b_{n,v-1}(t)\int_0^\infty b'_{n,v}(x)(x-t)^{m+2} dx \\
&\quad + \frac{1}{n}\sum_{v=1}^{\infty} b_{n,v-1}(t)\int_0^\infty t(1+t)b'_{n,v}(x)(x-t)^m dx \\
&\quad + (n+1)\mu_{n,m+1}(t) - mt(1+t)\mu_{n,m-1}(t) - \mu_{n,m}(t) \\
&= -(m+1)(1+2x)\mu_{n,m}(t) - (m+2)\mu_{n,m+1}(t) - mt(1+t)\mu_{n,m-1}(t) \\
&\quad + (n+1)\mu_{n,m+1}(t) - mt(1+t)\mu_{n,m-1}(t) - \mu_{n,m}(t).
\end{aligned}
$$

This leads to (2.2). ∎

Lemma 2.2. *Let the m^{th} order moment be defined as*

$$
T_{n,m}(x) = \beta_n\left((t-x)^m; x\right) = \frac{1}{n}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^\infty b_{n,v-1}(t)(t-x)^m dt + (-x)^m(1+x)^{-n-1}.
$$

Then

$$
T_{n,0}(x) = 1, \quad T_{n,1}(x) = \frac{2x}{(n-1)} \tag{2.3}
$$

and

$$
\begin{aligned}
(n-m-1)T_{n,m+1}(x) &= [m(1+2x)+2x]\,T_{n,m}(x) \\
&\quad + x(1+x)\left[T'_{n,m}(x) + 2mT_{n,m-1}(x)\right], \quad n > m+2.
\end{aligned} \tag{2.4}
$$

Further, for all $x \in [0, \infty)$, the sequences $T_{n,m}(x)$ has the following properties:

(i) $T_{n,m}(x)$ is a polynomial in x of degree m.

(ii) $T_{n,m}(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right)$.

Proof. We can easily verify (2.3) and the proof of (2.4) follows. We have

$$\begin{aligned} x(1+x)T'_{n,m}(x) = \frac{1}{n}\sum_{v=1}^{\infty} x(1+x)b'_{n,v}(x)\int_0^{\infty} b_{n,v-1}(t)(t-x)^m dt \\ - mx(1+x)T_{n,m-1}(x) + (n+1)(-x)^{m+1}(1+x)^{-n-1}. \end{aligned}$$

Using the relation which is mentioned in Lemma 2.1

$$\begin{aligned} &x(1+x)T'_{n,m}(x) \\ &= \frac{1}{n}\sum_{v=1}^{\infty}\{v-(n+1)x\}b_{n,v}(x)\int_0^{\infty} b_{n,v-1}(t)(t-x)^m dt \\ &\quad - mx(1+x)T_{n,m-1}(x) + (n+1)(-x)^{m+1}(1+x)^{-n-1} \\ &= \frac{1}{n}\sum_{v=1}^{\infty}\{v-(n+1)t+(n+1)(t-x)\}b_{n,v}(x)\int_0^{\infty} b_{n,v-1}(t)(t-x)^m dt \\ &\quad - mx(1+x)T_{n,m-1}(x) + (n+1)(-x)^{m+1}(1+x)^{-n-1} \\ &= \frac{1}{n}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty}\{(v-1)-(n+1)t\}b_{n,v-1}(t)(t-x)^m dt \\ &\quad - mx(1+x)T_{n,m-1}(x) + (n+1)T_{n,m+1}(x) \\ &\quad + \frac{1}{n}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty} b_{n,v-1}(t)(t-x)^m dt \\ &= \frac{1}{n}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty} t(1+t)b'_{n,v-1}(t)(t-x)^m dt + (n+1)T_{n,m+1}(x) \\ &\quad - mx(1+x)T_{n,m-1}(x) + \frac{1}{n}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty} b_{n,v-1}(t)(t-x)^m dt \\ &= \frac{1}{n}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty}[(1+2x)(t-x)+(t-x)^2+x(1+x)]b'_{n,v-1}(t)(t-x)^m dt \\ &\quad + \frac{1}{n}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty} b_{n,v-1}(t)(t-x)^m dt \\ &\quad - mx(1+x)T_{n,m-1}(x) + (n+1)T_{n,m+1}(x) \end{aligned}$$

$$= -(1+2x)(m+1)\frac{1}{n}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty} b_{n,v-1}(t)(t-x)^m dt$$
$$-(m+2)\frac{1}{n}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty} b_{n,v-1}(t)(t-x)^{m+1} dt$$
$$-mx(1+x)\frac{1}{n}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty} b_{n,v-1}(t)(t-x)^{m-1} dt + T_{n,m}(x)$$
$$-(-x)^m(1+x)^{-n-1} - mx(1+x)T_{n,m-1}(x) + (n+1)T_{n,m+1}(x)$$
$$= -[m(1+2x)+2x]T_{n,m}(x) + (n-m-1)T_{n,m+1}(x) - 2mx(1+x)T_{n,m-1}(x)$$

This leads to (2.4) ∎

Lemma 2.3. *For $m \in \mathbf{N}$ and n sufficiently large, there holds*

$$\beta_n\big((t-x)^m, k, x\big) = n^{-(k+1)}\{Q(m,k,x) + o(1)\},\quad for\ m \in \mathbf{N}$$

where $Q(m,k,x)$ is a certain polynomial in x of degree m and $x \in [0,\infty)$ is arbitrary but fixed.

Lemma 2.4. *Let $f \in BV(I_1)$ the following inequality holds:*

$$\left\|\beta_n\left(\chi(t)\int_x^t (t-z)^{2k+1} df(z); x\right)\right\|_{L_1(I_2)} \le Cn^{-(k+1)}\|f\|_{BV(I_1)},$$

where $\chi(t)$ is the characteristic function of I_1.

Proof. For each n there exists a nonnegative integer $r = r(n)$ such that

$$rn^{-1/2} \le max\{b_1 - a_{2,} b_2 - a_1\} \le (r+1)\, n^{-1/2}.$$

Then,

$$M := \left\|\beta_n\left(\int_x^t (t-z)^{2k+1} df(z)\chi(t); x\right)\right\|_{L_1(I_2)}$$
$$\le \sum_{i=0}^{r}\int_{a_2}^{b_2}\left\{\int_{x+in^{-1/2}}^{x+(i+1)n^{-1/2}} \chi(t)K_n(t,x)\big|t-x\big|^{2k+1}\left[\int_x^{x+(i+1)n^{-1/2}} \chi(z)\big|df(z)\big|\right] dt\right.$$
$$\left. + \int_{x-(i+1)n^{-1/2}}^{x-in^{-1/2}} \chi(t)K_n(t,x)\big|t-x\big|^{2k+1}\left(\int_{x-(i+1)n^{-1/2}}^{x}\chi(z)\big|df(z)\big|\right) dt\right\} dx.$$

Let $\chi_{x,c,k}(z)$ denote the characteristic function in the interval $\left[x-cn^{-1/2}, x+dn^{-1/2}\right]$ where c and d are nonnegative integers. Then we get

$$M \le \sum_{i=1}^{r}\left[n^2 i^{-4}\int_{a_2}^{b_2}\left\{\int_{x+in^{-1/2}}^{x+(i+1)n^{-1/2}} \chi(t)K_n(t,x)\big|t-x\big|^{2k+5}\left(\int_{a_1}^{b_1}\chi_{x,0,i+1}(z)\big|df(z)\big|\right) dt\right.\right.$$
$$\left.\left. + \int_{x-(i+1)n^{-1/2}}^{x-in^{-1/2}} \chi(t)K_n(t,x)\big|t-x\big|^{2k+5}\left(\int_{a_1}^{b_1}\chi_{x,i+1,0}(z)\big|df(z)\big|\right) dt\right\}\right]$$
$$+ \int_{a_2}^{b_2}\int_{a_2-n^{-1/2}}^{b_2+n^{-1/2}} \chi(t)K_n(t,x)\big|t-x\big|^{2k+1}\left(\int_{a_1}^{b_1}\chi_{x,01,1}(z)\big|df(z)\big|\right) dtdx.$$

By Fubini's Theorem and Lemma 2.1, we obtain

$$\begin{aligned} M &\leq Cn^{-(2k+1)/2}\Bigg[\sum_{i=1}^{r} i^{-4}\Bigg\{\int_{a_1}^{b_1}\Big(\int_{a_2}^{b_2}\chi_{x,0,i+1}(z)dx\Big)\big|df(z)\big| \\ &\quad + \int_{a_1}^{b_1}\Big(\int_{a_2}^{b_2}\chi_{x,i+1,0}(z)dx\Big)\big|df(z)\big|\Bigg\} + \int_{a_1}^{b_1}\Big(\int_{a_2}^{b_2}\chi_{x,1,1}(z)dx\Big)\big|df(z)\big|\Bigg] \\ &\leq Cn^{-(k+1)}\big\|f\big\|_{BV(I_1)}. \end{aligned}$$

This completes the proof. ∎

3. ERROR ESTIMATES

In this section, first we discuss the approximation in the smooth subspace $L_p^{(2k+2)}(I_1)$ of $L_p\,[0,\infty)$.

Theorem 3.1. *Let $f \in L_p^{(2k+2)}(I_1)$ and $1 < p < \infty$ then following inequality holds:*

(3.1) $$\big\|\beta_n(f,k,.) - f\big\|_{L_p(I_2)} \leq C_1 n^{-(k+1)}\Big\{\big\|f^{(2k+2)}\big\|_{L_p(I_1)} + \big\|f\big\|_{L_p[0,\infty)}\Big\},$$

where n is sufficiently large and $C_1 = C_1(k,p)$.

Proof. Let $p > 1$. For $t \in I_1$ and $x \in I_2$ we may write

$$f(t) = \sum_{j=0}^{2k+1}\frac{(t-x)^j}{j!}f^{(j)}(x) + \frac{1}{(2k+1)!}\int_x^t (t-z)^{2k+1} f^{(2k+2)}(z)dz.$$

Thus, if $\chi(t)$ is the characteristic function of I_1, then

$$\begin{aligned} f(t) = \sum_{j=0}^{2k+1}\frac{(t-x)^j}{j!}f^{(j)}(x) + \frac{1}{(2k+1)!}\int_x^t \chi(t)(t-z)^{2k+1} f^{(2k+2)}(z)dz \\ + F(t,x)\big(1-\chi(t)\big), \end{aligned}$$

where

$$F(t,x) = f(t) - \sum_{j=0}^{2k+1}\frac{(t-x)^j}{j!}f^{(j)}(x), \quad for\ all\ t \in [0,\infty)\ and\ x \in I_2.$$

For $\beta_n\,(1,k,x) = 1$, we get

$$\begin{aligned} \beta_n\,(f,k,x) - f(x) &= \sum_{j=1}^{2k+1}\frac{f^{(j)}(x)}{j!}\beta_n\left((t-x)^j,k,x\right) \\ &\quad + \frac{1}{(2k+1)!}\beta_n\left(\chi(t)\int_x^t (t-z)^{2k+1} f^{(2k+2)}(z)dz,k,x\right) \\ &\quad + \beta_n\big(F(t,x)(1-\chi(t)),k,x\big) \\ &=: E_1 + E_2 + E_3. \end{aligned}$$

Using Lemma 2.3,

$$\big\|E_1\big\|_{L_p(I_2)} \leq Cn^{-(k+1)}\big[\big\|f\big\|_{L_p(I_2)} + \big\|f^{(2k+2)}\big\|_{L_p(I_2)}\big].$$

To estimate E_2, let h_f be the Hardy-Littlewood majorant of $f^{(2k+2)}$ on I_1 (see [2], page 244). Applying the Hölder inequality and by Lemma 2.2, we obtain

$$\begin{aligned} Q_1 :=& \left| \beta_n \left(\chi(t) \int_x^t (t-z)^{2k+1} f^{(2k+2)}(z)dz; x \right) \right| \\ &\leq \beta_n \left(\chi(t)|t-x|^{2k+1} \left| \int_x^t |f^{(2k+2)}(z)|dz \right|; x \right) \\ &\leq \beta_n \left(\chi(t)(t-x)^{2k+2} |h_j(t)|; x \right) \\ &\leq \left\{ \beta_n \left(\chi(t)|t-x|^{(2k+2)q}; x \right) \right\}^{1/q} \{ \beta_n \left(\chi(t)|h_f(t)|^p; x \right) \}^{1/p} \\ &\leq Cn^{-(k+1)} \left\{ \int_{a_1}^{b_1} K_n(t,x) \left| h_f(t) \right|^p dt \right\}^{1/p}. \end{aligned}$$

Thus by Lemma 2.1, [2] (ch. 10, sec. 2, page 244) and using Fubini's theorem we obtain

$$\begin{aligned} \left\| Q_1 \right\|_{L_p(I_2)}^p &\leq Cn^{-(k+1)p} \int_{a_2}^{b_2} \int_{a_1}^{b_1} K_n(t,x) \left| h_f(t) \right|^p dt dx \\ &\leq Cn^{-(k+1)p} \int_{a_1}^{b_1} \left\{ \int_{a_2}^{b_2} K_n(t,x)\, dx \right\} \left| h_f(t) \right|^p dt \\ &\leq Cn^{-(k+1)p} \frac{n}{n-1} \int_{a_1}^{b_1} \left| h_f(t) \right|^p dt \\ &\leq Cn^{-(k+1)p} \| h_f \|_{L_p(I_1)}^p \qquad \text{(since } n \text{ is sufficiently large)} \\ &\leq Cn^{-(k+1)p} \| f^{(2k+2)} \|_{L_p(I_1)}^p. \end{aligned}$$

Consequently,

$$\left\| Q_1 \right\|_{L_p(I_2)} \leq Cn^{-(k+1)} \| f^{(2k+2)} \|_{L_p(I_1)}.$$

Hence, we get

$$\left\| E_2 \right\| \leq Cn^{-(k+1)} \| f^{(2k+2)} \|_{L_p(I_1)}.$$

For $t \in [0,\infty) \setminus I_1,\ x \in I_2\ \exists\ \delta > 0$ such that $\left| t-x \right| \geq \delta$, we get

$$\begin{aligned} \left| \beta_n (F(t,x)(1-\chi(t)); x) \right| &\leq \delta^{-(2k+2)} \Bigg[\beta_n \left(\left| f(t) \right| (t-x)^{2k+2}; x \right) \\ &\qquad + \sum_{j=0}^{2k+1} \frac{f^{(j)}(x)}{j!} \beta_n \left(\left| t-x \right|^{2k+j+2}; x \right) \Bigg] \\ &:= Q_2 + Q_3 \end{aligned}$$

Applying the Hölder inequality and by Lemma 2.2

$$\left| Q_2 \right| \leq Cn^{-(k+1)} \left\{ \beta_n \left(\left| f(t) \right|^p; x \right) \right\}^{1/p}.$$

Again, using the Fubini's theorem, for n sufficiently large, we obtain

$$\left| Q_2 \right|_{L_p(I_2)} \leq Cn^{-(k+1)} \| f \|_{L_p[0,\infty)}.$$

By Lemma 2.2, we get

$$\left\| Q_3 \right\|_{L_p(I_2)} \leq Cn^{-(k+1)} \left(\left\| f \right\|_{L_p(I_2)} + \left\| f^{(2k+2)} \right\|_{L_p(I_2)} \right).$$

Thus

$$\|E_3\|_{L_p(I_2)} \leq Cn^{-(k+1)}\big(\|f\|_{L_p[0,\infty)} + \|f^{(2k+2)}\|_{L_p(I_2)}\big).$$

The estimates of E_1 to E_3, leads to (3.1). ∎

Theorem 3.2. *Let $f \in L_1\,[0,\infty)$ and n is sufficiently large. If $f^{(2k+1)} \in I_1,\ f^{(2k)} \in AC(I_1)$ and $f^{(2k+1)} \in BV(I_1)$, then following inequality holds:*

(3.2) $$\|\beta_n(f,k,.) - f\|_{L_1(I_2)} \leq C_2 n^{-(k+1)}\Big\{\|f^{(2k+1)}\|_{BV(I_1)} + \|f^{(2k+1)}\|_{L_1(I_2)} + \|f\|_{L_1[0,\infty)}\Big\}$$

where $C_2 = C_2(k) > 0$.

Proof. Let $p = 1$ and $f \in L_1\,[0,\infty)$, for almost all $x \in I_2$ and for $t \in I_1$, we may write

$$f(t) = \sum_{j=0}^{2k+1} \frac{(t-x)^j}{j!} f^{(j)}(x) + \frac{1}{(2k+1)!}\int_x^t (t-z)^{2k+1} df^{(2k+1)}(z).$$

If $\chi(t)$ is the characteristic function of I_1 then

$$f(t) = \sum_{j=0}^{2k+1} \frac{(t-x)^j}{j!} f^{(j)}(x) + \frac{1}{(2k+1)!}\int_x^t (t-z)^{2k+1} df^{(2k+1)}(z)\chi(t)$$
$$+ F(t,x)\big(1-\chi(t)\big).$$

where $F(t,x) = f(t) - \sum\limits_{j=0}^{2k+1} \frac{(t-x)^j}{j!} f^{(j)}(x)$, for almost all $x \in I_2$ and for all $t \in [0,\infty)$. By operating β_n on the last equation, we get

$$\begin{aligned}\beta_n(f,k,x) - f(x) &= \sum_{j=1}^{2k+1} \frac{f^{(j)}(x)}{j!}\beta_n\big((t-x)^j,k,x\big)\\ &\quad + \frac{1}{(2k+1)!}\beta_n\left(\int_x^t (t-z)^{2k+1} df^{(2k+1)}(z)\chi(t),k,x\right)\\ &\quad + \beta_n\big(F(t,x)(1-\chi(t)),k,x\big)\\ &= E_1 + E_2 + E_3.\end{aligned}$$

Using Lemma 2.3, we get

$$\|E_1\|_{L_1(I_2)} \leq Cn^{-(k+1)}\big(\|f\|_{L_1(I_2)} + \|f^{(2k+1)}\|_{L_1(I_2)}\big).$$

By Lemma 2.4, we obtain

$$\|E_2\|_{L_1(I_2)} \leq Cn^{-(k+1)}\|f^{(2k+1)}\|_{BV(I_1)}.$$

Choosing $\delta > 0$ such that $|t-x| \geq \delta$, for all $x \in I_2,\ t \in [0,\infty)\setminus I_1$, then we get

$$\begin{aligned}&\|\beta_n\big(F(t,x)(1-\chi(t));x\big)\|_{L_1(I_2)}\\ &\leq \int_{a_2}^{b_2}\int_0^\infty K_n(t,x)|f(t)|\big(1-\chi(t)\big)dtdx\\ &\qquad + \sum_{j=0}^{2k+1}\frac{1}{j!}\int_{a_2}^{b_2}\int_0^\infty K_n(t,x)|f^{(j)}(x)||t-x|^j(1-\chi(t))dtdx\\ &= Q_1 + Q_2.\end{aligned}$$

If t is sufficiently large then we may obtain positive constant M and C_1 such that

$$\frac{(t-x)^{2k+2}}{t^{2k+2}+1} > C_1 \quad \text{for all } t \geq M, x \in I_2.$$

Now applying Fubini's theorem,

$$Q_1 = \left(\int_0^M \int_{a_2}^{b_2} + \int_M^\infty \int_{a_2}^{b_2} \right) K_n(t,x)\big|f(t)\big|(1-\chi(t))dxdt = Q_3 + Q_4.$$

Using Lemma 2.1, we obtain

$$\begin{aligned} Q_3 &= \delta^{-(2k+2)} \int_0^M \int_{a_2}^{b_2} K_n(t,x)\big|f(t)\big|(t-x)^{2k+2}dxdt \\ &\leq Cn^{-(k+1)} \int_0^M \big|f(t)\big|dt, \end{aligned}$$

and

$$\begin{aligned} Q_4 &\leq \frac{1}{C_1} \int_M^\infty \int_{a_2}^{b_2} K_n(t,x) \frac{(t-x)^{2k+2}}{t^{2k+2}+1} \big|f(t)\big|dxdt \\ &\leq Cn^{-(k+1)} \int_M^\infty \big|f(t)\big|dt, \quad \text{since } t \text{ is sufficiently large}. \end{aligned}$$

Combining estimates Q_3 and Q_4, we obtain

$$Q_1 \leq Cn^{-(k+1)} \big\|f\big\|_{L_1[0,\infty)}.$$

Using Lemma 2.2, we get

$$Q_2 \leq Cn^{-(k+1)} \big(\big\|f\big\|_{L_1(I_2)} + \big\|f^{(2k+1)}\big\|_{L_1(I_2)} \big).$$

Thus

$$\big\|\beta_n\big(F(t,x)(1-\chi(t));x\big)\big\|_{L_1(I_2)} \leq Cn^{-(k+1)} \big(\big\|f\big\|_{L_1[0,\infty)} + \big\|f^{(2k+1)}\big\|_{L_1(I_2)} \big).$$

Consequently,

$$\big\|E_3\big\|_{L_1(I_2)} \leq Cn^{-(k+1)} \big(\big\|f\big\|_{L_1[0,\infty)} + \big\|f^{(2k+1)}\big\|_{L_1(I_2)} \big).$$

Finally, using E_1, E_2 and E_3, we obtain (3.2). ∎

Theorem 3.3. *Suppose $f \in L_p[0,\infty)$, $1 \leq p < \infty$, then*

$$\big\|\beta_n(f,k,.) - f\big\|_{L_p(I_2)} \leq C\big(\omega_{2k+2}(f, n^{-1/2}, p, I_1) + n^{-(k+1)}\big\|f\big\|_{L_p[0,\infty)}\big), \tag{3.3}$$

where n is sufficiently large and C is a constant that independent of f and n, but is dependent on k and p.

Proof. Let $f_{\eta,2k+2}(t)$ be the Steklov mean of $(2k+2)^{th}$ corresponding to $f(t)$ over I_1 where $\eta > 0$ is sufficiently small and $f_{\eta,2k+2}(t)$ is defined as zero outside I_1, then we get

$$\begin{aligned} \big\|\beta_n(f,k,.) - f\big\|_{L_p(I_2)} &\leq \big\|\beta_n(f - f_{\eta,2k+2},k,.)\big\|_{L_p(I_2)} + \big\|\beta_n(f_{\eta,2k+2},k,.) - f_{\eta,2k+2}\big\|_{L_p(I_2)} \\ &\quad + \big\|f_{\eta,2k+2} - f\big\|_{L_p(I_2)} \\ &= E_1 + E_2 + E_3. \end{aligned}$$

Suppose $\chi(t)$ is the characteristic function of I_3, we get

$$\begin{aligned}\beta_n\big((f-f_{\eta,2k+2})(t);x\big) &= \beta_n\big(\chi(t)(f-f_{\eta,2k+2})(t);x\big)\\ &\quad+\beta_n\big((1-\chi(t))(f-f_{\eta,2k+2})(t);x\big)\\ &= Q_1+Q_2.\end{aligned}$$

From $p=1$ and $p>1$ the following inequality holds, now it follows for Hölder's inequaqlity

$$\int_{a_2}^{b_2}\big|Q_1\big|^p dx \le \int_{a_2}^{b_2}\int_{a_3}^{b_3} K_n(t,x)\big|(f-f_{\eta,2k+2})(t)\big|^p dtdx.$$

Applying Fubini's theorem and Lemma 2.1, we obtain

$$\big|Q_1\big|_{L_p(I_2)} \le 2\|f-f_{\eta,2k+2}\|_{L_p(I_3)}.$$

Similarly, for all $p\ge 1$

$$\big|Q_2\big|_{L_p(I_2)} \le Cn^{-(k+1)}\|f-f_{\eta,2k+2}\|_{L_p[0,\infty)}.$$

By property of Steklov means [4], (Theorem 18.17) or [1], (pp.163-165), we obtain

$$E_1 \le C\big(\omega_{2k+2}(f,\eta,p,I_1)+n^{-(k+1)}\big\|f\big\|_{L_p[0,\infty)}\big).$$

Since $\big\|f_{\eta,2k+2}^{(2k+1)}\big\|_{BV(I_3)} = \big\|f_{\eta,2k+2}^{(2k+1)}\big\|_{L_1(I_3)}$, by Theorem 3.1 and property of Steklov means, for all $p\ge 1$, we obtain

$$\begin{aligned}E_2 &\le Cn^{-(k+1)}\big(\big\|f_{\eta,2k+2}^{(2k+2)}\big\|_{L_p(I_3)}+\big\|f_{\eta,2k+2}\big\|_{L_p[0,\infty)}\big)\\ &\le Cn^{-(k+1)}\big(\eta^{-(2k+1)}\omega_{2k+1}(f,\eta,p,I_1)+\big\|f\big\|_{L_p[0,\infty)}\big).\end{aligned}$$

Again using property of Steklov means

$$E_3 \le C\omega_{2k+2}(f,\eta,p,I_1).$$

Choosing $\eta=n^{-1/2}$ and estimates of E_1 to E_3, this leads to prove of (3.3) ∎

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