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ESSENTIAL RANDOM FIXED POINT SET OF RANDOM OPERATORS

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ABSTRACT. We obtain necessary and sufficient conditions for the existence of essential random fixed point of a random operator defined on a compact metric space. The structure of the set of essential random fixed points is also studied.

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1. INTRODUCTION

The last fifty years have seen a dramatic expansion in the theory of random operators. Random techniques have become crucial in diverse areas, from pure mathematics to applied sciences including biology, physics, chemistry, engineering, and of course, random methods provide a convenient way of modelling many problems arising from economic theory, see for example [13] and references mentioned. The study of fixed points of random operators of various types is a lively and fascinating disipline for research lying at the intersection of nonlinear analysis and probability theory. Recently many authors ([1], [2], [3], [7], [11], [12], [15], [20], [21]) had studied the existence and various applications of random fixed points of single and set valued random operators. Fort [4] introduced the notion of essential fixed point of a continuous map T on a compact metric space X. Afterwards Kinoshita [8], Neill [10], Strother [16], Jiang [5], Tan, Yu and Yuan [17], Prete, Iorio and Naimpally [14] and many other authors have studied the properties and applications of essential fixed points. This concept of essentiality for fixed points is a stability property which is quite analogous to the concept of stable values of an operator. It has been successfully used in studying Nash equilibrium points in game theory (see [6], [9], [18], [22] and [23]) and it is further expected that this theory has applications in differential equations. A continuous map T may has no essential fixed point even though the space X has the fixed point property [4] for T. The aim of this paper is to introduce the notion of essential random fixed point of a random map T defined on a compact metric space X. We obtained conditions for the existence of the essential random fixed point of T. It is also shown that the set of all essential random fixed points of T is closed.

2. PRELIMINARIES

We first review some concepts for the convenience of the reader and state the notations used throughout this paper. (X, d) stands for a complete separable metric space and let (Ω, Σ) be a measurable space (i.e., Σ is a sigma algebra of subsets of Ω) . A function $\xi : \Omega \to X$ is said to be $\Sigma-$*measurable* if for any open subset B of X, $\xi^{-1}(B) \in \Sigma$. A mapping $T : \Omega \times X \to X$ is said to be a *random map* if and only if for each fixed $x \in X$, the mapping $T(., x) : \Omega \to X$ is measurable. The random map $T : \Omega \times X \to X$ is *continuous* if for each $\omega \in \Omega$, the mapping $T(\omega, .) : X \to X$ is continuous. A measurable mapping $\xi : \Omega \to X$ is a *random fixed point* of the random map $T : \Omega \times X \to X$ if and only if it is the stochastic solution of random operator equation $T(\omega, x(\omega)) = x(\omega)$ for each $\omega \in \Omega$; equivalently, ξ is a measurable selector of fixed point set function E_T of random operator T defined by $E(\omega) = \{x \in X : x = T(\omega, x)\}$. Note that, deterministic solvability of random operator equation $T(\omega, x(\omega)) = x(\omega)$ implies stochastic solvability of random operator equation $T(\omega, x(\omega)) = x(\omega)$, if $E(\omega)$ has measurable selector. For this way of studying the problem of the existence of random fixed point of certain random operator and measurability of E_T for various random operators, we refer to [21]. A complete separable metric space X is said to have *random fixed point property*, if each continuous random map $T : \Omega \times X \to X$, has a random fixed point. We denote by $M(\Omega, X)$ the set of all measurable functions from Ω into a compact metric space X. Define $m : M(\Omega, X) \times M(\Omega, X) \to R$ by $m(\xi, \eta) = \sup\{d(\xi(\omega), \eta(\omega)) : \omega \in \Omega\}$. Then $(M(\Omega, X), m)$ is a compact metric space. We also denote by $CR(X)$ the set of all continuous random operators $T : \Omega \times X \to X$, where X is a compact metric space having the random fixed point property. We define

$$\rho(T, S) = \sup_{\omega\in\Omega}\{\sup\{d(T(\omega, x), S(\omega, x)) : x \in X\}\}.$$

Then $(CR(X), \rho)$ is a metric space.

3. ESSENTIAL RANDOM FIXED POINT

Let X and Y be two metric spaces. A multifunction $E : X \rightsquigarrow Y$ is said to be *upper semicontinuous [lower semicontinuous]* at $p \in X$ if given $\epsilon > 0$ there exists $\delta > 0$ such that $E(B(p;\delta)) \subset B(E(p);\epsilon)$ [$E(p) \subset B(E(x);\epsilon)$ for all $x \in B(p;\delta)$]. Here $B(x;r)$ denotes the open sphere centred at x with radius r.The multifunction E is called *continuous* if it is upper and lower semicontinuous.

Definition 3.1. Let $T \in CR(X)$ and ξ be a random fixed point of T. The measurable mapping $\xi : \Omega \longrightarrow X$ is called *essential* ξ is random fixed point of T and for any given $\epsilon > 0$ there exists a $\delta > 0$ such that : If $S \in CR(X)$ and $\rho(T,S) < \epsilon$, then S has a random fixed point in the set W, where $W = \{\eta : \eta \in M(\Omega, X) \text{ and } m(\xi,\eta) < \delta\}$.

We define a multifunction $F : (CR(X), \rho) \rightsquigarrow (M(\Omega, X), m)$ by defining $F(T)$ to be the set of all random fixed point of T, where $T \in CR(X)$.

Proposition 3.1. *Let $T \in CR(X)$. The function F is upper semicontinuous at T.*

Proof. Let $T \in CR(X)$ and suppose $\epsilon > 0$. Define

$$\delta = \left\{ \begin{array}{ll} 1 & \text{if } M(\Omega,X) = B(F(T);\epsilon) \\ \inf\limits_{\omega\in\Omega}\{d(\eta(\omega), T(\omega,\eta(\omega))) : \eta \in M(\Omega,X) - B(F(T);\epsilon)\}, & \text{otherwise.} \end{array} \right\}$$

Clearly $\delta > 0$. If $\delta = 1$, then the proof follows immediately. For the second case, let $\xi \in M(\Omega, X)$ such that $m(\xi, F(T)) \geq \epsilon$. Then for $S \in CR(X)$ with $\rho(T,S) < \delta$, we have ξ is not a random fixed point of S. Thus $F(B(T,\delta)) \subset B(F(T);\epsilon)$. Hence F is upper semicontinuous at T. ∎

Remark 3.1. Proposition 3.1 is a random analogue of a result of Wehausen [19].

Theorem 3.2. *Let $T \in CR(X)$. Each random fixed point of T is an essential random fixed point if and only if T is a point of continuity of F.*

Proof. Let each random fixed point of T be an essential random fixed point. Let $\epsilon > 0$ be given. For each $\xi \in F(T)$ there is a neighborhood V_ξ of T such that if $S \in V_\xi$ then S has a random fixed point in the set $\{\eta : \eta \in M(\Omega, X)$ with $m(\eta,\xi) < \frac{\epsilon}{2}\}$.

There exists a finite set of $\xi_1, \xi_2, ...\xi_n$ of points of $F(T)$ such that each point of $F(T)$ is within distance $\frac{\epsilon}{2}$ of some one of the ξ_k. Choose a neighborhood V of T in $CR(X)$ which is contained in the intersection of $V_{\xi_1}, V_{\xi_2}, ...V_{\xi_n}$. For $S \in V$ we have $F(T) \subset B(F(S);\epsilon)$. It implies that F is lower semicontinuous at T. This together with Proposition 3.1 imply that F is continuous at T.

Conversely, let F be continuous at T. Let $\xi \in F(T)$ and $\epsilon > 0$ be given. Now choose $\delta > 0$ such that if $\rho(T,S) < \delta$ for some $S \in CR(X)$, then $H(F(T), F(S)) < \epsilon$ [Here H is the Hausdorff metric on $M(\Omega, X)$ induced by the metric m]. Thus if $S \in CR(X)$ and $\rho(T,S) < \delta$, then S has a random fixed point in $B(\xi;\epsilon)$. Hence ξ is an essential random fixed point of T. ∎

There may exists $T \in CR(X)$ which have no essential random fixed point. It is of interest to explore the conditions under which a random operator will have essential random fixed point. Our next theorem gives conditions for the existence of an essential random fixed point.

Theorem 3.3. *Let $T \in CR(X)$. If T has a single random fixed point ξ, then ξ is an essential random fixed point of T.*

Proof. Let $\epsilon > 0$. Since F is upper semicontinuous at T, there exists a $\delta > 0$ such that if $F(B(T;\delta)) \subset B(F(T);\epsilon)$. Since $F(T)$ is singleton, let $F(T) = \{\xi\}$. Thus if $S \in CR(X)$ and $\rho(T,S) < \delta$,then $F(S) \subseteq F(B(T;\delta)$. Now any random fixed point of S lies in $B(\xi;\delta)$. It implies that ξ is an essential random fixed point of T. ∎

Theorem 3.4. *Let $T \in CR(X)$. The set $EF(T)$ of all essential random fixed point of T is closed.*

Proof. Let $\{\xi_n\}$ be the sequence of essential random fixed points of T and $\xi_n \longrightarrow \xi$ as $n \longrightarrow \infty$, then for every ω in Ω we have, $\xi_n(\omega) = T(\omega, \xi_n(\omega))$ for any n. Now continuity of T and separability of space X implies ξ is random fixed point of T. For any $\epsilon > 0$, we may choose $\delta > 0$ such that if $S \in CR(X)$ and $\rho(T,S) < \delta$, we obtain the sequence $\{\zeta_n\}$ of random fixed points of S in $B(\xi_n;\delta)$, Combining this fact with the convergence of $\{\xi_n\}$ to ξ, we obtain S has a random fixed point ζ in $B(\xi;\delta)$. Therefore ξ is an essential random fixed point of T. Hence $EF(T)$ is closed. ∎

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