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INVERSE PROBLEMS FOR PARABOLIC EQUATIONS

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ABSTRACT. Let $u_t - \nabla^2 u = f(x) := \sum_{m=1}^{M} a_m \delta(x - x_m)$ in $D \times [0, \infty)$, where $D \subset R^3$ is a bounded domain with a smooth connected boundary S, $a_m = const$, $\delta(x - x_m)$ is the delta-function. Assume that $u(x, 0) = 0$, $u = 0$ on S. Given the extra data $u(y_k, t) := b_k(t)$, $1 \leq k \leq K$, can one find M, a_m, and x_m? Here K is some number. An answer to this question and a method for finding M, a_m, and x_m are given.

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1. INTRODUCTION

Let $D \subset \mathbb{R}^3$ be a bounded domain with a smooth connected boundary S. Consider the problem

$$u_t - \nabla^2 u = f(x) = \sum_{m=1}^{M} a_m \delta(x - x_m) \qquad (x,t) \in D \times [0, \infty), \tag{1.1}$$

$$u(x,0) = 0, \qquad u|_S = 0, \tag{1.2}$$

where $M,\ a_m$ and x_m are not known, a_m are some numbers. Thus, we want to recover a special type of source from some extra data. The standard data from which the time-independent source f of general type can be recovered uniquely are the data $u(x,T)$, where $T > 0$ is an arbitrary fixed number. Let us first describe the solution to this problem, which is known, but we present this solution in the form useful for our purposes. Let $Lu = \nabla^2$ be the Dirichlet Laplacian in D, and let v_j be its normalized in $L^2(D)$ eigenfunctions:

$$Lv_j + \lambda_j v_j = 0 \quad \text{in } D, \quad v_j|_S = 0,$$

where $\lambda_j > 0$ are the eigenvalues, $\lim_{j\to\infty} \lambda_j = \infty$. The unique solution to problem (1.1)-(1.2) is

$$u(x,t) = \sum_{j=1}^{\infty} \frac{1 - e^{-\lambda_j t}}{\lambda_j} \sum_{p=1}^{P(j)} v_{jp}(x) c_{jp}, \tag{1.3}$$

where $P(j)$ is the multiplicity of the eigenvalue λ_j, $\{v_{jp}\}_{1\leq p\leq P(j)}$ is the orthonormal eigenbasis in the eigenspace corresponding to the eigenvalue λ_j, and $c_{jp} = (f, v_{jp})$, where (h,g) is the inner product in $L^2(D)$. If the numbers c_{jp} are known, then the unknown f can be uniquely recovered by equation (1), namely,

$$f = u_t - Lu,$$

where u is given by formula (1.3). Therefore if

$$u(x,T) = \sum_{j=1}^{\infty} \frac{1 - e^{\lambda_j T}}{\lambda_j} \sum_{p=1}^{P(j)} v_{jp}(x) c_{jp}$$

is known, then the coefficients

$$(u(x,T), v_{jp}) = \frac{1 - e^{\lambda_j T}}{\lambda_j} c_{jp}$$

can be calculated, then the numbers c_{jp} can be calculated, then $u(x,t)$ can be calculated by formula (1.3) for all $x \in D$ and all $t > 0$, and, finally, f can be calculated by equation (1.1) (see, e.g., [1], where other inverse problems for parabolic equations are also considered).

The aim of this note is to consider different data, the data which are often easier to measure in practice. Namely, we assume that the extra data are the functions $b_k(t)$, $1 \leq k \leq K$, where K is some number, and $b_k(t) := u(y_k, t)$ are known for all sufficiently large $t > 0$ at some points $y_k \in D$, which we can choose and at which we can measure the temperature at all times $t \geq 0$.

The inverse problem is:

Can one determine M, a_m and x_m from the data $b_k(t)$ known for all $t \geq 0$ and all $k = 1, 2, \dots K$?

How large should one choose K?

Our main result is an answer to these questions. Let us assume that

$$K := \max_j P(j) < \infty. \tag{1.4}$$

This assumption is valid not for all domains. For example, if D is a ball, then $P(j) = 2j + 1$, so that $P(j) \to \infty$ as $j \to \infty$. While generically one expects $K < \infty$, even $K = 1$, but for domains with certain symmetry the multiplicity of eigenvalues may be not uniformly bounded as a function of j, as the example of a ball shows.

For a general source $f(x)$, which is a function of three variables, x_1, x_2, x_3, intuitively one does not expect that unique recovery of f is possible from the data $\{b_k(t)\}_{1\le k\le K}$ consisting of finitely many functions of one variable t. However, for the special source f, defined in equation (1.1), it is possible to recover f, i.e., the number M, the M numbers a_m, and the M points (vectors) x_m from the above data provided that (1.4) holds.

Moreover, if $K = 1$, then one can determine an arbitrary source $f \in C_0^2(D)$ from *just one* data function $u(y_1, t)$ known for all sufficiently large $t > 0$ at a suitably chosen point y_1. Thus, if $P(j) = 1$ for all j, then one function $u(y_1, t)$ of one variable t determines a function f of three variables $x = (x_1, x_2, x_3)$ uniquely provided that $y_1 \in D$ is suitably chosen.

Indeed, if $P(j) = 1$ for all j, then

$$u(y_1, t) = \sum_{j=1}^{\infty} \frac{1 - e^{-\lambda_j}}{\lambda_j} v_j(y_1) c_j.$$

As $t \to \infty$, one can determine uniquely from the above data the numbers $v_j(y_1)c_j$ for all j. If y_1 is chosen so that $v_j(y_1) \neq 0$ for all j, then the numbers c_j are uniquely determined for all j. Consequently, the function

$$u(x, t) = \sum_{j=1}^{\infty} \frac{1 - e^{-\lambda_j t}}{\lambda_j} v_j(x) c_j$$

is uniquely determined for all $x \in D$ because the eigenfunctions v_j are known. Therefore, by equation (1.1), the source f is uniquely determined.

Theorem 1.1. *If (1.4) holds then one can choose points y_k so that the data $\{u(y_k, t)\}_{1\le k\le K, \forall t\ge 0}$ determine M, a_m, and x_m uniquely.*

We will outline a method for finding M, a_m, and x_m. The main tool in the proof are the following two lemmas.

Lemma 1.2. *If v_p, $1 \le p \le K$, is a linearly independent system of continuous functions, defined in a domain D, then there exist points $y_k \in D$, $1 \le k \le K$, such that the matrix $[v_p(y_k)]_{1\le p,k\le K}$ is nonsingular.*

Lemma 1.3. *The data $\{b_k(t)\}_{1\le k\le K, \forall t\ge 0}$ determine the numbers c_{jp} in (1.3) uniquely.*

Remark 1.1. Our results and proofs can be easily extended to the case of other selfadjoint boundary conditions, inhomogeneous boundary conditions, non-zero initial conditions, and a general selfadjoint second-order elliptic operators L.

In Section 2 proofs are given.

2. PROOFS

Proof of Lemma 1.2. If $K = 1$ then the conclusion of Lemma 1.2 is true, because if it fails, then $v_1(y) = 0$ for every $y \in D$, which contradicts the linear independence. Assume that Lemma 1.2 holds for all $k \le K$ and let us prove that it holds for $k \le K + 1$. Denote $y_{K+1} := y$. Assuming that Lemma 1.2 fails, one gets $\Delta_{K+1} := \det_{1\le p,k\le K+1} v_p(y_k) = 0$ for all $y_{K+1} = y$, and $\det_{1\le p,k\le K} v_p(y_k) := \Delta_K \neq 0$. A cofactor expansion of the determinant Δ_{K+1} yields:

$$0 = \Delta_{K+1} = \Delta_K v_{K+1}(y) + \sum_{p=1}^{K} v_p(y) A_p, \quad \forall y \in D, \tag{2.1}$$

where A_p are cofactors which do not depend on y. Because $\Delta_K \neq 0$ does not depend on y, equation (2.1) implies that $v_{K+1}(y)$ is a linear combination of $v_p(y), 1 \le p \le K$. This contradicts the linear independence of the system $v_1, \dots v_{K+1}$. This contradiction proves Lemma 1.2. ∎

Proof of Lemma 1.3. Using (1.3), one gets

$$u(y_k, t) = \sum_{j=1}^{\infty} \frac{1 - e^{-\lambda_j t}}{\lambda_j} \sum_{p=1}^{P(j)} v_{jp}(y_k) c_{jp}. \tag{2.2}$$

One has $\lambda_1 < \lambda_2 < \dots$, $\lim_{j\to\infty} \lambda_j = \infty$. Therefore the data

$$B_k(t) := u_t(y_k, t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{p=1}^{P(j)} v_{jp}(y_k) c_{jp} \tag{2.3}$$

allow one to find

$$b_{jk} := \sum_{p=1}^{P(j)} v_{jp}(y_k) c_{jp}$$

for all j and all $k = 1, 2, \dots K$. Namely:

$$b_{jk} = \lim_{t\to\infty} e^{\lambda_j t} [u_t(y_k, t) - \sum_{i=1}^{j-1} e^{-\lambda_i t} \sum_{p=1}^{P(i)} v_{ip}(y_k) c_{ip}]. \tag{2.4}$$

For a fixed j we have a linear algebraic system for finding the numbers c_{jp}:

$$b_{jk} = \sum_{p=1}^{P(j)} v_{jp}(y_k) c_{jp}. \tag{2.5}$$

We have assumed that $P(j) \le K < \infty$. By Lemma 1.2, we can choose y_k such that for any fixed j the determinant $\det v_{jp}(y_k) \neq 0$, so that the numbers c_{jp} are uniquely determined by linear algebraic system (2.5). Lemma 1.3 is proved. ∎

Proof of Theorem 1.1. From formulas (1.1) and (1.3) one gets:

$$\sum_{j=1}^{\infty} \sum_{p=1}^{P(j)} v_{jp}(x) c_{jp} = \sum_{m=1}^{M} a_m \delta(x - x_m), \tag{2.6}$$

so that

$$c_{jp} = \sum_{m=1}^{M} a_m v_{jp}(x_m). \tag{2.7}$$

If the numbers c_{jp} are known for all j and all p, $1 \le p \le P(j)$, then $u(x,t)$ is uniquely determined by formula (1.3), and then formula (1.1) determines uniquely M, a_m and x_m. Thus, Theorem 1.1 will be proved if we prove that all the numbers c_{jp} are uniquely determined by the data. But this is the conclusion of Lemma 1.3. Theorem 1.1 is proved. ∎

Remark 2.1. A concrete choice of y_k can be made for eigenfunctions known analytically. For example, if D is a cylinder $0 \leq \rho \leq 1,\ 0 \leq z \leq \pi$, then $v_{j1} = \kappa_{mns}\cos(m\phi)J_m(\nu_{mn}\rho)\sin(sz)$, where κ_{mns} is the norming constant, $J_m(\rho)$ is the Bessel function, ν_{mn} are its zeros, $s = 1, 2, \dots, m = 0, 1, 2, \dots$, (ρ, ϕ) are polar coordinates on the plane x_1, x_2, $z = x_3$, v_{j2} is defined similarly to v_{j1} with $\cos(m\phi)$ replaced by $\sin(m\phi)$, and $m = 1, 2, \dots$, the role of j is taken by the set $j = \{m, n, s\}$, $\lambda_j = \nu^2_{mn} + s^2$, and $K = 2$ in this example. The choice of y_k can be done in many ways. For example, choose $y_k = (\rho_k, \phi_k, z_k)$, $k = 1, 2$, so that $\sin[m(\phi_1 - \phi_2)] \neq 0$ for all $m = 1, 2, \dots$, $z_1 = z_2 = \zeta$ and $\sin(s\zeta) \neq 0$ for all $s = 1, 2, \dots$, $\rho_1 = \rho_2 = r$ and $J_m(\nu_{mn}r) \neq 0$ for all $m = 0, 1, 2 \dots$ and all $n = 1, 2, \dots$. Then the determinant used in the proof of Theorem 1.1 does not vanish.

Our arguments show that in this example, when $K = 2$, one can recover uniquely M, a_m, and x_m from two data functions $u(y_k, t)$, known for all $t > 0$ and for a suitably chosen points y_k, $k = 1, 2$.

In the proof of Theorem 1.1 we assume that the data are exact. The inverse problem under discussion is ill-posed: small perturbation of the data may threw the data out of the set of admissible data. For example, the solution $u(x, t)$ is infinitely differentiable (even analytic) with respect to t in the region $t > 0$, while the perturbed data $u(y_k, t)$ do not have this property, in general. The ill-posedness of the inverse problem is also seen in the algorithm for finding the quantities M, a_m, and x_m. For example, although it is possible to find a point ζ such that $\sin(s\zeta) \neq 0$ for all $s = 1, 2, \dots$, i.e., $s\zeta \neq q\pi$, where q is an integer, but the set of points $\frac{q\pi}{s}$ is dense in $(0, \pi)$ when s and q run through the set of all positive integers. Therefore, for large s, n, m, one has a small divisors problem.

Similarly one can consider the case when D is a box with sides $a_i\pi$, $i = 1, 2, 3$, where a_i are such that the equation $\lambda_j = \sum_{i=1}^{3} \frac{n_i^2}{a_i^2}$ has a unique solution in integers n_i. This happens if, for example, the numbers a_i are mutually incommensurate. In this case the equation

$$\sum_{i=1}^{3} \frac{n_i^2}{a_i^2} = \sum_{i=1}^{3} \frac{m_i^2}{a_i^2},$$

where n_i and m_i are integers, implies $n_i = m_i$. Consequently, $P(j) = 1$, and one extra data $u(y_1, t)$ determines uniquely the source. The eigenfunctions of the Dirichlet Laplacian in this case are known:

$$v_j := v_{n_1,n_2,n_3} = \mu_{n_1,n_2,n_3} \sin(\frac{n_1 x_1}{a_1}) \sin(\frac{n_2 x_2}{a_2}) \sin(\frac{n_3 x_3}{a_3}),$$

where μ_{n_1,n_2,n_3} are the known norming constants. Thus, in this case one can choose y_1, such that $v_j(y_1) \neq 0$ for all j, i.e., for all n_1, n_2 and n_3.

After this paper has been finished, the author learned of a recent paper "Identification of a point source in a linear advection-dispersion-reaction equation: application to a pollution source problem" by A.Badia et. al., Inverse Problems, 21, (2005), 1121-1136. In this paper a simpler problem with one-dimensional heat operator is considered and different arguments are used to investigate this simpler problem. If the heat operator is one-dimensional, then the difficulty with the multiplicity $K > 1$ of the eigenfunctions does not arise.

REFERENCES

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