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SOME GENERALIZATIONS OF STEFFENSEN'S INEQUALITY

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ABSTRACT. Some generalization of Steffensen's inequalities are given.

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1. INTRODUCTION

Steffensen's inequality reads as follows:
Assume that two integrable functions $f(t)$ and $g(t)$ are defined on the interval (a,b), that $f(t)$ never increases and that $0 \le g(t) \le 1$ in (a,b). Then

$$\int_{a-\lambda}^{b} f(t)\,dt \le \int_{a}^{b} f(t)\,g(t)\,dt \le \int_{a}^{a+\lambda} f(t)\,dt, \tag{1.1}$$

$where\ \lambda = \int_a^b g(t)\,dt.$

A generalization via new proof of Steffensen's inequality has been given by Bellman in [2] (see also [3], p. 41; [4], p. 11). It can be stated as follows:

Let $f(t)$ *be a nonnegative and monotone decreasing function in* $[a,b]$ *and* $f \in L^p[a,b]$ *and let* $g(t) \ge 0\ in\ [a,b]\ and\ \int_a^b g(t)^q\,dt \le 1, where\ p > 1\ and\ 1/p + 1/q = 1.\ Then$

$$\left(\int_a^b f(t)\,g(t)\,dt\right)^p \le \int_a^{a+\lambda} f(t)^p\,dt, \tag{1.2}$$

$where\ \lambda = \left(\int_a^b g(t)\,dt\right)^p.$

Bellman's result, is incorrect, as has been pointed out by Godunova and Levin [6] , where a generalization is made for $p \le 1$. Notice that for $p \ge 1$, a similar result to inequality (1.2) is given in [6]. Pečarić [7], however, through some modification, gives the following generalization.

Theorem 1.1. *Let $f : [0,1] \to R$ be a nonnegative and nonincreasing function and let $g : [0,1] \to R$ be an integrable function such that $0 \le g(t) \le 1$ for all $t \in [0,1]$. If $p \ge 1$, then*

$$\left(\int_0^1 f(t)\,g(t)\,dt\right)^p \le \int_0^{\lambda} f(t)^p\,dt, \tag{1.3}$$

where $\lambda = \left(\int_0^1 g(t)\,dt\right)^p.$

The aim of this paper is to give a correct version of Bellman's result, as well as, some other new results.

2. A GENERALIZATION AND SOME NEW RESULTS

Theorem 2.1. *Let $f, g, h : [a,b] \to R$ be nonnegative, f is nonincreasing, g and h are integrable functions satisfying:*

1. $\lambda^{1-1/p} g(t) \le h(t)$, *where* $\lambda = \left(\int_a^b g(t)\,dt\right)^p$ $p > 0$.

2. $\int\limits_{k_2(\lambda)}^{b} h(t)\,dt \le \lambda \le \int\limits_{a}^{k_1(\lambda)} h(t)\,dt$, *for some* k_1, k_2 *functions of* λ *with* $a \le k_1(\lambda)$; $k_2(\lambda) \le b$.

Then, we have

$$\int\limits_{k_2(\lambda)}^{b} f^p(t)\,h(t)\,dt \le \lambda^{1-1/p} \int\limits_{0}^{b} f^p(t)\,g(t)\,dt \le \int\limits_{a}^{k_1(\lambda)} f^p(t)\,h(t)\,dt. \tag{2.1}$$

Proof. Observe that

$$\begin{aligned}
&\int\limits_{a}^{k_1(\lambda)} f^p(t)\,h(t)\,dt - \lambda^{1-1/p} \int\limits_{a}^{b} f^p(t)\,g(t)\,dt \\
={}& \int\limits_{a}^{k_1(\lambda)} f^p(t)\left[h(t) - \lambda^{1-1/p} g(t)\right] dt - \lambda^{1-1/p} \int\limits_{k_1(\lambda)}^{b} f^p(t)\,g(t)\,dt \\
\ge{}& f^p(k_1(\lambda)) \int\limits_{a}^{k_1(\lambda)} \left[h(t) - \lambda^{1-1/p} g(t)\right] dt - \lambda^{1-1/p} \int\limits_{k_1(\lambda)}^{b} f^p(t)\,g(t)\,dt \\
={}& f^p(k_1(\lambda)) \left[\int\limits_{a}^{k_1(\lambda)} h(t)\;dt - \lambda^{1-1/p} \left(\int\limits_{a}^{b} - \int\limits_{k_1(\lambda)}^{b}\right) g(t)\right] - \lambda^{1-1/p} \int\limits_{k_1(\lambda)}^{b} f^p(t)\,g(t)\,dt \\
={}& f^p(k_1(\lambda)) \left[\int\limits_{a}^{k_1(\lambda)} h(t)\;dt - \lambda + \lambda^{1-1/p} \int\limits_{k_1(\lambda)}^{b} g(t)\,dt\right] - \lambda^{1-1/p} \int\limits_{k_1(\lambda)}^{b} f^p(t)\,g(t)\,dt \\
={}& f^p(k_1(\lambda)) \left[\int\limits_{a}^{k_1(\lambda)} h(t)\,dt - \lambda\right] + \lambda^{1-1/p} \int\limits_{k_1(\lambda)}^{b} \left[f^p(k_1(\lambda)) - f^p(t)\right] g(t)\,dt \\
\ge{}& 0,
\end{aligned}$$

which proves the concerned inequality in (2.1). The left inequality can be done similarly , and therefore is omitted. ∎

Theorem 2.2. *Let* $f, g, h : [a, b] \to R$ *be nonnegative,* f *is nonincreasing,* g *and* h *are integrable functions satisfying* $\lambda^{1-1/p} g(t) \le h(t)$ *, where* $\lambda = \left(\int\limits_{a}^{b} g(t)\,dt\right)^p$*, p>0 .*

1. *If* $\lambda \le \int\limits_{a}^{k(\lambda)} h(t)\,dt$ *, for a function* k *of* λ *with* $a \le k(\lambda) \le b$*, then*

$$\left(\int\limits_{a}^{b} f(t)\,g(t)\,dt\right)^p \le \int\limits_{a}^{k(\lambda)} f^p(t)\,h(t)\,dt, p \ge 1. \tag{2.2}$$

2. *If* $\lambda \geq \int\limits_{k(\lambda)}^{b} h(t)\,dt$, $0 < p \leq 1$, *then, with the same assumptions for* k,

$$\left(\int_a^b f(t)\,g(t)\,dt\right)^p \geq \int_{k(\lambda)}^{b} f^p(t)\,h(t)\,dt. \tag{2.3}$$

Proof. Since, by Hölder's inequality,

$$\int_a^b f(t)\,g(t)\,dt = \int_a^b f(t)\,[g(t)]^{\frac{1}{p}}\,[g(t)]^{\frac{p-1}{p}}\,dt$$

$$\leq \left(\int_a^b f^p(t)\,g(t)\,dt\right)^{\frac{1}{p}} \left(\int_a^b g(t)\,dt\right)^{\frac{p-1}{p}}$$

hence,

$$\left(\int_a^b f(t)\,g(t)\,dt\right)^p \leq \lambda^{1-1/p} \int_a^b f^p(t)\,g(t)\,dt.$$

The proof of (2.2) follows from (2.1).
The proof of (2.3) is similar and therefore omitted. ∎

Corollary 2.3. *Let* $f, g : [a,b] \to R$ *be nonnegative,* f *is nonincreasing,* g *is integrable function satisfying* $\lambda^{1-1/p} g(t) \leq c$, $where\ \lambda = \left(\int\limits_a^b g(t)\,dt\right)^p$, $p \geq 1$. *Then we have*

$$\left(\int_a^b f(t)\,g(t)\,dt\right)^p \leq c \int_a^{a+\lambda/c} f^p(t)\,dt. \tag{2.4}$$

Proof. Follows from Theorem 2.2, by choosing $h(t) = c$ and $k(\lambda) = a + \lambda/c$. The definition of $k(\lambda)$ is correct as $\lambda^{1-1/p} g(t) \leq c \Rightarrow \lambda \leq c(b-a)$. ∎

Corollary 2.4. *Let* $f, g : [a,b] \to R$ *be nonnegative,* f *is nonincreasing,* g *is integrable function satisfying* $\lambda^{1-1/p} g(t) \leq c$, *where* $\lambda = \left(\int\limits_a^b g(t)\,dt\right)^p$, $0 < p \leq 1$. *Then we have*

$$\left(\int_a^b f(t)\,g(t)\,dt\right)^p \geq c \int_{b-\lambda/c}^{b} f^p(t)\,dt. \tag{2.5}$$

Proof. Follows from Theorem 2.2, by putting $h(t) = c$ and $k(\lambda) = b - \lambda/c$. ∎

Corollary 2.5. ([7]) *Let* $f : [0,1] \to R$ *be a nonnegative and nonincreasing function and let* $g : [0,1] \to R$ *be an integrable function such that* $0 \leq g(x) \leq 1$ $(\forall x \in [0,1])$. *If* $p \geq 1$, *then*

$$\left(\int_0^1 f(t)\,g(t)\,dt\right)^p \leq \int_0^{\lambda} f^p(t)\,dt, \tag{2.6}$$

where

$$\lambda = \left(\int_0^1 g(t)\ dt \right)^p .$$

Proof. Follows from Corollary 2.3, by putting $a = 0, b = c = 1$. ∎

Corollary 2.6. *Let $f(t, u) : [a, b] \times [a, b] \to R$ be nonnegative, nonincreasing with respect to both t and u. Let $g, h : [a, b] \to R$ be nonnegative integrable functions satisfying*

$$\lambda_1^{1-1/p} g(t) \le c_1, \lambda_2^{1-1/p} h(u) \le c_2,$$

where

$$\lambda_1 = \left(\int_a^b g(t)\ dt \right)^p \quad , \quad \lambda_2 = \left(\int_a^b h(t)\, dt \right)^p , p \ge 1.$$

Then, we have

$$\left(\int_a^b \int_a^b f(t, u)\, g(t)\, h(u) dt du \right)^{p^2} \le (c_1 c_2) (\lambda_1 \lambda_2)^{p-1} \left(\int_a^{a+\lambda_1/c_1} \int_a^{a+\lambda_2/c_2} f^{p^2}(t, u)\, du dt \right). \tag{2.7}$$

Proof. We have succesively

$$\begin{aligned}
&\int_a^b \int_a^b f(t, u)\, g(t)\, h(u)\, dt du \\
&= \int_a^b \left(\int_a^b f(t, u)\, g(t)\, dt \right) h(u)\, du \le \int_a^b \left(c_1 \int_a^{a+\lambda_1/c_1} f^p(t, u)\, dt \right)^{1/p} h(u)\, du \\
&= c_1^{1/p} \int_a^b \left(\int_a^{a+\lambda_1/c_1} f^p(t, u)\, dt \right)^{1/p} [h(u)]^{1/p} [h(u)]^{\frac{p-1}{p}}\ du \\
&\le c_1^{1/p} \left(\int_a^b \int_a^{a+\lambda_1/c_1} f^p(t, u)\, h(u)\, dt du \right)^{1/p} \left(\int_a^b h(u)\, du \right)^{\frac{p-1}{p}}
\end{aligned}$$

$$\begin{aligned}
&= c_1^{1/p}\left[\int_a^{a+\lambda_1/c_1}\left(\int_a^b f^p(t,u)\,h(u)\,du\right)dt\right]^{1/p}\lambda_2^{\frac{p-1}{p^2}}\\
&\leq c_1^{1/p}\left[\int_a^{a+\lambda_{1_1}/c_1}\left(c_2\int_a^{a+\lambda_2/c_2} f^{p^2}(t,u)\;du\right)^{1/p}dt\right]^{1/p}\lambda_2^{\frac{p-1}{p^2}}\\
&\leq c_1^{1/p}c_2^{1/p^2}\left[\left(\int_a^{a+\lambda_1/c_1}\int_a^{a+\lambda_2/c_2} f^{p^2}(t,u)\,dudt\right)^{1/p}\left(\int_a^{a+\lambda_1/c_1} dt\right)^{\frac{p-1}{p}}\right]^{1/p}\lambda_2^{\frac{p-1}{p^2}}\\
&= c_1^{1/p}c_2^{1/p^2}\lambda_2^{\frac{p-1}{p^2}}\left(\frac{\lambda_1}{c_1}\right)^{\frac{p-1}{p^2}}\left(\int_a^{a+\lambda_1/c_1}\int_a^{a+\lambda_2/c_2} f^{p^2}(t,u)\,dtdu\right)^{1/p^2}\\
&= (c_1c_2)^{\frac{1}{p^2}}(\lambda_1\lambda_2)^{\frac{p-1}{p^2}}\left(\int_a^{a+\lambda_1/c_1}\int_a^{a+\lambda_2/c_2} f^{p^2}(t,u)\,dtdu\right)^{\frac{1}{p^2}}.
\end{aligned}$$

The desired inequality (2.7) is thus obtained. ∎

Lemma 2.7. *Define*

$$f(x) := \int_0^x \frac{u^{\frac{\lambda}{2}-1}}{(1+u)^{\lambda}}du, \lambda > 0, x \geq 1.$$

Then

$$f(x) \leq \beta\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(1-\frac{1}{2}x^{\lambda/2}\right) \tag{2.8}$$

for any $x \geq 1$, where β is the Euler's Beta function.

Proof. Observe that

$$\begin{aligned}
f(x) &= \left(\int_0^\infty - \int_x^\infty\right)\frac{u^{\frac{\lambda}{2}-1}}{(1+u)^{\lambda}}du\\
&= \beta\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) - x^{\frac{\lambda}{2}}y^{\frac{\lambda}{2}}\int_0^y \frac{u^{\frac{\lambda}{2}-1}}{(1+u)^{\lambda}}du \quad \left(y=\frac{1}{x}\right)\\
&= \beta\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) - x^{\frac{\lambda}{2}}g(y)
\end{aligned}$$

, say.

Now

$$
\begin{aligned}
g'(y) &= \frac{y^{\lambda-1}}{(1+y)^{\lambda}} - \left(\frac{\lambda}{2}\right) y^{\frac{\lambda}{2}-1} \int_0^y \frac{u^{\frac{\lambda}{2}-1}}{(1+u)^{\lambda}} du \\
&\leq \frac{y^{\lambda-1}}{(1+y)^{\lambda}} - \left(\frac{\lambda}{2}\right) \frac{y^{\frac{\lambda}{2}-1}}{(1+y)^{\lambda}} \int_0^y u^{\frac{\lambda}{2}-1} du \\
&= 0
\end{aligned}
$$

This shows that g is nonincreasing which implies $g(y) \geq g(1)$ for $y \leq 1$. Therefore

$$
f(x) \leq \beta\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) - x^{\frac{\lambda}{2}} g(1) = \beta\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\left(1 - \frac{1}{2} x^{\frac{\lambda}{2}}\right),
$$

and the inequality (2.8) is proved. ∎

Corollary 2.8. *Let* $f(t,u) : [0,1] \times [0,1] \to R$ *be nonnegative, nonincreasing with respect to both* t *and* u*. Let* $g, h : [0,1] \to R$ *be nonnegative integrable functions such that* $0 \leq g(t) \leq 1, 0 \leq h(t) \leq 1$*. Define*

$$
\lambda_1 = \left(\int_0^1 g(t)\, dt\right)^p, \lambda_2 = \left(\int_0^1 h(t)\, dt\right)^p, p > 1.
$$

Then we have, for $\lambda > 0$,

$$
\begin{aligned}
(2.9) \qquad & \left(\int_0^1 \int_0^1 \frac{f(t,u)\, g(t)\, h(u)}{(t+u)^{\lambda}} dt du\right)^p \\
\leq\ & (\lambda_1 \lambda_2)^{p-1} \beta\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left(\int_0^{\lambda_1} t^{(1-\lambda/2)p/q - \lambda/2} \left(1 - \frac{1}{2} t^{-\frac{\lambda}{2}}\right) f^{\frac{p^3}{2}}(t,0)\, dt\right)^{1/p} \\
\times\ & \left(\int_0^{\lambda_2} t^{(1-\lambda/2)q/p - \lambda/2} \left(1 - \frac{1}{2} t^{-\frac{\lambda}{2}}\right) f^{\frac{p^2 q}{2}}(0,t)\, dt\right)^{1/q},
\end{aligned}
$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By Corollary 2.6, with $a = 0$, $c_1 = c_2 = 1, \frac{1}{p} + \frac{1}{q} = 1$, we may state that

$$
\begin{aligned}
\left(\int_0^1 \int_0^1 \frac{f(t,u)\, g(t)\, h(u)}{(t+u)^{\lambda}} dt du\right)^p &\leq (\lambda_1 \lambda_2)^{p-1} \int_0^{\lambda_1} \int_0^{\lambda_2} \frac{f^{p^2}(t,u)}{(t+u)^{\lambda}} dt du \\
= (\lambda_1 \lambda_2)^{p-1} \int_0^{\lambda_1} \int_0^{\lambda_2} & \frac{f^{p^2/2}(t,u) \left(\frac{u^{1/p}}{t^{1/q}}\right)^{\left(\frac{\lambda}{2}-1\right)}}{(t+u)^{\lambda/p}} \frac{f^{p^2/2}(t,u) \left(\frac{t^{1/q}}{u^{1/p}}\right)^{\left(\frac{\lambda}{2}-1\right)}}{(t+u)^{\lambda/q}} dt du
\end{aligned}
$$

$$
\begin{aligned}
&\le (\lambda_1\lambda_2)^{p-1}\left(\int_0^{\lambda_1}\int_0^{\lambda_2}\frac{f^{p^3/2}(t,u)\left(\frac{u}{t^{p/q}}\right)^{\left(\frac{\lambda}{2}-1\right)}}{(t+u)^{\lambda}}dtdu\right)^{1/p}\\
&\times\left(\int_0^{\lambda_1}\int_0^{\lambda_2}\frac{f^{p^2q/2}(t,u)\left(\frac{t}{u^{q/p}}\right)^{\left(\frac{\lambda}{2}-1\right)}}{(t+u)^{\lambda}}dtdu\right)^{1/q}\\
&= (\lambda_1\lambda_2)^{p-1}M^{1/p}N^{1/q},
\end{aligned}
$$

say. Observe by Lemma 2.7 that

$$
\begin{aligned}
M &\le \int_0^{\lambda_1} f^{\frac{p^3}{2}}(t,0)\,t^{(1-\lambda/2)p/q-\lambda/2}dt\int_0^{\lambda_2}\frac{\left(\frac{u}{t}\right)^{\frac{\lambda}{2}-1}\frac{1}{t}}{\left(1+\frac{u}{t}\right)^{\lambda}}du\\
&= \int_0^{\lambda_1} t^{(1-\lambda/2)p/q-\lambda/2}f^{\frac{p^3}{2}}(t,0)\,dt\int_0^{\frac{\lambda_2}{t}}\frac{u^{\frac{\lambda}{2}-1}}{(1+u)^{\lambda}}du\\
&\le \int_0^{\lambda_1} t^{(1-\lambda/2)p/q-\lambda/2}f^{\frac{p^3}{2}}(t,0)\,dt\int_0^{\frac{1}{t}}\frac{u^{\frac{\lambda}{2}-1}}{(1+u)^{\lambda}}du\\
&\le \beta\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\int_0^{\lambda_1} t^{(1-\lambda/2)p/q-\lambda/2}f^{\frac{p^3}{2}}(t,0)\left(1-\frac{1}{2}t^{-\frac{\lambda}{2}}\right)dt.
\end{aligned}
$$

Similarly,

$$
N \le \beta\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\int_0^{\lambda_2} u^{(1-\lambda/2)q/p-\lambda/2}f^{\frac{q^3}{2}}(0,u)\left(1-\frac{1}{2}u^{-\frac{\lambda}{2}}\right)du.
$$

This completes the proof. ∎

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