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# WEAK SOLUTION FOR HYPERBOLIC EQUATIONS WITH A NON-LOCAL CONDITION

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ABSTRACT. In this paper, we study hyperbolic equations with a non-local condition. We prove the existence and uniqueness of weak solutions, using energy inequality and the density of the range of the operator generated by the problem.

*Key words and phrases:* Non-local condition, A priori estimate, Weak solution.

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## 1. INTRODUCTION

Various problems arising in heat conduction [5], [6], [8], chemical engineering [7], thermo-elasticity [14], and plasma physics [12] can be reduced to the non-local problems with integral boundary conditions. This type of boundary value problems has been investigated in [1], [3], [5], [6], [7], [8], [14], [16] for parabolic equations and in [2], [11], [15] for hyperbolic equations. Boundary value problems with integral conditions constitute a very interesting and important class of problems. For comments on their importance, we refer the reader to the above papers. This paper is a continuation of the mentioned papers, our goal is to prove the existence and uniqueness of weak solutions for one-dimensional wave equations with a non-local boundary condition.

Consider the equation

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} + \frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} + U = F(t,x), \tag{1.1}$$

in the rectangular domain $\Omega = (0,T) \times (0,1)$.

To equation (1.1) we attach the initial conditions

$$U(0,x) = \Phi(x), \tag{1.2}$$

$$U_t(0,x) = \Psi(x), \tag{1.3}$$

Dirichlet boundary condition

$$U(t,1) - U(t,0) = 0, \tag{1.4}$$

and the non-local boundary condition

$$\int_0^1 U(t,x)dx = 0. \tag{1.5}$$

We assume that $\Phi(x), \Psi(x) \in L_2(0,1)$ are known functions and satisfy the compatibility conditions

$$\Phi(1) - \Phi(0) = 0, \Psi(1) - \Psi(0) = 0 \text{ and } \int_0^1 \Phi(x)dx = \int_0^1 \Psi(x)dx = 0.$$

Such equations become more complicated when studied with a non-local boundary condition. For that, we reduce (1.1)-(1.5) to an equivalent problem.

**Lemma 1.1.** *Problem (1.1)-(1.5) is equivalent to the following problem*

$$(PR) \qquad \begin{cases} \frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} + \frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} + U &= F(t,x), \\ U(0,x) &= \Phi(x), \\ U_t(0,x) &= \Psi(x), \\ U(t,1) - U(t,0) &= 0, \\ U_x(t,1) - U_x(t,0) &= -\int_0^1 F(t,x)dx. \end{cases}$$

*Proof.* Let $U(t,x)$ be a solution of (1.1)-(1.5). Integrating (1.1) with respect to $x$ over $(0,1)$, and taking in account (1.5)-(1.5), we obtain

$$U_x(t,1) - U_x(t,0) = -\int_0^1 F(t,x)dx. \tag{1.6}$$

Let now $U(t,x)$ be a solution of $(PR)$, we are required to show that

$$\int_0^1 U(t,x)dx = 0, \forall t \in (0,T).$$

For this end we integrate again (1.1) with respect to $x$ and obtain

$$\frac{d^2}{dt^2}\int_0^1 U(t,x)dx + \frac{d}{dt}\int_0^1 U(t,x)dx + \int_0^1 U(t,x)dx = 0, \forall t \in (0,T),$$

by virtue of the compatibility conditions

$$\int_0^1 U(0,x)dx = 0 \text{ and } \int_0^1 U_t(0,x)dx = 0,$$

we get

$$\int_0^1 U(t,x)dx = 0.$$

∎

Introduce now the new unknown function $u(t,x) = U(t,x) - w(t,x)$, where

$$w(t,x) = \frac{x(1-x)}{2}\int_0^1 F(t,x)dx.$$

Then $(PR)$ is transformed into

$$(\mathrm{Pr}) \qquad \begin{cases} \ell u \equiv \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} + u = f(t,x), \\ \ell_0 u \equiv u(0,x) = \varphi(x), \\ \ell_1 u \equiv u_t(0,x) = \psi(x), \\ u(t,1) - u(t,0) = 0, \\ u_x(t,1) - u_x(t,0) = 0, \end{cases}$$

where

$$\begin{aligned} f(t,x) &= F(t,x) - \frac{x(1-x)}{2}\int_0^1 F_{tt}(t,x)dx \\ &\qquad - \frac{x(1-x)}{2}\int_0^1 F_t(t,x)dx + \frac{(x^2+x-3)}{2}\int_0^1 F(t,x)dx, \\ \varphi(x) &= \Phi(x) - \frac{x(1-x)}{2}\int_0^1 F(0,x)dx, \end{aligned}$$

and

$$\psi(x) = \Psi(x) - \frac{x(1-x)}{2}\int_0^1 F_t(0,x)dx.$$

## 2. ABSTRACT FORMULATION OF THE BOUNDARY PROBLEM

We consider the problem $(\mathrm{Pr})$ as a solution of the operational equation

$$Lu = \mathcal{F},$$

where $L = (\ell, \ell_0, \ell_1)$ with domain of definition $D(L)$ consisting of functions $u$ belonging to the Sobolev space $H^2(\Omega)$ and satisfying the boundary conditions of $(\mathrm{Pr})$. The operator $L$ is considered from $E$ to $W$, where $E$ is the Banach space consisting of functions $u \in L_2(\Omega)$ having the finite norm

$$||u||_E^2 = \sup_{0 \le \tau \le T} \left[ \int_0^1 (u^2 + u_t^2 + u_x^2)(\tau, x) dx \right] + \int_\Omega u_t^2(t, x)\, dtdx < \infty$$

and satisfying the boundary conditions of $(\mathrm{Pr})$ and $W$ is the Hilbert space obtained by completion of $L_2(\Omega) \times H^1(0,1) \times L_2(0,1)$ with respect to the norm

$$||\mathcal{F}||_W^2 = \int_\Omega f^2(t,x) dtdx + \int_0^1 \left[\varphi^2(x) + \varphi'^2(x)\right] dx + \int_0^1 \psi^2(x) dx.$$

The inner product in $W$ is defined by:

$$(\mathcal{F}, Z)_W = (f, w)_{0,\Omega} + (\varphi, w_0)_{1,(0,1)} + (\psi, w_1)_{0,(0,1)}\,,$$

where $\mathcal{F} = (f, \varphi, \psi)$, $Z = (w, w_0, w_1)$ belongs to $W$ and $(\cdot,\cdot)_{0,\Omega}$ , $(\cdot,\cdot)_{0,(0,1)}$ and $(\cdot,\cdot)_{1,(0,1)}$ denote the inner product in $L_2(\Omega)$ , $L_2(0,1)$ and $H^1(0,1)$ respectively.

## 3. A PRIORI ESTIMATES

Here we establish an energy inequality which ensures the uniqueness of the weak solution.

**Theorem 3.1.** *For the problem* $(\mathrm{Pr})$, *we have*

(3.1) $$||u||_E \le c_0 ||Lu||_W, \forall u \in D(L),$$

*where* $c_0 > 0$ *is independent on* $u$.

*Proof.* Define the operator

$$Mu = 2u_t,$$

and consider the scalar product $(\ell u, Mu)_{0,\Omega^\tau}$, where $0 \le \tau \le T$, and $\Omega^\tau = (0,\tau) \times (0,1)$. Employing integration by parts, we obtain

$$\begin{aligned} 2\left(\ell u, u_t\right)_{0,\Omega^\tau} &= \int_0^1 \left[u^2 + u_t^2 + u_x^2\right](\tau, x) dx + 2\int_{\Omega^\tau} u_t^2(t,x)\, dtdx \\ &\quad + 2\int_{\Omega^\tau} \left[u_x(t,x) \times u_t(t,x)\right] dtdx - \int_0^1 u^2(0,x) dx \\ &\quad - \int_0^1 u_t^2(0,x) dx - \int_0^1 u_x^2(0,x) dx \\ &\quad - 2\int_0^\tau \left[u_x(t,1) \times u_t(t,1) - u_x(t,0) \times u_t(t,0)\right] dt. \end{aligned}$$

Taking into account the initial and boundary conditions of $(\mathrm{Pr})$, we see that

(3.2) $$\begin{aligned} \int_0^1 [u^2 + u_t^2 \;\; + \;\; u_x^2](\tau, x) dx + 2\int_{\Omega^\tau} u_t^2(t,x)\, dtdx &= 2\left(\ell u, u_t\right)_{0,\Omega^\tau} \\ &\quad - 2\int_{\Omega^\tau} \left[u_x(t,x) \times u_t(t,x)\right]\, dtdx \\ &\quad + \int_0^1 \psi^2(x) dx + \int_0^1 \left[\varphi^2(x) + \varphi'^2(x)\right] dx. \end{aligned}$$

We now apply the $\varepsilon$-inequality $2|ab| \leq \varepsilon a^2 + \frac{1}{\varepsilon}b^2, \varepsilon > 0$ to the first and second terms on the right-hand side of (3.2) and employ Gronwall's Lemma (see e.g. Lemma 3.4 [13]), we get the inequality

$$\int_0^1 (u^2 + u_t^2 + u_x^2)(\tau, x)dx + \int_{\Omega^\tau} u_t^2(t,x)\, dtdx \leq c_0^2 |||Lu||_W^2 \tag{3.3}$$

where

$$||Lu||_W^2 = \int_\Omega f^2(t,x)dtdx + \int_0^1 \left[\varphi^2(x) + \varphi'^2(x)\right] dx + \int_0^1 \psi^2(x)dx.$$

Now, as the right-hand side of (3.3) is independent of $\tau$, replacing the left-hand side by its upper bound with respect to $\tau$ in the interval $(0,T)$, we obtain the desired inequality. This completes the proof. ∎

## 4. EXISTENCE AND UNIQUENESS

For existence of the weak solution for $(\mathrm{Pr})$, we shall prove that the range $\Re(L)$ is dense in $W'$, where $W' = E^* \times H^{-1}(0,1) \times L_2(0,1)$, $W \subset W'$ and $E^*$ is the dual space of $E$ with respect to the canonical bilinear form $\langle u, v\rangle, u \in E$ and $v \in E^*$, which is the extension by continuity of the bilinear form $(u,v)$, where $u \in L_2(\Omega)$ and $v \in E$. First consider $u \in D_0(L)$ where $D_0(L) = \{u \in D(L) \;/\; \ell_0 u = \ell_1 u = 0\}$, then $(\mathrm{Pr})$ becomes

$$(\mathrm{Pr})_0 \qquad \begin{cases} \ell u \equiv \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} + u = f(t,x), \\ \ell_0 u \equiv u(0,x) = 0, \\ \ell_1 u \equiv u_t(0,x) = 0, \\ \quad u(t,1) - u(t,0) = 0, \\ \quad u_x(t,1) - u_x(t,0) = 0. \end{cases}$$

Our aim here is to prove existence of weak solutions of $(\mathrm{Pr})_0$. The proof is based on an energy inequality and the density of the range of the operators generated by the studied problem.

Analogous to the problem $(\mathrm{Pr})_0$, we consider its dual problem. We denote by $\ell^*$ the formal dual of the operator $\ell$, which is defined with respect to the inner product in the space $L_2(\Omega)$ using

$$(\ell u, v) = (u, \ell^* v) \quad \text{for all } u, v \in C_0^2(\Omega), \tag{4.1}$$

where $(\cdot,\cdot)$ stands for the inner product in $L_2(\Omega)$.

Let

$$(\mathrm{Pr})_0^* \qquad \begin{cases} \ell^* v \equiv \dfrac{\partial^2 v}{\partial t^2} - \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x} + v = g(t,x), \\ \ell_0^* v \equiv v(T,x) = 0, \\ \ell_1^* v \equiv v_t(T,x) = 0, \\ \quad v(t,1) - v(t,0) = 0, \\ \quad v_x(t,1) - v_x(t,0) = 0. \end{cases}$$

The solution of $(\mathrm{Pr})_0$ will be considered as a solution of the operational equation:

$$\ell u = f, \;\; u \in D(\ell), \tag{4.2}$$

and the solution of $(\mathrm{Pr})_0^*$ as a solution of the operational equation:

$$\ell^* v = g, \; v \in D(\ell^*). \tag{4.3}$$

To solve the equation (4.2) for every $f \in E^*$, we establish the following existence and uniqueness theorems of weak solutions for problems $(\mathrm{Pr})_0$ and $(\mathrm{Pr}\,.)_0^*$.

**Theorem 4.1.** *For the problem* $(\mathrm{Pr})_0$*(resp.* $(\mathrm{Pr})_0^*$*) we have*

$$||u||_E \leq c_1||\ell u||_{E^*}, \forall u \in E \tag{4.4}$$

$$||v||_E \leq c_1^*||\ell^* v||_{E^*}, \forall v \in E^*. \tag{4.5}$$

*where the constants* $c_1 > 0$ *and* $c_1^* > 0$ *are independent on* $u$ *and* $v$*.*

*Proof.* An application of (3.1) gives (4.4) for $u \in D(\ell)$. For $u \in E$, we use the regularization operators of Friedreichs [9], [10] to conclude that (4.4) holds true. ∎

**Theorem 4.2.** *For all functions* $f \in E^*$ *(resp.* $g \in E^*$*) there exists one and only one weak solution of the problem* $(\mathrm{Pr})_0$ *(resp.* $(\mathrm{Pr})_0^*$*).*

*Proof.* We mention that from the inequality (4.4) follows immediately the uniqueness of the solutions. It also ensures the closure of the range set $\Re(\ell)$ of the operator $\ell$.

An application of the Theorem II.19 in [4] with the inequality (4.5) give $\Re(\ell) = E^*$. ∎

The second part of Theorems 4.1-4.2 can be proved in a similar way by using the operator $M^*v = 2v_t$.

Now we need the following Lemma.

**Lemma 4.3.** *If* $w \in E$ *and for all* $u \in D_0(L)$*, we have*

$$\langle \ell u, w \rangle_{E^*,E} = 0,$$

*then* $w = 0$.

*Proof.* It sufficient to show that $\Re(\ell)$ is dense in $E^*$. The fact that $\Re(\ell) = E^*$; results directly from Theorem 4.1. It remains to prove that the inclusion $\overline{\Re(\ell)} \subseteq \Re(\ell)$. Indeed, let $\{f_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence in the space $E^*$, which consists of elements of set $\Re(\ell)$. Then it corresponds to a sequence $\{u_k\}_{k\in\mathbb{N}} \subseteq D(\ell)$ such that: $\ell u_k = f_k, k \in \mathbb{N}$.

From the inequality (3.1), we conclude that the sequence $\{u_k\}$ is also a Cauchy sequence in the space $E$ and converges to an element $u$ in $E$, we define the element

$$\ell u = f (f = \lim_{k\to\infty} f_k).$$

This establishes the inclusion $\overline{\Re(\ell)} \subseteq \Re(\ell)$. ∎

To this end, we show that the following existence theorem:

**Theorem 4.4.** *The range* $\Re(L)$ *of the operator* $L$ *is dense in* $W'$.

*Proof.* Since $W'$ is a Hilbert space, the density of $\Re(L)$ in $W'$ is equivalent to the orthogonality of the vector $Z = (w, w_0, w_1) \in W'$ to the set $\Re(L)$, i.e. the equality

$$\langle \ell u, w \rangle_{E^*,E} + (\varphi, w_0)_{1,(0,1)} + (\psi, w_1)_{0,(0,1)} = 0, \tag{4.6}$$

implying that $Z = 0$. In particular, put $u \in D_0(L)$ in (4.6). Then, $\langle \ell u, w \rangle_{E^*,E} = 0$ and we conclude by Lemma 4.3 that $w = 0$, so it follows from (4.6) that $(\varphi, w_0)_{1,(0,1)} + (\psi, w_1)_{0,(0,1)} = 0$. But since the range of the trace operators $\ell_0$ and $\ell_1$ are dense in $H^1(0,1)$ and $L_2(0,1)$ respectively, thus $w_0 = w_1 = 0$. Consequently, $Z = 0$. Hence $\overline{\Re(L)} = W'$. ∎

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